On classes of functions related to starlike functions with respect to symmetric conjugate points defined by a fractional differential operator

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- Definitions
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Starlike functions with respect to symmetrical points

ST_s Sakaguchi 1959

$$Re\,\frac{zf'(z)}{f(z)-f(-z)} > 0, \quad \text{for } z \in E.$$

Starlike functions with respect to symmetrical conjugate points

ST_{sc} El-Ashwah and Tomas 1987

$$Re \frac{zf'(z)}{f(z) - \overline{f(-\bar{z})}} > 0, \quad \text{for } z \in E.$$

λ-Starlike functions with respect to 2m-symmetric conjugate points

$ST_m(\lambda)$. Al-Amiri et al 1995

$$Re\,\frac{(1-\lambda)zf'(z)+\lambda z\left(zf'(z)\right)'}{(1-\lambda)f_m(z)+\lambda z f'_m(z)} > 0, \quad \text{in } E, \text{for } \lambda \geq 0,$$

$$f_m(z) = \frac{1}{2m}\sum_{k=0}^{m-1}\left[\omega^{-k} f\left(\omega^k z\right) + \omega^k \overline{f(\omega^k \bar{z})}\right]$$

$$\omega = \exp\left(2\pi i/(m)\right).$$

linear fractional differential operator $D_{\lambda}^{n,\alpha}$

Al-Oboudi and Al-Amoudi 2008

$$D_{\lambda}^{0,0} f(z) = f(z), \lambda \geq 0$$

$$D_{\lambda}^{1,\alpha} f(z) = (1-\lambda)\Omega^{\alpha} f(z) + \lambda z(\Omega^{\alpha} f(z))', \; 0 \leq \alpha < 1$$

$$= D_{\lambda}^{\alpha} f(z)$$

$$D_{\lambda}^{2,\alpha} f(z) = D_{\lambda}^{\alpha}\left(D_{\lambda}^{1,\alpha} f(z)\right)$$

$$\vdots$$

$$D_{\lambda}^{n,\alpha} f(z) = D_{\lambda}^{\alpha}\left(D_{\lambda}^{n-1,\alpha} f(z)\right), n \in N$$

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

$$\Omega^{\alpha} f(z) = \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} f(z), \qquad \alpha \neq 2,3,4.$$

$$= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} a_k z^k$$

$$= \varphi(2, 2-\alpha; z) * f(z)$$

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$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\alpha}} dt, \quad 0 \le \alpha < 1,$$

$$D_{\lambda}^{n,\alpha} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} (1+\lambda(k-1)) \right)^{n} a_k z^k .$$

$$D_{\lambda}^{n,\alpha} f(z) = [\underbrace{(\varphi(2,2-\alpha;z) * g_{\lambda}(z) * \ldots * \varphi(2,2-\alpha;z) * g_{\lambda}(z))}_{n-times}] * f(z)$$

$$g_{\lambda}(z) = \frac{z-(1-\lambda)z^2}{(1-z)^2} = z + \sum_{k=2}^{\infty} [1+\lambda(k-1)] z^k$$

Definitions

$$D_\lambda^{n,\alpha} f_m(z) = \frac{1}{2m}\sum_{k=0}^{m-1}[\omega^{-k}\ D_\lambda^{n,\alpha} f(\omega^k z)\omega^k + \omega^k \overline{D_\lambda^{n,\alpha} f(\omega^k \bar{z})}]$$

$$z(\ D_\lambda^{n,\alpha} f_m(z))' = \frac{1}{2m}\sum_{k=0}^{m-1}[\omega^{-k}\ z(D_\lambda^{n,\alpha} f(\omega^k z))' + \omega^k \overline{z(D_\lambda^{n,\alpha} f(\omega^k \bar{z}))'}]$$

$$D_\lambda^{n,\alpha} f_m(\omega^j z) = \omega^j D_\lambda^{n,\alpha} f_m(z), \qquad D_\lambda^{n,\alpha} f_m(\bar{z}) = \overline{D_\lambda^{n,\alpha} f_m(z)}$$

$$\omega = \exp\left(2\pi i/(m)\right).$$

Definition 1 $ST_{m,\lambda}^{n,\alpha}(h)$

$$\frac{z\left(D_{\lambda}^{n,\alpha} f(z)\right)'}{D_{\lambda}^{n,\alpha} f_m(z)} \prec h(z), \qquad z \in E,$$

h is a convex function in E, with $h(0) = 1$

n $\in N_0 = N \cup \{0\}, m \in N$

$0 \leq \alpha < 1$ and $\lambda \geq 0$

Definition 2 $k^{n,\alpha}_{m,\lambda}(h)$

$$\frac{z\left(D^{n,\alpha}_{\lambda} f(z)\right)'}{D^{n,\alpha}_{\lambda} g_m(z)} \prec h(z), \quad z \in E,$$

