Discussion Session for Open Problems

(Chairman: C. D. Minda)

The following open problems have been discussed in a special session chaired by C. D. Minda.

S. S. Bhoosnurmath, India:

Problem: Nevanlinna theory studies the properties of meromorphic functions in the plane. Not much is known about the properties of the meromorphic functions (using Nevanlinna theory) in the unit disc $\mathbb{D}$.

Using the Poincaré metric for $\mathbb{D}$, is it possible to get more information about the meromorphic functions in $\mathbb{D}$?

D. A. Herron, USA:

Problem 1:

(a) When is a domain $\Omega \subset \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$ quasiconformally equivalent to a uniform domain?

(b) Is there a proper subdomain $\Omega \subset \mathbb{R}^n$ with (Ω, k) Gromov hyperbolic, but $f(\Omega)$ not uniform for any quasiconformal map of f?

Problem 2: In 1962 Gehring and Hayman proved that there exists an absolute constant K with the property that for any simply connected plane domain Ω,

$$(*) \qquad \forall\, x, y \in \Omega: \quad \ell(\gamma) \leq K\ell(\alpha)$$

where γ is the hyperbolic geodesic in Ω joining x and y, and α can be any curve joining x and y in Ω.

(a) What is the best such constant K?

(b) Describe the (non-simply connected) plane domains which possess property $(*)$.

(c) What about $\mathbb{R}^n$?

W. Ma and C. D. Minda, USA:

Suppose $\Omega \subset \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\lambda_\Omega(z)\,|dz|$ is the hyperbolic metric on Ω. Define

$$\mu_\Omega(z) = \frac{\lambda_\Omega(z)\,|dz|}{\lambda_{\mathbb{D}}(z)\,|dz|}.$$

Problem: What is the geometric characterization for the hyperbolic convexity of $\log \mu_\Omega(z(s))$?

A real-valued function $F(w)$ on Ω is hyperbolic convex if $F(w(s))$ is convex along every hyperbolic geodesic with hyperbolic parameter s.

S. S. Miller, USA:

Problem 1: If $p, p + zp'$ are univalent in the unit disc, then

$$z \prec p(z) + zp'(z) \;\Rightarrow\; h_1(z) = \frac{z}{2} \prec p(z),$$

where $h_1(z) = z/2$ is the best subordinant. How does this generalize, for instant what is the best subordinant $h_n(z)$ so that

$$z \prec p(z) + zp'(z) + \cdots + z^n p^{(n)}(z) \;\Rightarrow\; h_n(z) \prec p(z)?$$

Problem 2: Find conditions on the function $g\colon \mathbb{R}^2 \times \mathbb{C} \to \mathbb{C}$ such that

$$f(z) = z + \int_0^1 \int_0^1 g(r, s, z)\, dr\, ds = z + a_2 z^2 + \cdots$$

is starlike, convex, etc.

A. K. Mishra, India:

Problem: With a fixed rational number n, $f \in S^*(\alpha)$ and $0 \le \alpha < 1$.

$$\begin{aligned}(f(z))^{-n} &= (z + a_2 z^2 + a_3 z^3 + \cdots)^{-n}, \quad z \in \mathbb{D} = \{z \in \mathbb{C} :\, |z| < 1\}\\ &= \frac{1}{z^n} + \frac{a_{-n+1}^{(-n)}}{z^{n+1}} + \frac{a_{-n+2}^{(-n)}}{z^{n+2}} + \cdots a_0^{(-n)} + \cdots .\end{aligned}$$

Find the sharp estimate for $|a_{-n+\theta}^{(-n)}|$, $\theta = 1, 2, 3, \ldots$.

As a consequence if

$$f^{-1}(w) = w + A_1 w + A_2 w^2 + \cdots \quad |w| < 1/4,$$

then the sharp bound for $|A_n|$ can be obtained. In particular sharp bounds for $|A_2|, |A_3|$ are already established (Krzyż, Libera, Zlokiewicz 1976).

T. Nayak, India:

Problem 1: Let γ be an arc in $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and $f\colon \mathbb{C} \to \mathbb{C}$ be a transcendental meromorphic function such that $f^n(z) \neq \infty$ for any $n \in \mathbb{N}$ when $z \in \gamma, |f'(z)| > 1$ for all $z \in \gamma$ and $f(\gamma) \subseteq \gamma$. Then

(i) Is it always true that γ is unbounded?
(ii) Is it always true that γ is not rectifiable, if it is bounded?

