

Starlikeness and convexity of generalized Bessel functions

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Introduction

Inequalities

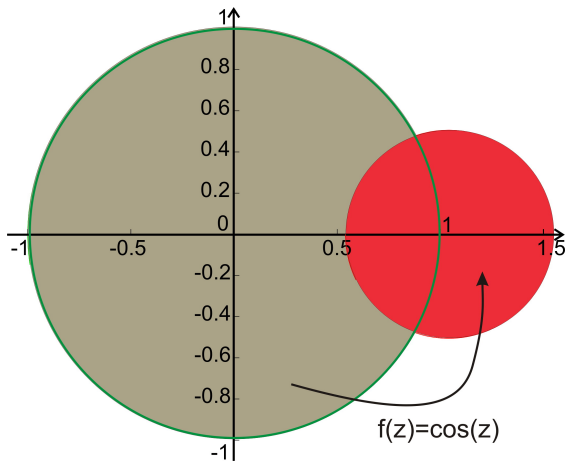
Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. Then the following inequalities are valid **[Mi]**

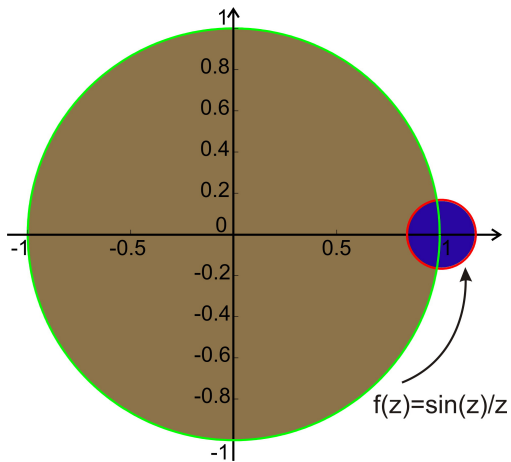
- $|\cos z| < 2$ for all $z \in \mathbb{D}$.
- $\left| \frac{\sin z}{z} \right| < \frac{6}{5}$ for all $z \in \mathbb{D}$.

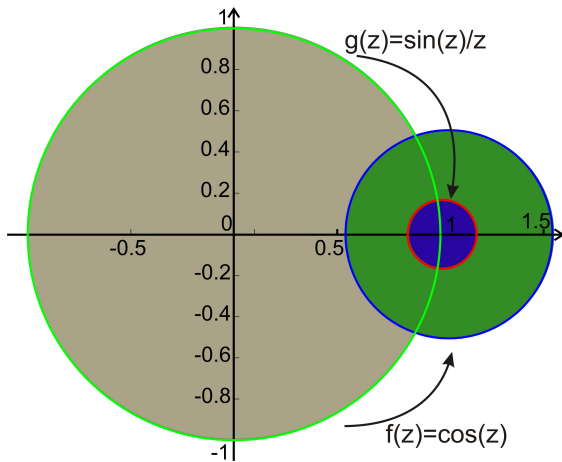
As we can see below the constants 2 and $6/5$ are far from being the best possible. In fact the following sharp inequalities are valid **[AB]**

- $|\cos z| < \cosh 1 \approx 1.543$ for all $z \in \mathbb{D}$.
- $\left| \frac{\sin z}{z} \right| < \sinh 1 \approx 1.175$ for all $z \in \mathbb{D}$.









Let us consider now the function $\mathcal{J}_\rho : \mathbb{C} \rightarrow \mathbb{C}$, defined by

$$\mathcal{J}_\rho(z) = 2^\rho \Gamma(\rho + 1) z^{-\rho} J_\rho(z),$$

where Γ is the well-known Euler gamma function and J_ρ is the Bessel function of the first kind of order ρ . It is worth mentioning that in particular we have

