On Mori's theorem for quasiconformal maps in the *n*-space

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Abstract. This talk deals with

A joint research with Matti Vuorinen

We discuss here the Hölder continuity of the K-quasiconformal mappings of the unit ball with respect to the euclidean and hyperbolic metrics.

[BV] B. A. Bhayo and M. Vuorinen: On Mori's theorem for quasiconformal maps in the n-space. Manuscript arXiv:0906.2853v2 [math.CA] 23 Jun 2009.

Notation

- **The one point compactification of $\mathbb{R}^n$ ($n \geq 2$) is denoted by $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ (the Möbius space $\overline{\mathbb{R}}^n$).**
- **We denote by $B^n(x, r)$ and $S^{n-1}(x, r)$, the euclidean ball and sphere with radius r centered at x, respectively. We set $B^n(r) := B^n(0, r)$ and $S^{n-1}(r) := S^{n-1}(0, r)$.**

Introduction

Definition of M(K,n)

Let QC_K, $K \geq 1$, stand for the family of all K-quasiconformal maps of the unit ball $\mathbf{B}^n$ onto itself keeping the origin pointwise fixed. Then for all $K \geq 1$, $n \geq 2$, there exists a least constant $M(n, K) \geq 1$ such that

$$|f(x) - f(y)| \leq M(n, K)|x - y|^{\alpha}, \quad \alpha = K^{1/(1-n)},$$

for all $f \in QC_K, x, y \in \mathbf{B}^n$.

L. V. Ahlfors [A1] proved in 1954 the Hölder continuity for plane quasiconformal maps $M(2, K) < \infty$ and this property was refined by A. Mori [M] in 1956 to the effect that $M(2, K) \leq 16$ and that 16 cannot be replaced by a smaller constant independent of K.

The Mori Conjecture

$M(2,K) = 16^{1-1/K}$.

O. Lehto and K.I. Virtanen demonstrated in 1973 [LV, pp. 68] that $M(2,K) \geq 16^{1-1/K}$.

It is natural to expect that for a fixed $n \geq 2$, $M(n,K) \to 1$ **when** $K \to 1$ **and this fact was proved by R. Fehlmann and M. Vuorinen [FV].**

R. Fehlmann and M. Vuorinen's theorem (1988)

[FV, Theorem 1.3]: Let f be a K-quasiconformal mapping of $\mathbf{B}^n$ onto $\mathbf{B}^n$, $n \geq 2$, $f(0) = 0$. Then

$$|f(x) - f(y)| \leq M(n, K)|x - y|^\alpha$$

for all $x, y \in \mathbf{B}^n$ where $\alpha = K^{1/(1-n)}$ and the constant $M(n, K)$ has the following three properties:

1. $M(n, K) \to 1$ **as** $K \to 1$**, uniformly in** n **,**
2. $M(n, K)$ **remains bounded for fixed** K **and varying** n **,**
3. $M(n, K)$ **remains bounded for fixed** n **and varying** K **.**

For $n = 2$, the first majorants with the convergence property in [FV, Theorem 1.3(1)] were proved by several authors (cited [FV]) only in the mid 1980s and for $n \geq 3$ in [FV].

As far as we know, the best upper bound known today for $n = 2$ is $M(2, K) \leq 48^{1-1/K}$ due to S.-L. Qiu [Q] (1997).

G. Anderson and M. Vamanamurthy's bound (1988) [AV]

For $n \geq 2, K \geq 1$,

$$M(n, K) \leq 4\lambda_n^{2(1-\alpha)},$$

where $\alpha = K^{1/(1-n)}$ and $\lambda_n \in [4, 2e^{n-1})$, $\lambda_2 = 4$, is the Grötzsch ring constant [Vu1, p.89].

Theorem

For $n \geq 2, K \geq 1$, $M(n,K) \leq T(n,K)$

$$T(n,K) \leq \inf\{h(t) : t \geq 1\}, \quad h(t) = (3 + \lambda_n^{\beta-1} t^{\beta}) t^{-\alpha} \lambda_n^{2(1-\alpha)},$$

where $\alpha = K^{1/(1-n)} = 1/\beta$. **There exists a number** $K_1 > 1$ **such that for all** $K \in (1, K_1)$ **the function** h **has a minimum at a point** t_1 **with** $t_1 > 1$ **and**

$$T(n,K) \leq h(t_1) = (A+B)\lambda_n^{2(1-\alpha)},$$

here $A = \dfrac{3^{1-\alpha}(\beta-\alpha)^{\alpha^2}}{\alpha^{\alpha}} \lambda_n^{\alpha-\alpha^2}$ **and** $B = \lambda_n^{\beta-1}\left(\dfrac{3\alpha\lambda_n^{\alpha-1}}{(\beta-\alpha)^{\alpha}}\right)^{\beta-\alpha}$.