(iii) What kind of set (uniform perfect, smooth, etc) γ will be?

Motivation for Problem 1: If $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable and $|f'(x)| > 1\ \forall\, x \in \mathbb{R}$, then for any (a,b), $-\infty < a < b < \infty$ and $x_0 \in \mathbb{R}, \exists\, n \in \mathbb{N}$ such that $x_0 \in f^n\big((a,b)\big)$.

Problem 2: Given a transcendental meromorphic or entire function f, can it be checked whether for any $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$, $\lambda_1 f$ and $\lambda_2 f$ are quasiconformally conjugate? That is whether there exists a quasiconformal map $\phi\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that

$$\lambda_1 f \circ \phi(z) = \phi \circ \lambda_2 f(z) \quad \forall\, z \in \hat{\mathbb{C}}.$$

A special case of Problem 2: It is known that for $\lambda > 3$ and $f(z) = \tanh(e^z)$,

(a) there exists a unique $a_\lambda \in \mathbb{C}$ such that $\lambda f(a_\lambda) = a_\lambda$ and $|\lambda f'(a_\lambda)| < 1$ such that

$$\begin{aligned} f(\lambda f) = \text{ Fatou set of } \lambda f : &= \big\{z \in \mathbb{C} :\ \{f^n(z)\}_{n\geq 0} \text{ is normal}\big\} \\ &= \{z \in \mathbb{C} :\ \lim_{n\to\infty} f^n(z) = a_\lambda\}. \end{aligned}$$

(b) $J(\lambda f) \equiv$ Julia set of $\lambda f := \hat{\mathbb{C}} \setminus F(\lambda f)$ has Lebesgue measure zero.
(c) $J(\lambda f)$ has infinitely many maximally connected subsets and $f(\lambda f)$ is connected.
(d) λf does not assume the values λ and $-\lambda$ in the complex plane.

Can it be shown that $\lambda_1 f$ and $\lambda_2 f$ are quasiconformally conjugate at least $\lambda_1, \lambda_2 > 3$?

Possible Approach: It is known that $\lambda_1 f$ and $\lambda_2 f$ are quasiconformally conjugate in neighbourhoods of $a\lambda_1$ and $a\lambda_2$, where $\lambda_i f(a_{\lambda_i}) = a_{\lambda_i}$ and $|\lambda_i f'(a_{\lambda_i})| < 1$ for $i = 1, 2$.

By using pulling back technique of Beltrami forms and Measurable Riemann Mapping Theorem, can this quasiconformal map be extended to the whole of $\hat{\mathbb{C}}$?

Problem 3: Let

$$S = \{A \subset \mathbb{C} :\ A \text{ is perfect and has empty interior}\}$$

and

$$S' = \{B \in S :\ B \text{ is connected}\}.$$

Let us call $B_1, B_2 \in S'$ quasiconformally equivalent if there exists a quasiconformal map $\phi\colon B_1 \to B_2$.

Can all the equivalence classes of S' be identified? Can all the equivalence classes of some subclass of S' be identified?

J. Patel, India:

Problem: Let $p = 2, 3, \dots$ and $0 \le \alpha < p$. Define

$$\mathcal{S}_p^*(\alpha) = \Big\{ f(z) = z^p + a_{p+1}z^{p+1} + \cdots \text{ is analytic in } |z| < 1 \\ \text{and } \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \text{ in } |z| < 1 \Big\}$$

Let $\mathcal{K}_p(\alpha)$ denote the class of p-valently convex functions of order α.

It has been shown by Nunokawa that

$$\mathcal{K}_p(0) \not\subset \mathcal{S}_p^*(\beta), \ \beta > 0.$$

On the other hand, MacGregor has shown that

$$\mathcal{K}(\alpha) \subseteq \mathcal{S}^*(\beta), \quad 0 \le \alpha < 1, \ \beta = 0,$$

where $\mathcal{K}(\alpha) := \mathcal{K}_1(\alpha)$ and similarly for $\mathcal{S}^*(\beta)$.