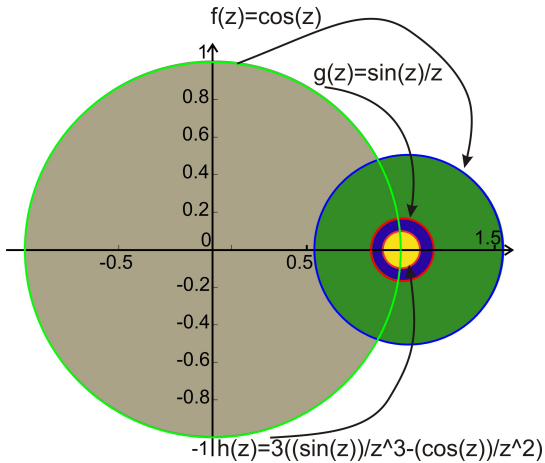
$$\mathcal{J}_{-1/2}(z) = \sqrt{\pi/2} \cdot z^{1/2} J_{-1/2}(z) = \cos z,$$

$$\mathcal{J}_{1/2}(z) = \sqrt{\pi/2} \cdot z^{-1/2} J_{1/2}(z) = \frac{\sin z}{z},$$

$$\mathcal{J}_{3/2}(z) = 3\sqrt{\pi/2} \cdot z^{-3/2} J_{3/2}(z) = 3 \left(\frac{\sin z}{z^3} - \frac{\cos z}{z^2} \right),$$

respectively. The last figure shows that in fact $\mathcal{J}_{1/2}(\mathbb{D}) \subset \mathcal{J}_{-1/2}(\mathbb{D})$. Moreover, as we can see below the following is also valid $\mathcal{J}_{3/2}(\mathbb{D}) \subset \mathcal{J}_{1/2}(\mathbb{D}) \subset \mathcal{J}_{-1/2}(\mathbb{D})$. We note that although the domains $\mathcal{J}_{3/2}(\mathbb{D})$, $\mathcal{J}_{1/2}(\mathbb{D})$, $\mathcal{J}_{-1/2}(\mathbb{D})$ are symmetric with respect to the real axis, they are not disks with center in 1. They are just some “egg-shaped” or disk-like domains.





In view of the above figures and based on some computer experiments we may ask the followings:

- Is it true that if $p > -1$ increase, then the image region $\mathcal{J}_p(\mathbb{D})$ decrease, more precisely if $p \geq q > -1$, then $\mathcal{J}_p(\mathbb{D}) \subseteq \mathcal{J}_q(\mathbb{D})$?
- What is the radius r_p (depending on p) of the smallest disk centered at 1 which contains the image region $\mathcal{J}_p(\mathbb{D})$ for $p > -1$?
- What can we say about the convexity of $z \mapsto \mathcal{J}_p(\sqrt{z})$ in \mathbb{D} ?
- What can we say about the starlikeness (of order 1/2) of $z \mapsto z\mathcal{J}_p(\sqrt{z})$ in \mathbb{D} ?



Generalized Bessel functions

Definitions

The generalized Bessel function **[Ba1]** $w = w_{p,b,c}$ of the first kind and order p is defined as the particular solution of the second-order linear homogenous differential equation

$$z^2 w''(z) + b z w'(z) + [c z^2 - p^2 + (1 - b)p] w(z) = 0, \quad (1)$$

which is a natural generalization of Bessel's equation. This function has the familiar infinite sum representation

$$w(z) = w_{p,b,c}(z) = \sum_{n \geq 0} \frac{(-1)^n c^n}{n! \Gamma(p + n + \frac{b+1}{2})} \left(\frac{z}{2}\right)^{2n+p}, \quad z \in \mathbb{C}.$$

The differential equation (1) permits the study of Bessel, modified Bessel, spherical Bessel and modified spherical Bessel functions all together. In particular, when $b = c = 1$ we reobtain the Bessel function $w_{p,1,1} = J_p$, and for $c = -b = -1$ the function $w_{p,1,-1} = I_p$ becomes the modified Bessel function.



Now, consider the function $u_{p,b,c}$ defined by the transformation

$$u_{p,b,c}(z) = 2^p \Gamma\left(p + \frac{b+1}{2}\right) \cdot z^{-p/2} w_{p,b,c}(\sqrt{z}).$$