Moreover, for $\beta \in (1,2)$ we have

$$h(t_1) \le 3^{\beta-\alpha} 2^{1-\alpha} K^5 \left(\frac{3}{2} \sqrt[4]{\beta-\alpha} + \exp(\sqrt{\beta^2-1}) \right).$$

In particular, $h(t_1) \to 1$ when $K \to 1$.

Theorem

If $f : \mathbf{B}^2 \to \mathbf{R}^2$ is a non-constant K-quasiregular mapping with $f\mathbf{B}^2 \subset \mathbf{B}^2$, and ρ is the hyperbolic metric of $\mathbf{B}^2$, then

$$\rho(f(x), f(y)) \le c(K) \max\{\rho(x,y), \rho(x,y)^{1/K}\}$$

for all $x, y \in \mathbf{B}^2$ where $c(K) = 2\mathrm{arth}(\varphi_K(\mathrm{th}\frac{1}{2}))$ and

$$1.5412(K-1)+1 \le \log(\mathrm{ch}(K\mathrm{arch}(e))) \le c(K) \le 1.3507(K-1)+K.$$

In particular, $c(1) = 1$. (We use the notation ch, th, arch and arth to denote the hyperbolic cosine, tangent and their inverse functions, respectively.)

K-quasiconformal mapping

Let D and D' be domains in $\overline{\mathbf{R}}^n$, $K \geq 1$, and let $f : D \to D'$ be a homeomorphism. Then f is K-quasiconformal if

$$M(\Gamma)/K \leq M(f\Gamma) \leq KM(\Gamma)$$

for every curve family Γ in D.

Ring

For subsets $E, F, D \subset \overline{\mathbf{R}}^n$ we denote by $\Delta(E, F; D)$ the family of all curves joining E and F in D. A ring is a domain in $\mathbf{R}^n$, whose complement consists of two compact and connected sets. If these sets are E and F, then ring is denoted by $R(E, F)$. The capacity of a ring $R(E, F)$ is

$$\mathrm{cap} R(E, F) = M(\Delta(E, F; \mathbf{R}^n)).$$

Grötzsch and Teichmüller rings

The complementary components of the Grötzsch ring $R_{G,n}(s)$ are $\overline{\mathbf{B}}^n$ and $[se_1, \infty], s > 1$, while those of the Teichmüller ring $R_{T,n}(t)$ are $[-e_1, 0]$ and $[te_1, \infty], t > 0$. The conformal capacities of $R_{G,n}(s)$ and $R_{T,n}(t)$ are denoted by

$$\begin{cases} \gamma_n(s) = M(\Delta(\overline{\mathbf{B}}^n, [se_1, \infty])) \\ \tau_n(t) = M(\Delta([-e_1, 0], [te_1, \infty])) \end{cases}$$

respectively. Here $\gamma_n : (1, \infty) \to (0, \infty)$ and $\tau_n : (0, \infty) \to (0, \infty)$ are decreasing homeomorphisms and they satisfy the fundamental identity

$$\gamma_n(s) = 2^{n-1}\tau_n(s^2 - 1), \quad s > 1\,.$$

Special functions [Vu1, Theorem 7.47]

For $n \geq 2$ **and** $K > 0$**, the distortion function** $\varphi_{K,n} : [0,1] \to [0,1]$ **defined by**

$$\varphi_{K,n}(t) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/t))}, \quad t \in (0,1), \tag{1}$$

and $\varphi_{K,n}(0) = 0$ **and** $\varphi_{K,n}(1) = 1$ **is a homeomorphism. For** $n \geq 2, K \geq 1$ **and** $0 \leq r \leq 1$

$$\varphi_{K,n}(r) \leq \lambda_n^{1-\alpha} r^{\alpha}, \quad \alpha = K^{1/(1-n)},$$

$$\varphi_{1/K,n}(r) \geq \lambda_n^{1-\beta} r^{\beta}, \quad \beta = K^{1/(n-1)}.$$

Gehring Lemma [Vu1, Lemma 7.35]

Let $R = R(E, F)$ **be a ring in** $\overline{\mathbf{R}}^n$ **and let** $a, b \in E, c, d \in F$ **be distinct points. Then**

$$\mathbf{cap}R = M(\Delta(E, F)) \geq \tau_n \left(\frac{|a-c||b-d|}{|a-b||c-d|} \right).$$

Equality holds if $b = t_1 e_1, a = t_2 e_1, c = t_3 e_1, d = t_4 e_1$ **and** $t_1 < t_2 < t_3 < t_4$**.**

Teichmüller's extremal problem

For $x \in \mathbf{R}^n \setminus \{0, e_1\}, n \geq 2$**, define**

$$p_n(x) = \inf_{E,F} M(\Delta(E, F))$$

where the infimum is taken over all the pairs of continua E **and** F **in** $\overline{\mathbf{R}}^n$ **with** $0, e_1 \in E$ **and** $x, \infty \in F$.