If $0 < \alpha < \frac{p-1}{2}$, then find a sharp value of $\beta > 0$ such that

$$\mathcal{K}_p(\alpha) \subset \mathcal{S}_p^*(\beta).$$

Note that the last relation is known for α satisfying $\frac{p-1}{2} \le \alpha < p$.

K. Pearce, USA:

Problem: Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}(\mathbb{D})$ denote the class of all normalized functions that are analytic in the unit disk $\mathbb{D}$. Define

$$K = \{f \in \mathcal{A}(\mathbb{D}) : f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ f(\mathbb{D}) \text{ is convex}\}.$$

For $\frac{1}{2} < r < 1$, let $\ell_f(r)$ be the linear measure of the circle $|z| = r$ which lies outside of $f(\mathbb{D})$, that is,

$$\ell_f(r) = 2\pi r - m\big(f(\mathbb{D}) \cap \{|z| = r\}\big).$$

Let $\ell_K(r) = \sup_{f\in K} \ell_f(r)$. Then find

(i) $\ell_K(r)$;
(ii) the function $f \in K$ which is extremal for (i).

C. M. Pokhrel, Nepal:

Problem 1: Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Define

$$\mathcal{A} = \{f : f \text{ is holomorphic in } \mathbb{D}\}$$

and

$$C\{\phi\} = \{f : f \text{ is convex in the direction of } e^{i\phi}\}.$$

A domain $D \subset \mathbb{C}$ is said to be convex in one direction if $D \cap L_\phi$ is either empty or connected.

A function $g \in \mathcal{A}$ is said to be in the class DCP (direction convexity preserving) if $g * f \in C(\phi)$ for all $f \in C(\phi)$.

For $\lambda > 0$ and $z \in \mathbb{D}$ we define

$$V_\lambda(z) = \frac{\lambda z}{\lambda + 1}\, {}_2F_1(1, 1-\lambda; 2+\lambda; -z).$$

Then $V_\lambda \in DCP$ for $\lambda \geq \frac{1}{2}$ (proved).

Find a necessary condition that

$$\overline{CO}\left\{V_\lambda(z) : \lambda_k \geq \frac{1}{2}\right\} \subseteq DCP.$$

One sufficient condition has been obtained by Chinta Mani Pokhrel.

Problem 2: Define

$$SDCP = \{f : e^{i\alpha} f \in DCP, \forall\, \alpha \in \mathbb{R}\}.$$

Find the condition that

$$g \tilde{*} h \in K_h \;\forall\, h \in C_h \quad \text{and} \quad g \in SDCP,$$

where K_h is the class of convex harmonic functions, C_h is the class of close to convex harmonic functions. The symbol $\tilde{*}$ is defined as follows. If $h = f + \bar{k}$ is harmonic, then

$$g \tilde{*} h = (g * f) + \overline{(g * k)},$$

where $*$ is the usual Hadamard product.

S. Ponnusamy and S. K. Sahoo, India:

A domain $D \subset \mathbb{R}^n$ is said to be *uniform* if for any pair $x, y \in D$ there exist a rectifiable path $\gamma \subset D$ joining x and y; and a constant c (independent of x and y) such that

1. $\ell(\gamma) \leq c\,|x - y|$; and
2. $\min\{\ell(\gamma[x,z]), \ell(\gamma[z,y])\} \leq c\, d(z, \partial D)$ for all $z \in \gamma$.

Here $\gamma[x,z]$ means the part of γ from x to z and $d(z, \partial D)$ denotes the minimum distance from z to the boundary ∂D of D.

In general, the union of two uniform domains may or may not be uniform. There are examples for both the facts. The same is true for the case of intersection as well. So it is natural to raise a problem as follows:

Problem: Let A and B be two uniform domains in $\mathbb{R}^n$. Then find necessary and sufficient conditions so that $A \cup B$ (or $A \cap B$) is uniform.

M. Vuorinen, Finland:

Problem: Let $\Omega \subset \mathbb{C}$ be a domain and $W(\Omega)$ its circular width (cf. Sugawa's talk), and let T be a linear mapping of the plane (in the sense of linear algebra). Does there exist a $C \in [1, \infty)$ such that

$$\frac{1}{C} \leq W(T(\Omega))/W(\Omega) \leq C?$$