By using the well-known Pochhammer (or Appell) symbol, defined in terms of the Euler gamma function for $a \neq 0, -1, -2, \dots$ by $(a)_n = \Gamma(a+n)/\Gamma(a) = a(a+1)\cdots(a+n-1)$ and $(a)_0 = 1$, we obtain for the function $u_{p,b,c}$ the following representation

$$u_{p,b,c}(z) = \sum_{n \geq 0} \frac{(-c/4)^n z^n}{(\kappa)_n n!},$$

where $\kappa = p + (b+1)/2 \neq 0, -1, -2, \dots$. Finally, let us consider the function $\lambda_{p,b,c}$, defined by $\lambda_{p,b,c}(z) = u_{p,b,c}(z^2)$, which we will denote in particular as $\lambda_{p,1,1} = \mathcal{J}_p$ and $\lambda_{p,1,-1} = \mathcal{I}_p$. Observe that these functions have the following forms

$$\mathcal{J}_p(z) = 2^p \Gamma(p+1) z^{-p} J_p(z) \quad \text{and} \quad \mathcal{I}_p(z) = 2^p \Gamma(p+1) z^{-p} I_p(z),$$

where J_p and I_p stand for the Bessel and modified Bessel functions of the first kind and order p , respectively.



Lemma ([AB])

Let $p, q, b, c \in \mathbb{C}$ and let $2\kappa_p = 2p + b + 1$, $2\kappa_q = 2q + b + 1$. If $\operatorname{Re} \kappa_p > \operatorname{Re} \kappa_q > 0$, then the following integral representation formula holds:

$$u_{p,b,c}(z) = \int_0^1 u_{q,b,c}(tz) d\mu_{p,q,b}(t),$$

where $d\mu_{p,q,b}(t) = \mu_{p,q,b}(t)dt$ and

$$\mu_{p,q,b}(t) = \frac{t^{\kappa_q-1}(1-t)^{\kappa_p-\kappa_q-1}}{B(\kappa_q, \kappa_p - \kappa_q)}$$

is a probability measure on $[0, 1]$.



Lemma ([BP])

If $p, b \in \mathbb{R}$ and $c \in \mathbb{C}$ such that $\kappa > \max\{0, c_0\}$, where $c_0 = |c|/4 - 1$, then the function $u_{p,b,c} : \mathbb{D} \rightarrow \mathbb{C}$ satisfies the following inequalities for all $z \in \mathbb{D}$

$$\frac{4\kappa(\kappa + 1) - (2\kappa + 1)|c| + |c|^2/8}{\kappa [4(\kappa + 1) - |c|]} \leq |u_{p,b,c}(z)| \leq \frac{32\kappa^2 - |c|^2}{8\kappa [4\kappa - |c|]}.$$

Idea of the proof: use the triangle inequality, and the inequality $(\kappa)_n n! \geq 2\kappa^n$, which holds for all $n \geq 2$.



Theorem ([BP])

Let $p, b \in \mathbb{R}$ and $c \in \mathbb{C} \setminus \{0\}$ such that $\kappa \geq c_1$ and $\kappa \neq 0$, where $c_1 = \left[3|c| - 12 + \sqrt{7|c|^2 + 8|c| + 16} \right] / 8$. Then the function $u_{p,b,c}$ is convex in \mathbb{D} and is starlike of order $1/2$ with respect to 1. Moreover, the function $z \mapsto zu_{p+1,b,c}(z)$ is starlike in \mathbb{D} .

- We note that recently by using the method of differential subordinations Á.B. [Ba1] proved that if $c \neq 0$ and $\kappa \geq c_0 + 3/2$, then the function $u_{p,b,c}$ is convex in \mathbb{D} . If $|c| \leq 3.815870196 \dots$, then $c_0 + 3/2 \geq c_1$, and in this case the above Theorem improves the above mentioned result from [Ba1].
- In 1995 V. Selinger [Se] by using the method of admissible functions proved that $z \mapsto \mathcal{J}_p(\sqrt{z})$ is convex in \mathbb{D} for all $p \geq -0.25$. Recently, R. Szász and P. Kupán [SK1] improved this result by showing that the function $z \mapsto \mathcal{J}_p(\sqrt{z})$ is convex in \mathbb{D} for all $p \neq -1$ and $p \geq p_0 \simeq -1.406929669 \dots$, where p_0 is the root of the equation $4p^2 + 17p + 16 = 0$.