Lemma

Let $f : \mathbf{B}^n \to \mathbf{B}^n$ be a K-quasiconformal mapping with $f\mathbf{B}^n = \mathbf{B}^n$, $f(0) = 0$, let $h : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ be the inversion $h(x) = x/|x|^2$, and let $g : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ be defined by $g(x) = f(x)$ for $x \in \mathbf{B}^n, g(x) = h(f(h(x)))$ for $x \in \mathbf{R}^n \setminus \overline{\mathbf{B}}^n$ and

$$g(x) = \lim_{z \to x} f(z) \text{ for } x \in \partial \mathbf{B}^n, g(\infty) = \infty.$$

Then g is a K-quasiconformal mapping, and we have for $x \in \mathbf{B}^n$

$$\varphi_{1/K,n}(|x|) \leq |f(x)| \leq \varphi_{K,n}(|x|). \tag{2}$$

and for $x \in \mathbf{R}^n \setminus \overline{\mathbf{B}}^n$

$$1/\varphi_{K,n}(1/|x|) \leq |g(x)| \leq 1/\varphi_{1/K,n}(1/|x|). \tag{3}$$

proof

It is well-known that the above definition defines g as a K-quasiconformal homeomorphism. The formula (2) is well-known (see [AVV2, Theorem 4.2]) and (3) follows easily.

Lemma [Vu2, Theorem 1.5]

For $z \in \mathbf{R}^n, |z| > 1$, the following inequalities hold:

$$\tau_n(|z|) = p_n(-|z|e_1) \le p_n(z) \le p_n(|z|e_1) = p_n(|z| - 1)$$

where $p_n(z)$ is the Teichmüller function. Furthermore, for $z \in \mathbf{R}^n \setminus \{0, e_1\}$, there exists a circular arc E with $0, e_1 \in E$ and a ray F with $z, \infty \in F$ such that

$$p_n(z) \le \tau_n\left(\frac{|z| + |z - e_1| - 1}{2}\right) = M(\Delta(E, F; \mathbf{R}^n)) \le \tau_n(|z| - 1)$$

with equality in the first inequality both for $z = -se_1, s > 0$, $z = se_1, s > 1$.

Theorem 1

For $n \geq 2, K \geq 1$, **let** $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ **be a** K**-quasiconformal mapping, with** $f\mathbf{B}^n \subset \mathbf{B}^n$, $f(0) = 0$ **and** $f(\infty) = \infty$**. Then for** $t \geq 1$ **we have**

$$\begin{aligned} |f(x) - f(y)| &\leq (3 + \varphi_{1/K,n}(1/t)^{-1})\varphi^2_{K,n}\left(\left(\frac{2|x-y|}{s_1 + |x-y|}\right)^{1/2}\right) \\ &\leq (3 + \lambda_n^{(\beta-1)} t^{\beta})\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{s_1 + |x-y|}\right)^{\alpha} \end{aligned}$$

where $\alpha = K^{1/(1-n)} = 1/\beta$ **and**

$s_1 = \max\{a, b\}, a = t + |x| + |y + t\dfrac{x}{|x|}|, b = t + |y| + |x + t\dfrac{y}{|y|}|$**,**

for all $x, y \in \mathbf{B}^n$**.**

Corollary

For $n \geq 2, K \geq 1$**, let** $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ **be a** K**-quasiconformal mapping, with** $f\mathbf{B}^n \subset \mathbf{B}^n$**,** $f(0) = 0$ **and** $f(\infty) = \infty$**. Then**

$$|f(x) - f(y)| \leq 4\lambda_n^{2(1-\alpha)} \left(\frac{2|x-y|}{s + |x-y|} \right)^{\alpha} .$$

Here $\alpha = K^{1/(1-n)}$ **and**
$s = \max\{a, b\}, a = 1 + |x| + |y + \frac{x}{|x|}|, b = 1 + |y| + |x + \frac{y}{|y|}|$**, for all** $x, y \in \mathbf{B}^n$**.**