for some $g \in ST^{n,\alpha}_{m,\lambda}(h)$

h is a convex function in E, with $h(0) = 1$

$n \in N_0 = N \cup \{0\}, m \in N$

$0 \leq \alpha < 1$ and $\lambda \geq 0$

Special Cases

1. For $n = 0,\ m = 1, ST^{0,\alpha}_{1,\lambda}(h) \equiv ST_c(h)$ Ravichandran

2. For $n = 1,\ m = 1$, and $h(z) = \frac{1+z}{1-z}, ST^{1,\alpha}_{1,\lambda}\left(\frac{1+z}{1-z}\right) \equiv ST_c(\lambda)$ Radha

3. For $n = 1, \lambda = 1,\ \alpha = 0, m = 1$, and $h(z) = \frac{1+z}{1-z}, ST^{1,0}_{1,1}\left(\frac{1+z}{1-z}\right) \equiv ST_c$

El-Ashawah and Thomas

4. For $n = 1, \alpha = 0$ and $h(z) = \frac{1+z}{1-z},\ ST^{0,0}_{m,\lambda}\left(\frac{1+z}{1-z}\right) \equiv ST_m(\lambda)$.

Al-Amiri et al

5. For $n=1, \alpha=0, m=1$, and $h(z)=\frac{1+z}{1-z}$, $ST_{1,\lambda}^{1,0}\left(\frac{1+z}{1-z}\right) \equiv ST_c(\lambda)$.

Radha

6. For $n=0$ or $\lambda=0$ and $h(z)=\frac{1+z}{1-z}$, $ST_{m,\lambda}^{0,\alpha}\left(\frac{1+z}{1-z}\right) \equiv ST_{m,0}^{n,\alpha}\left(\frac{1+z}{1-z}\right) \equiv ST_m(0)$

Al-Amiri et al

7. For $n=0$ and $h(z)=\frac{1+(1-2\beta)z}{1-z}$, $ST_{m,\lambda}^{0,\alpha}\left(\frac{1+(1-2\beta)z}{1-z}\right) \equiv ST_c^{m,\beta}$,

Kasi and Ravichandran

8. For $n = 1,\ \lambda = 1,\ \alpha = 0$ and $h(z) = \frac{1+z}{1-z}$, $ST_{m,1}^{1,0}\left(\frac{1+z}{1-z}\right) \equiv ST_m(1)$

Al-Amiri et al

9. For $n = 1, \alpha = 0$, , and $h(z) = \frac{1+z}{1-z}$, $K_{m,\lambda}^{1,0}\left(\frac{1+z}{1-z}\right) \equiv K_m(\lambda)$

Al-Amiri et al

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10. For $n = 1, \alpha = 0$, $m = 1$, and $h(z) = \frac{1+z}{1-z}$, $K_{1,\lambda}^{1,0}\left(\frac{1+z}{1-z}\right) \equiv K_c(\lambda)$

Radha

Lemmas

- **Lemma1**

S. S. Miller and P. T. Mocanu,

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\,dt.$$

$$zf'(z)/f(z) \prec h(z), \qquad Re\, h(z) + c > 0 \text{ in } E.$$

$$\frac{zF'(z)}{F(z)} \prec q(z) \prec h(z),$$

$$q + \frac{zq'(z)}{q(z)+c} = h(z).$$

Lemma 2

St. Ruscheweyh,

Let f and g, respectively, be in the classes CV and ST

$F \in A,$

$$\frac{f(z) * g(z)F(z)}{f(z) * g(z)} \in \overline{co}\, F(E), \quad z \in E.$$

Lemma 3

S. S. Miller and P. T. Mocanu,

$$Re\, P(z) > 0 \qquad p(0) = h(0)$$

$$p(z) + P(z) z p'(z) \prec h(z),$$

$$\Rightarrow \quad p(z) \prec h(z).$$

Lemma 4

St. Ruscheweyh, and T. Sheil-Small,

Let f and g be starlike functions of order ½

so is $f * g$

Main Theorem

$f \in ST_{m,\lambda}^{n,\alpha}(h)$ $\lambda > 0$ $h(0) = 1, \overline{h(\bar{z})} = h(z)$

$$\Rightarrow \quad \frac{z\left(D_{\lambda}^{n,\alpha} f_m(z)\right)'}{D_{\lambda}^{n,\alpha} f_m(z)} \prec h(z).$$