Corollary ([BP])

If $p \neq -1$ and $p \geq p_1 \simeq -1.429029454\dots$, where p_1 is the root of the quadratic equation $4p^2 + 17p + 129/8 = 0$, then the function $z \mapsto \mathcal{J}_p(\sqrt{z})$ is convex in \mathbb{D} and starlike of order $1/2$ with respect to 1 . Moreover, the function $z \mapsto z\mathcal{J}_{p+1}(\sqrt{z})$ is starlike in \mathbb{D} .

Corollary ([BP])

If $p \neq -1$ and $p \geq p_2 \simeq -1.432619635\dots$ or $-1.875 \leq p < p_3 \simeq -1.719669914\dots$, where p_2 is the greatest root of the cubic equation $128p^3 + 928p^2 + 2148p + 1549 = 0$, while p_3 is the greatest root of the equation $32p^2 + 144p + 153 = 0$, then the function $z \mapsto \mathcal{J}_p(\sqrt{z})$ is convex in \mathbb{D} and starlike of order $1/2$ with respect to 1 . Moreover, the function $z \mapsto z\mathcal{J}_{p+1}(\sqrt{z})$ is starlike in \mathbb{D} . In addition, the constant -1.875 is the best possible in the sense that it is the smallest value of p for which the function $z \mapsto \mathcal{J}_p(\sqrt{z})$ is convex in \mathbb{D} and the function $z \mapsto z\mathcal{J}_{p+1}(\sqrt{z})$ is starlike in \mathbb{D} .



Corollary ([BP])

Let p, q, b be real numbers, $c \in \mathbb{C} \setminus \{0\}$ and $2\kappa_p = 2p + b + 1$, $2\kappa_q = 2q + b + 1$. If $\kappa_p \geq \kappa_q > \max\{0, c_1\}$, then $\lambda_{p,b,c}(\mathbb{D}) \subseteq \lambda_{q,b,c}(\mathbb{D})$. In particular, if $p \geq q > -1$, then $\mathcal{J}_p(\mathbb{D}) \subseteq \mathcal{J}_q(\mathbb{D})$.

For $p \geq q \geq -1/4$ the inclusion $\mathcal{J}_p(\mathbb{D}) \subseteq \mathcal{J}_q(\mathbb{D})$ has been verified by S. András and Á.B. [AB].

Corollary ([AB])

The radius r_p , $p > -1$ of the smallest disk centered at 1 which contains the domain $\mathcal{J}_p(\mathbb{D})$ is $\mathcal{I}_p(1) - 1$. Moreover if $p, q \in \mathbb{C}$ such that $\operatorname{Re} p \geq \operatorname{Re} q \geq -1/4 + [\operatorname{Im} q]^2/6$, then $\mathcal{J}_p(\mathbb{D}) \subseteq \mathcal{J}_q(\mathbb{D})$ holds. In particular, if p, q are real numbers such that $p \geq q \geq -1/4$, then $\mathcal{J}_p(\mathbb{D}) \subseteq \mathcal{J}_q(\mathbb{D})$.



Theorem ([BP])

Let $p, b \in \mathbb{R}$ and $c \in \mathbb{C}$ such that $\kappa \geq c_2$, where $c_2 = \left[4|c| - 4 + \sqrt{14|c|^2 + 16|c| + 16} \right] / 8$. Then the function $z \mapsto zU_{p,b,c}(z)$ is starlike of order $1/2$ in \mathbb{D} , and consequently the function $z \mapsto z^{1-p}W_{p,b,c}(z)$ is starlike in \mathbb{D} .

- Recently, Á.B. [Ba1] proved that if $\kappa \geq |c|/2 + 1$, then the function $z \mapsto zU_{p,b,c}(z)$ is starlike of order $1/2$ in \mathbb{D} , and consequently the function $z \mapsto z^{1-p}W_{p,b,c}(z)$ is starlike in \mathbb{D} . If $|c| \leq 2.506329\dots$, then the above Theorem improves the above mentioned result from [Ba1].
- In 1995 V. Selinger [Se] by using the technique of differential subordinations proved that the function $z \mapsto z\mathcal{J}_p(\sqrt{z})$ is starlike of order $1/2$ in \mathbb{D} for all $p \geq 0.5$. Recently, R. Szász and P. Kupán [SK1] proved that the normalized Bessel function $z \mapsto z\mathcal{J}_p(\sqrt{z})$ is actually starlike of order $1/2$ in \mathbb{D} for all $p \geq p_4 \simeq -0.1339745962\dots$, where p_4 is the greatest root of the equation $4p^2 + 8p + 1 = 0$.