Corollary

For $n \geq 2, K \geq 1, t \geq 1$**, let** f **be as in Theorem 1. Then**

$$|f(x)-f(y)| \leq (3+\lambda_n^{(\beta-1)} t^{\beta})\lambda_n^{2(1-\alpha)} \left(\frac{2|x-y|}{2t+||x|-|y||+|x-y|} \right)^{\alpha},$$

for all $x, y \in \mathbf{B}^n$,

$$|f(x)-f(y)| \leq (3+\lambda_n^{(\beta-1)} t^{\beta})\lambda_n^{2(1-\alpha)} \left(\frac{|x-y|}{\max\{t+|x|, t+|y|\}} \right)^{\alpha}, \tag{4}$$

for all $x, y \in \mathbf{B}^n$, **and**

$$|f(x)-f(y)| \leq (3+\lambda_n^{(\beta-1)} t^{\beta})\lambda_n^{2(1-\alpha)} \left(\frac{|x-y|}{t+|x|+(|x-y|)/2} \right)^{\alpha},$$

if $|y+t\frac{x}{|x|}| > t+|x|,\ x, y \in \mathbf{B}^n$.

Theorem

For $n \geq 2, K \geq 1$, $M(n,K) \leq T(n,K)$

$$T(n,K) \leq \inf\{h(t) : t \geq 1\}, \quad h(t) = (3 + \lambda_n^{\beta-1} t^{\beta}) t^{-\alpha} \lambda_n^{2(1-\alpha)}, \tag{5}$$

where $\alpha = K^{1/(1-n)} = 1/\beta, t \geq 1$ **and** λ_n **is as in [AV]. There exists a number** $K_1 > 1$ **such that for all** $K \in (1, K_1)$ **the function** h **has a minimum at the point** $t_1 > 1$ **and**

$$T(n,K) \leq h(t_1) = (A+B)\lambda_n^{2(1-\alpha)}, \tag{6}$$

here $A = \dfrac{3^{1-\alpha}(\beta-\alpha)^{\alpha^2}}{\alpha^{\alpha}} \lambda_n^{\alpha-\alpha^2}$ **and** $B = \lambda_n^{\beta-1} \left(\dfrac{3\alpha\lambda_n^{\alpha-1}}{(\beta-\alpha)^{\alpha}} \right)^{\beta-\alpha}$.

Moreover, for $\beta \in (1,2)$ we have

$$h(t_1) \leq 3^{\beta-\alpha} 2^{1-\alpha} K^5 \left(\frac{3}{2} \sqrt[4]{\beta - \alpha} + \exp(\sqrt{\beta^2 - 1}) \right) . \qquad (7)$$

In particular, $h(t_1) \to 1$ when $K \to 1$.

Proof

(1) This statement follows easily from the inequality (4).
(2) We see that the function h has a local minimum at $t_1 = 3\alpha\lambda_n^{\alpha-1}(\beta-\alpha)^{-\alpha}$. If $t_1 \geq 1$, then the inequality (4) yields the desired conclusion. The upper bound for $T(n,K)$ follows by substituting the argument t_1 in the expression of h .

Graphical and numerical comparision of various bounds

Solving numerically the equation $4 \cdot 16^{1-1/K} = h(t_1)$ for K we obtain $K = 1.3044$ and 8.9105. We give numerical and graphical comparison of the various upper bounds for Mori's constant when $n = 2$ and $\lambda_2 = 4$ as a function of K:
(a) Mori's conjectured bound $16^{1-1/K}$
(b) the Anderson-Vamanamurthy bound $4 \cdot 16^{1-1/K}$,
(c) the bound from (6).
For $K \in (1, 1.3044)$ and $K > 8.9105$ the upper bound in (6) is better than the Anderson-Vamanamurthy bound.

For graphing and tabulation purposes we use the logarithmic scale. Note that the upper bound for $M(2, K)$ given in [FV, Theorem 2.29 (1)] also has the desiderable property that it converges to 1 when $K \to 1$ (see Figure 2).