Moreover if $Re[h(z) + \frac{1}{\lambda} - 1] > 0$, in E

$$\Rightarrow \quad \frac{z\left(D_{\lambda}^{n-1,\alpha}(\Omega^{\alpha} f_m(z)\right)'}{D_{\lambda}^{n-1,\alpha}(\Omega^{\alpha} f_m(z))} \prec q(z) \prec h(z),$$

$$q + \frac{zq'(z)}{q(z) + \frac{1}{\lambda} - 1} = h(z), \qquad q(0) = 1.$$

The key of the proof

$$D_\lambda^{n,\alpha} f_m(z) = (1-\lambda)\, D_\lambda^{n-1,\alpha}\big(\Omega^\alpha f_m(z)\big) + \lambda z\big(D_\lambda^{n-1,\alpha}(\Omega^\alpha f_m(z))\big)',$$

$$\Longrightarrow \quad D_\lambda^{n-1,\alpha}\big(\Omega^\alpha f_m(z)\big) = \frac{1}{\lambda z^{\frac{1}{\lambda}-1}} \int_0^z t^{\frac{1}{\lambda}-2} D_\lambda^{n,\alpha} f_m(t)\, dt.$$

Corollary 1

$$f \in ST^{n,\alpha}_{\lambda,m}(h), Reh(z) > 0,$$

$$\Rightarrow \quad D^{n,\alpha}_{\lambda} f_m \in ST,$$

$\Rightarrow$ $D^{n,\alpha}_{\lambda} f$ is a close to convex function

Inclusion Relations

- $ST^{n+1,\alpha}_{\lambda,m}(h) \subset ST^{n,\alpha}_{\lambda,m}(h),$
- $ST^{n,\alpha}_{m,\lambda}(h) \subset ST^{n,\mu}_{m,\lambda}(h), \qquad 0 \le \mu \le \alpha < 1$

Theorem 2

Let $f \in ST_{\lambda,m}^{n+1,\alpha}(h)$,

$Re\, h(z) > 0$, and $\overline{h(\bar{z})} = h(z)$

$\Rightarrow$ $f \in ST_{\lambda,m}^{n,\alpha}(h)$

Lemma 5

Let $\Omega^{\alpha} f \in ST_{\lambda,m}^{n,\alpha}(h)$

$Re\, h > 0$, and $\overline{h(\bar{z})} = h(z)$

$f \in ST_{\lambda,m}^{n,\alpha}(h)$

Proof of Lemma 5

$$\Omega^{\alpha} f \in ST_{\lambda,m}^{n,\alpha}(h), Re\, h > 0,$$

$$D_{\lambda}^{n,\alpha}(\Omega^{\alpha} f_m(z)) \in ST$$

$$D_{\lambda}^{n,\alpha} f_m(z) = \varphi(2-\alpha, 2; z) * D_{\lambda}^{n,\alpha}(\Omega^{\alpha} f_m(z)),$$

Where $\varphi(2-\alpha, 2; z) \in CV$.

$$z(D_{\lambda}^{n,\alpha} f(z))' = \varphi(2-\alpha, 2; z) * z(D_{\lambda}^{n,\alpha}(\Omega^{\alpha} f(z)))'$$

$$\frac{z\left(D_\lambda^{n,\alpha} f(z)\right)'}{D_\lambda^{n,\alpha} f_m(z)} = \frac{\varphi(2-\alpha,2;z) * \left(z\left(D_\lambda^{n,\alpha}\left(\Omega^\alpha f(z)\right)\right)' / D_\lambda^{n,\alpha}\left(\Omega^\alpha f_m(z)\right)\right) D_\lambda^{n,\alpha}\left(\Omega^\alpha f_m(z)\right)}{\varphi(2-\alpha,2;z) * D_\lambda^{n,\alpha}\left(\Omega^\alpha f_m(z)\right)}$$

$$\in \overline{Co}\left(\frac{z\left(D_\lambda^{n,\alpha}(\Omega^\alpha f)\right)'}{D_\lambda^{n,\alpha}(\Omega^\alpha f_m)}\right)(\mathrm{E}) \subseteq \mathrm{h}(\mathrm{E})$$

Remark 1

n=1, $\alpha = 0$, in Theorem 2.2,

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Corollary

$$ST_{m,\lambda}^{n+1,\alpha}\left(\frac{1+z}{1-z}\right) \subset ST_{m,\lambda}^{n,\alpha}\left(\frac{1+z}{1-z}\right) \subset \cdots \subset ST_{m,\lambda}^{0,\alpha}\left(\frac{1+z}{1-z}\right) = ST_m(0),$$