Corollary ([BP])

If $p \geq p_5 \simeq -0.1522087521 \dots$, where p_5 is the root of the equation $32p^2 + 64p + 9 = 0$, then the function $z \mapsto z\mathcal{J}_p(\sqrt{z})$ is starlike of order $1/2$ in \mathbb{D} , and consequently, the function $z \mapsto z^{1-p}J_p(z)$ is starlike in \mathbb{D} .

Corollary ([BP])

If $p \geq p_6 \simeq -0.1572242627 \dots$ or $-0.875 \leq p < p_7 \simeq -0.7196699141 \dots$, where p_6 is the greatest root of the cubic equation $64p^3 + 256p^2 + 274p + 37 = 0$, while p_7 is the root of the equation $32p^2 + 80p + 41 = 0$, then the function $z \mapsto z\mathcal{J}_p(\sqrt{z})$ is starlike of order $1/2$ in \mathbb{D} , and consequently, the function $z \mapsto z^{1-p}J_p(z)$ is starlike in \mathbb{D} . In addition, the constant -0.875 is the best possible in the sense that it is the smallest value of p for which the function $z \mapsto \mathcal{J}_p(\sqrt{z})$ is starlike of order $1/2$ in \mathbb{D} .



Lemma ([BP])

If $p, b \in \mathbb{R}$ and $c \in \mathbb{C}$ such that $\kappa > \max\{0, c_3\}$, where $c_3 = |c|/8 - 1$, then the function $u_{p,b,c} : \mathbb{D} \rightarrow \mathbb{C}$ satisfies the following inequalities for all $z \in \mathbb{D}$

$$|u_{p,b,c}(z)| \geq \frac{8\kappa(\kappa + 1) - (2\kappa + 3)|c|}{\kappa [8(\kappa + 1) - |c|]},$$

and

$$|u'_{p,b,c}(z)| \leq \frac{|c|}{4\kappa} \cdot \frac{8(\kappa + 1) + |c|}{8(\kappa + 1) - |c|},$$

or equivalently,

$$|u_{p,b,c}(z)| \leq \frac{8\kappa + |c|}{8\kappa - |c|} \quad \text{for } \kappa > |c|/8.$$

Idea of the proof: use the triangle inequality, and the inequality $2(\kappa)_n n! \geq (2\kappa)^n$, which holds for all $n \geq 1$.



Lemma ([BP])

Let $a > 0$, $n \in \{3, 4, 5, \dots\}$ and let $\alpha_0 \simeq 1.302775637 \dots$ be the greatest root of the quadratic equation $\alpha^2 + \alpha - 3 = 0$. Then the following inequality holds

$$\Gamma(a+n)/\Gamma(a+1) > (a+\alpha_0)^{n-1} \iff (a)_n > a(a+\alpha_0)^{n-1}.$$

Lemma ([BP])

If $p, b \in \mathbb{R}$ and $c \in \mathbb{C}$ such that $\kappa > \max\{0, c_4\}$, where $c_4 = |c|/[4(1+\alpha_0)] - \alpha_0$, then the function $u_{p,b,c} : \mathbb{D} \rightarrow \mathbb{C}$ satisfies the following inequalities for all $z \in \mathbb{D}$

$$|u_{p,b,c}(z)| > 1 + \frac{|c|^2}{16(1+\alpha_0)\kappa(\kappa+\alpha_0)} - \frac{|c|^2}{32\kappa(\kappa+1)} - \frac{|c|}{\kappa} \cdot \frac{(1+\alpha_0)(\kappa+\alpha_0)}{4(1+\alpha_0)(\kappa+\alpha_0) - |c|}$$

$$|u'_{p,b,c}(z)| < \frac{|c|}{4\kappa} \cdot \left[1 - \frac{|c|^2}{16(1+\alpha_0)(\kappa+1)(\kappa+\alpha_0+1)} + \frac{|c|^2}{32(\kappa+1)(\kappa+2)} + \frac{|c|}{\kappa+1} \cdot \frac{(1+\alpha_0)(\kappa+\alpha_0+1)}{4(1+\alpha_0)(\kappa+\alpha_0+1) - |c|} \right].$$