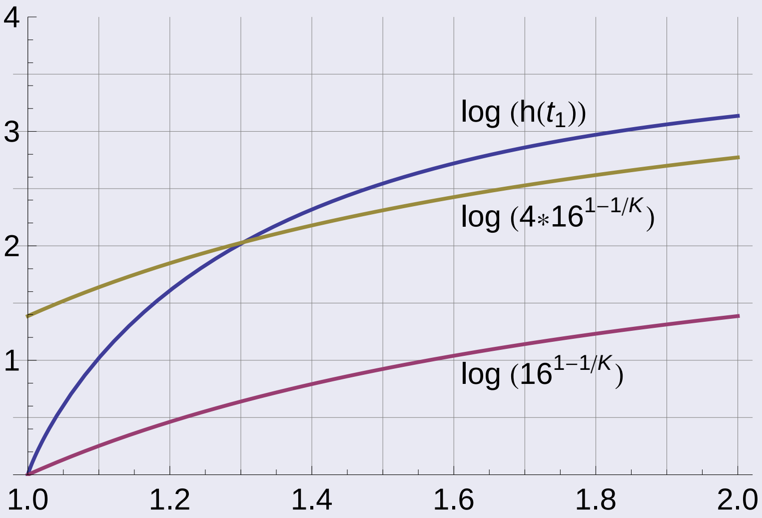

Figure: 1

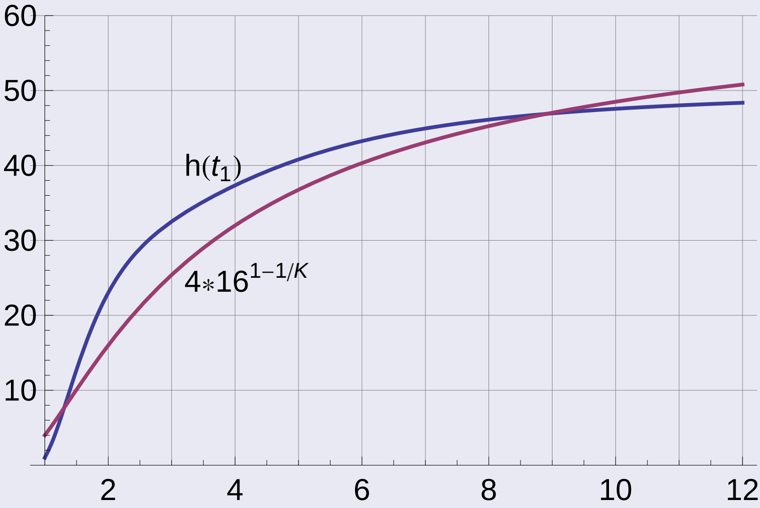

Figure: 2

Numerical comparision

K	$\log(16^{1-1/K})$	$\log(4 \cdot 16^{1-1/K})$	$\log(FV)$	$\log(h(t_1))$
1.1	0.2520	1.6383	0.7051	1.0194
1.2	0.4620	1.8483	1.2484	1.6093
1.3	0.6398	2.0261	1.7046	2.0181
1.4	0.7921	2.1784	2.0912	2.3176
1.5	0.9241	2.3104	2.4221	2.5445
1.6	1.0397	2.4260	2.7093	2.7204
1.7	1.1416	2.5279	2.9633	2.8593
1.8	1.2322	2.6185	3.1920	2.9707
1.9	1.3133	2.6996	3.4020	3.0612
2.0	1.3862	2.7725	3.5978	3.1355

The upper bound in (6) is better than the Fehlmann-Vuorinen bound for $K > 1.6097$.

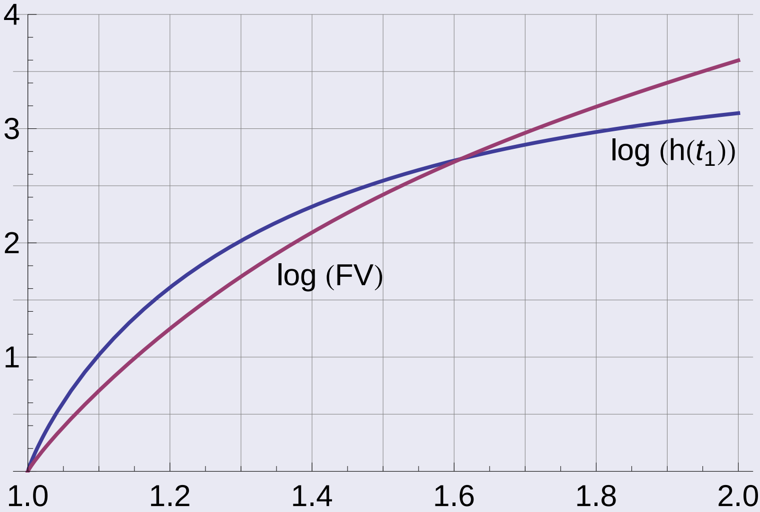

Figure: 3

Graphical illustration of FV and (6) the various upper bounds when $n = 2$ and $\lambda_2 = 4$:(a) the bound from (6). (b) the Fehlmann and Vuorinen bound [FV, Theorem 2.29]

$$M(2,K) \leq \left(1 + \varphi_{K,2}\left(\frac{K^2-1}{K^2+1}\right)\right) 2^{2K-3/K} \frac{(K^2+1)^{(K+1/K)/2}}{(K^2-1)^{(K-1/K)/2}}.$$

For $K > 1.6097$ the upper bound in (6) is better than the Fehlmann-Vuorinen bound.