Theorem 3

Let $0 \leq \mu \leq \alpha < 1$, and let $Re h(z) > \frac{1}{2}$

$$ST_{\lambda,m}^{n,\alpha}(h) \subset ST_{\lambda,m}^{n,\mu}(h).$$

Proof

$$D_\lambda^{n,\mu} f(z) = [\underbrace{(\varphi(2,2-\mu;z) * g_\lambda(z)\,)\ldots(\varphi(2,2-\mu;z) * g_\lambda(z)\,)}_{n-times}] * f(z)$$

$$= \underbrace{[(\varphi(2-\alpha,2-\mu;z) * \ldots * \varphi(2-\alpha,2-\mu;z))]}_{n-times} * D_\lambda^{n,\alpha} f(z).$$

$$z(D_\lambda^{n,\mu} f(z))' = \underbrace{[(\varphi(2-\alpha,2-\mu;z) * \ldots * (\varphi(2-\alpha,2-\mu;z)\ \ ]}_{n-times} * z(D_\lambda^{n,\alpha} f(z))'$$

$$D_\lambda^{n,\mu} f_m(z) = \underbrace{[(\varphi(2-\alpha,2-\mu;z) * \ldots * \varphi(2-\alpha,2-\mu;z))]}_{n-times} * D_\lambda^{n,\alpha} f_m(z).$$

$$\varphi(2-\alpha,2-\mu;z) \in ST\left(\frac{1}{2}\right)$$

$$\underbrace{[(\varphi(2-\alpha,2-\mu;z) * \dots * (\varphi(2-\alpha,2-\mu;z)\ ]}_{n-times} \in ST\left(\frac{1}{2}\right).$$

$$D_\lambda^{n,\mu} f_m(z) \in ST\left(\frac{1}{2}\right)$$

$$\frac{z(D_\lambda^{n,\mu} f(z))'}{D_\lambda^{n,\mu} f_m(z)}$$

$$= \frac{\underbrace{[(\varphi(2-\alpha,2-\mu;z) * \dots * (\varphi(2-\alpha,2-\mu;z)\ ]}_{n-times} * (z(D_\lambda^{n,\alpha} f(z))'/D_\lambda^{n,\alpha} f_m(z))D_\lambda^{n,\alpha} f_m(z)}{\underbrace{[(\varphi(2-\alpha,2-\mu;z) * \dots * \varphi(2-\alpha,2-\mu;z))]}_{n-times} * D_\lambda^{n,\alpha} f_m(z)}$$

$$\in \overline{co}\left(\frac{z\left(D_\lambda^{n,\alpha} f\right)'}{D_\lambda^{n,\alpha} f_m}(E)\right) \subseteq h(E).$$

Theorem 4

$$f \in ST_{\lambda,m}^{n,\alpha}(h), Reh(z) > 0$$

φ is a convex functions with real coefficients in E

$$f * \varphi \in ST_{\lambda,m}^{n,\alpha}(h)$$

Corollary 2

$$ST^{n,\alpha}_{\lambda,m}(h), \quad Reh(z) > 0$$

is invariant under the following operators

$$F_1(z) = \int_0^z \frac{f(\xi)}{\xi}\, d\xi = (f * \varphi_1)(z),$$

$$F_2(z) = \frac{2}{z}\int_0^z f(\xi)\, d\xi = (f * \varphi_2)(z),$$

$$F_3(z) = \frac{2}{z}\int_0^z \frac{f(\xi) - f(x\xi)}{\xi - x\xi}\, d\xi, -1 \leq x < 1$$

$$= (f * \varphi_3)(z),$$

$$F_4(z) = \frac{1+c}{z^c}\int_0^z \xi^{c-1} f(\xi)\, d\xi, \qquad c > -1$$

$$= (f * \varphi_4)(z),$$

$$\varphi_1(z) = \sum_{k=1}^{\infty} \frac{1}{k} z^k = -\log(1-z),$$

$$\varphi_2(z) = \sum_{k=1}^{\infty} \frac{2}{k+1} z^k = \frac{-2[z+\log(1-z)]}{z},$$

$$\varphi_3(z) = \sum_{k=1}^{\infty} \frac{1-x^k}{k(1-x)} = \frac{1}{1-z} \log \frac{1-xz}{1-z}, -1 \le x < 1,$$

$$\varphi_4(z) = \sum_{k=1}^{\infty} \frac{1+c}{k+c} k, \qquad c > -1.$$

Remark 2

$$f(z) = \frac{z}{(1-z)^2} \ and \ \varphi(z) = \frac{z}{(1-iz)}.$$

It is clear that $f \in ST_{1,\lambda}^{0,\alpha}\left(\frac{1+z}{1-z}\right)$

and φ is convex

but $f * \varphi$ does not belong to $ST_{1,\lambda}^{0,\alpha}\left(\frac{1+z}{1-z}\right)$

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Thank You

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