Although the above bounds are quite tight they does not give us better results in particular for the convexity and starlikeness of the functions $z \mapsto \mathcal{J}_\rho(\sqrt{z})$ and $z \mapsto z\mathcal{J}_\rho(\sqrt{z})$. The reason for this is that for example for the starlikeness of the function $f(z) = z\mathcal{J}_\rho(\sqrt{z})$ instead of

$$\operatorname{Re} \left[\frac{zf'(z)}{f(z)} \right] > 0$$

we prove the stronger statement that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1$$

and for better bounds we do not have necessarily the best range for p .



All the same, by using some known results on modified Bessel functions we can obtain the following:

Theorem ([BP])

The following inequalities hold for all $p > -1$ and $z \in \mathbb{D}$

$$|\mathcal{J}_p(\sqrt{z})| > 2 - \mathcal{I}_p(1) \quad \text{and} \quad |\mathcal{J}'_p(\sqrt{z})| < \frac{\mathcal{I}_{p+1}(1)}{2(p+1)}.$$

Moreover, the following assertions are true:

- 1 If $p \neq -1$ and $p \geq p_{13} \simeq -1.4373468657\dots$, where p_{13} is the unique root of the transcendental equation $2^p \Gamma(p+2) [I_{p+2}(1) + 2I_{p+1}(1)] = 2$, then the function $z \mapsto \mathcal{J}_p(\sqrt{z})$ is convex in \mathbb{D} and starlike of order $1/2$ with respect to 1 . Moreover, the functions $z \mapsto z\mathcal{J}_{p+1}(\sqrt{z})$ is starlike in \mathbb{D} .
- 2 If $p \geq p_{14} \simeq -0.1614476422\dots$, where p_{14} is the unique root of $2^p \Gamma(p+1) [I_{p+1}(1) + I_p(1)] = 2$, then the function $z \mapsto z\mathcal{J}_p(\sqrt{z})$ is starlike of order $1/2$ in \mathbb{D} , and consequently, the function $z \mapsto z^{1-p} J_p(z)$ is starlike in \mathbb{D} .



Conjectures and further comments

- The function $z \mapsto \mathcal{J}_\rho(\sqrt{z})$ is convex in \mathbb{D} if and only if $\rho \neq -1$ and $\rho \geq -1.875$.
- The function $z \mapsto z\mathcal{J}_\rho(\sqrt{z})$ is starlike in \mathbb{D} if and only if $\rho \geq -0.875$.
- The function $z \mapsto z\mathcal{J}_\rho(\sqrt{z})$ is starlike of order $1/2$ in \mathbb{D} if and only if $\rho \geq -0.875$.
- Question: what happens if ρ is complex?
- Radius of convexity of $z \mapsto \mathcal{J}_\rho(\sqrt{z})$?
- Radius of starlikeness of $z \mapsto z\mathcal{J}_\rho(\sqrt{z})$?
- E.O.A. Kreyszig and J. Todd [KT], 1958
- R.K. Brown [Br], 1960 – by using a method of M.S. Robertson, 1954
- R. Szász and P. Kupán [SK2] – by using the infinite product representation of Bessel functions



- Recently, T. Pogány **[Po]** proved the following result:

Lemma (**[Po]**)

If $\lambda \in (0, 1)$ and $p > -1$, then

$$|\mathcal{J}_p(z)| \leq \exp\left(-\lambda \frac{|z|^2}{4(p+1)}\right)$$

for all $z \in \mathbb{C}_{p,\lambda}$, where $\mathbb{C}_{p,\lambda}$ is the closed Cassinian oval






$$\mathbb{C}_{p,\lambda} = \left\{ z \in \mathbb{C} : |z^2 - j_{p,1}^2| \leq j_{p,1}^2 \frac{1-\lambda}{1+\lambda} \right\}$$

and $j_{p,1}$ is the first positive zero of the Bessel function J_p .







The above result has been used to study the convergence of some generalized Kapteyn-type series **[Po]**. It would be of interest to see what happens with the monotonicity of the image region $\mathcal{J}_p(\mathbb{C}_{p,\lambda})$.



References

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