Comparison of estimates for the Hölder quotient

For a K-quasiconformal mapping $f : \mathbf{B}^n \to f\mathbf{B}^n = \mathbf{B}^n$ we call the expression

$$HQ(f) = \sup\{|f(x) - f(y)|/|x-y|^\alpha : \; x, y \in \mathbf{B}^n\}, \; x \neq y$$

$\alpha = K^{1/(1-n)}$**, the Hölder coefficient of f. Clearly $HQ(f) \leq M(n, K)$. Theorem 3 yields, after dividing the both sides of the inequality in Theorem 3 by $|x-y|^\alpha$, the upper bound $HQ(f) \leq HQ(K)$ for the Hölder quotient with**

$$HQ(K) = \sup\{\inf\{U(t, x, y) : \; t \geq 1\} : x, y \in \mathbf{B}^n\} \,. \tag{8}$$

$$U(t, x, y) = (3 + \varphi_{1/K,n}(1/t)^{-1})\varphi^2_{K,n}\left(\left(\frac{2|x-y|}{s_1 + |x-y|}\right)^{1/2}\right)\frac{1}{|x-y|^\alpha}$$

For $n = 2$ we compare $HQ(K)$ to several other bounds
(a) Mori's conjectured bound,
(b) the FV bound,
(c) the AV bound
and give the results as a table and Figure 4. Because the supremum and infimum in (8) cannot be explicitly found, we use numerical methods that come with Mathematica software. For the numerical tests we used for the supremum a sample of $100,000$ random points of the unit disk.

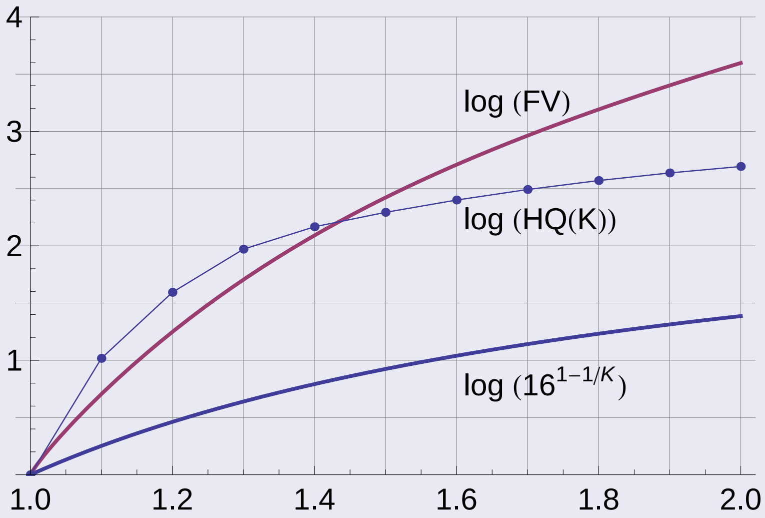

Figure: 4

K	$\log(16^{1-1/K})$	$\log(4 \cdot 16^{1-1/K})$	$\log(FV)$	$\log(HQ(K))$
1.1	0.2521	1.6384	0.7051	1.0171
1.2	0.4621	1.8484	1.2485	1.5940
1.3	0.6398	2.0261	1.7046	1.9712
1.4	0.7922	2.1785	2.0913	2.1668
1.5	0.9242	2.3105	2.4221	2.2928
1.6	1.0397	2.4260	2.7094	2.4003
1.7	1.1417	2.5280	2.9633	2.4922
1.8	1.2323	2.6186	3.1921	2.5706
1.9	1.3133	2.6996	3.4020	2.6371
2.0	1.3863	2.7726	3.5979	2.6934

An explicit form of the Schwarz lemma

Hyperbolic metric

Recall that the hyperbolic metric $\rho(x,y), x,y \in \mathbf{B}^n$, of the unit ball is given by (cf. [Vu1])

$$\mathrm{th}^2 \frac{\rho(x,y)}{2} = \frac{|x-y|^2}{|x-y|^2+t^2}, \quad t^2 = (1-|x|^2)(1-|y|^2). \tag{9}$$

Theorem (1988) [Vu1, 11.2]

Let $f : \mathbf{B}^n \to \mathbf{R}^n$ be a nonconstant K-quasiregular mapping with $f\mathbf{B}^n \subset \mathbf{B}^n$. Then

$$\mathrm{th}\frac{\rho(f(x),f(y))}{2} \le \varphi_{K,n}\left(\mathrm{th}\frac{\rho(x,y)}{2}\right), \tag{10}$$

$$\rho(f(x),f(y)) \le K(\rho(x,y)+\log 4), \tag{11}$$

for all $x,y \in \mathbf{B}^n$.

Lemma 2

For $K > 1$ **the function**

$$t \mapsto \frac{2\operatorname{arth}(\varphi_K(\operatorname{th}\frac{t}{2}))}{\max\{t, t^{1/K}\}},$$

is monotone increasing on $(0,1)$ **and decreasing on** $(1,\infty)$.

Bounds for the constant $c(K)$.

$$K \le u(K-1)+1 \le \log(\operatorname{ch}(K\operatorname{arch}(e))) \le c(K) \le v(K-1)+K,$$

here

$$u = \operatorname{arch}(\mathrm{e})\operatorname{th}(K\operatorname{arch}(e)) > 1.5412$$

and

$$v = \log(2(1+\sqrt{1-e^{-2}})) < 1.3507$$

.

$\log(\mathrm{ch}(K\,\mathrm{arch}(e))) \le c(K) \le 1.3507(K-1)+K$

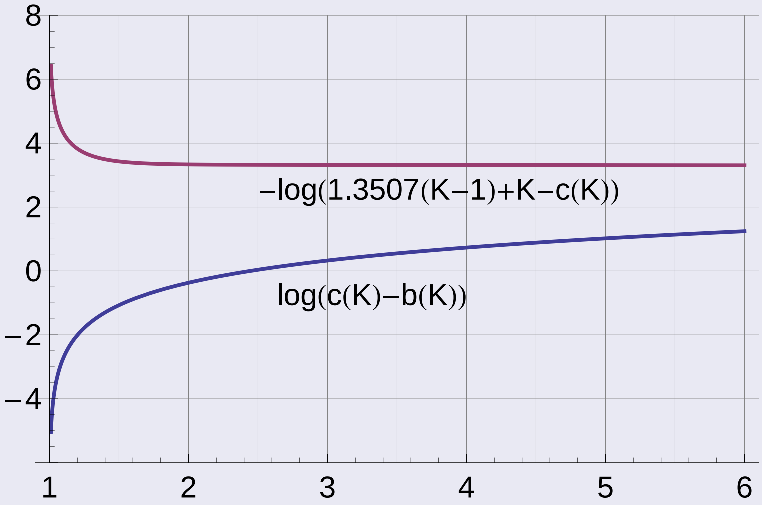

Figure: 5, here $b(K) = \log(\mathrm{ch}(K\,\mathrm{arch}(e)))$.

Theorem

If $f : \mathbf{B}^2 \to \mathbf{R}^2$ is a non-constant K-quasiregular mapping with $f\mathbf{B}^2 \subset \mathbf{B}^2$, and ρ is the hyperbolic metric of $\mathbf{B}^2$, then

$$\rho(f(x), f(y)) \le c(K) \max\{\rho(x, y), \rho(x, y)^{1/K}\}$$

for all $x, y \in \mathbf{B}^2$ where $c(K) = 2\mathrm{arth}(\varphi_K(\mathrm{th}\frac{1}{2}))$ and

$$\begin{aligned} K \le 1.5412(K-1)+1 &\le \log(\mathrm{ch}(K\mathrm{arch}(e))) \\ &\le c(K) \le 1.3506(K-1)+K. \end{aligned}$$

In particular, $c(1) = 1$.

Proof

The maximum value of the function considered in Lemma 2 is $c(K) = 2\mathrm{arth}(\varphi_K(\mathrm{th}\frac{1}{2}))$.

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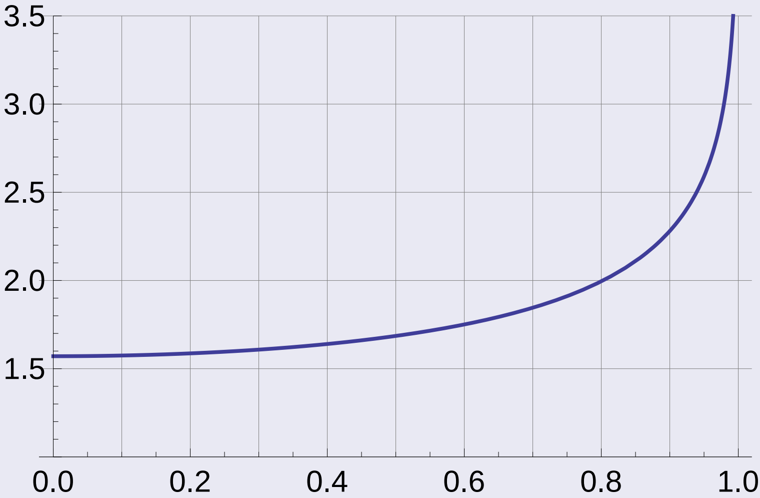
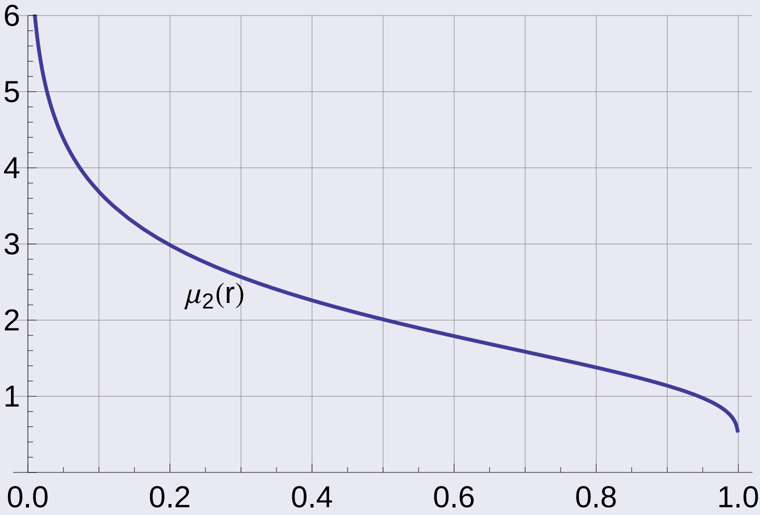
Thank you very much for your kind attention.

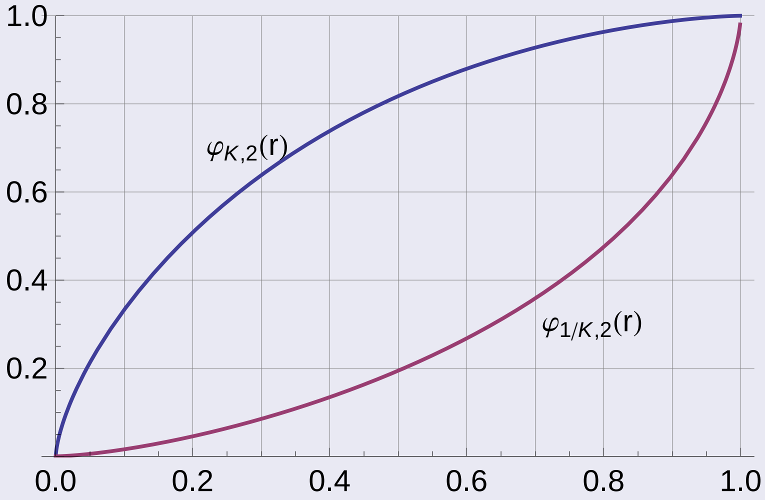
Elliptic integral $\mathcal{K}(r)$, $r \in (0, 1)$



$\mu(r) = \frac{\mathcal{K}(r')}{\mathcal{K}(r)}, \; r \in (0,1)$

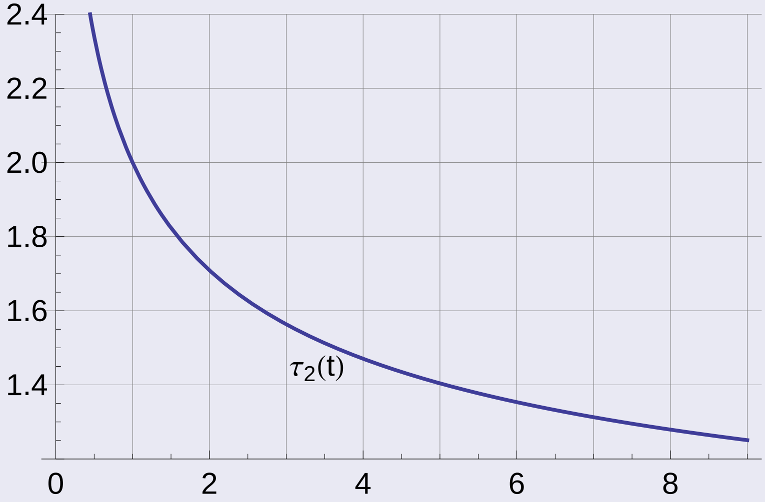


$\varphi_{K,2}(r)$ and $\varphi_{1/K,2}(r)$, $K = 1.5$, $r \in (0,1)$



$\gamma_2(s), \; s > 1, \;\; s \in (1.001, 10)$

$\tau_2(t),\ t > 0,\ \ t \in (0.001, 9)$



$h(t)$ for $n = 2$ and $K = 4/3$

$T(t) = t \mapsto \frac{2\mathrm{arth}(\varphi_K(\mathrm{th}\frac{t}{2}))}{\max\{t, t^{1/K}\}}$ is monotone increasing on $(0,1)$, here K=1.5