Computing Moduli of Rings and Quadrilaterals with hp-FEM

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- Joint work with
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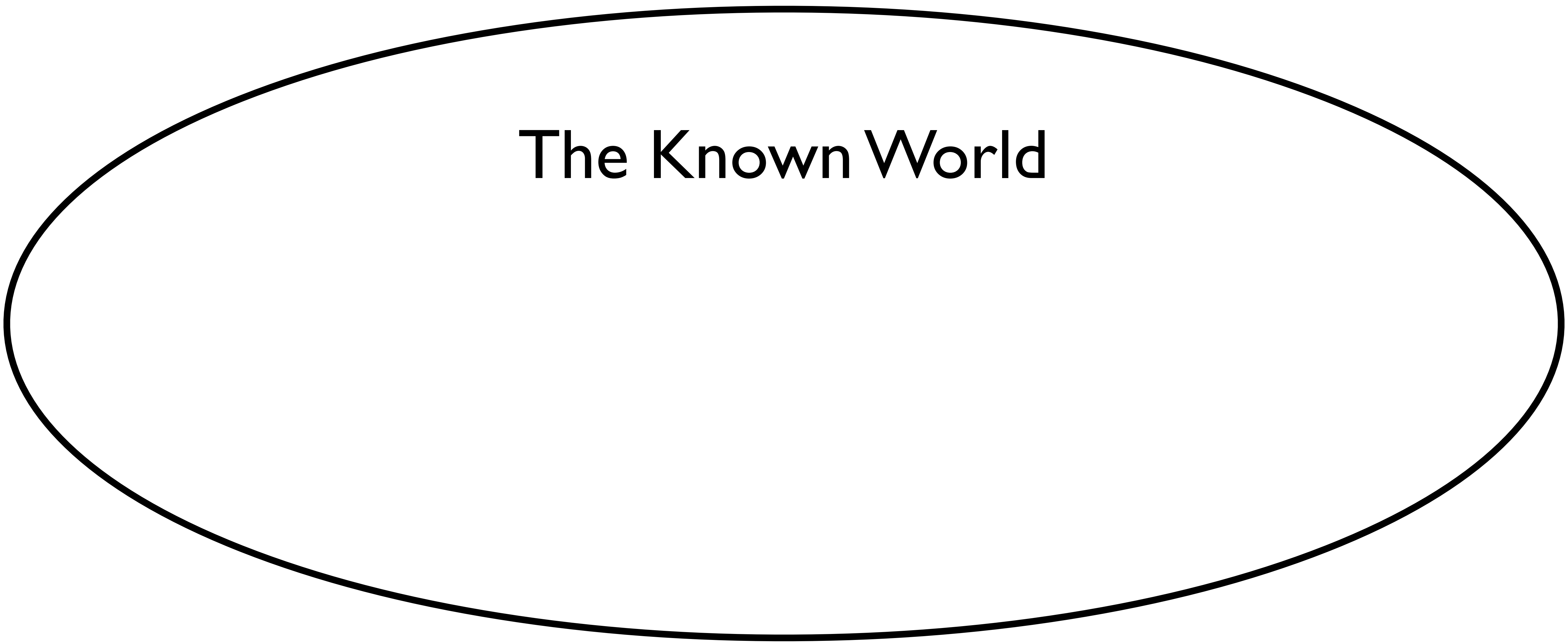
  - Matti Vuorinen

Flat-Earth Outline



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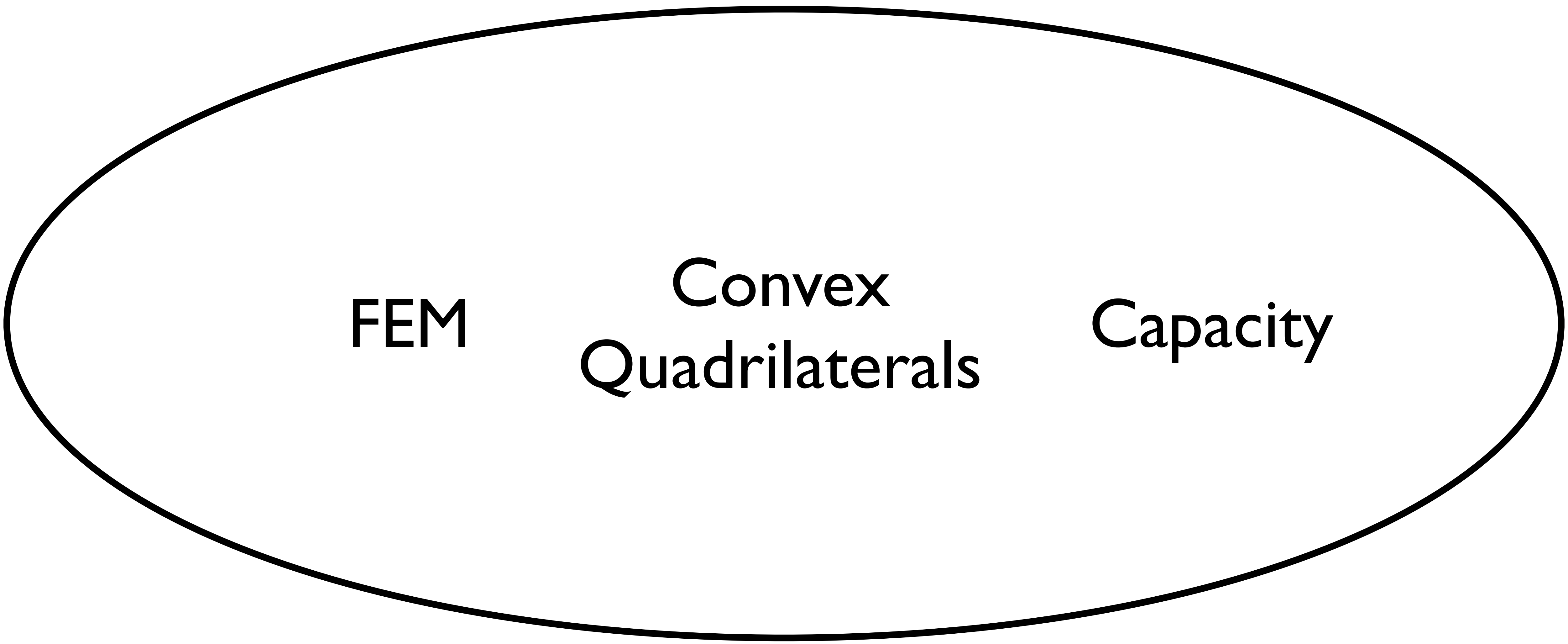

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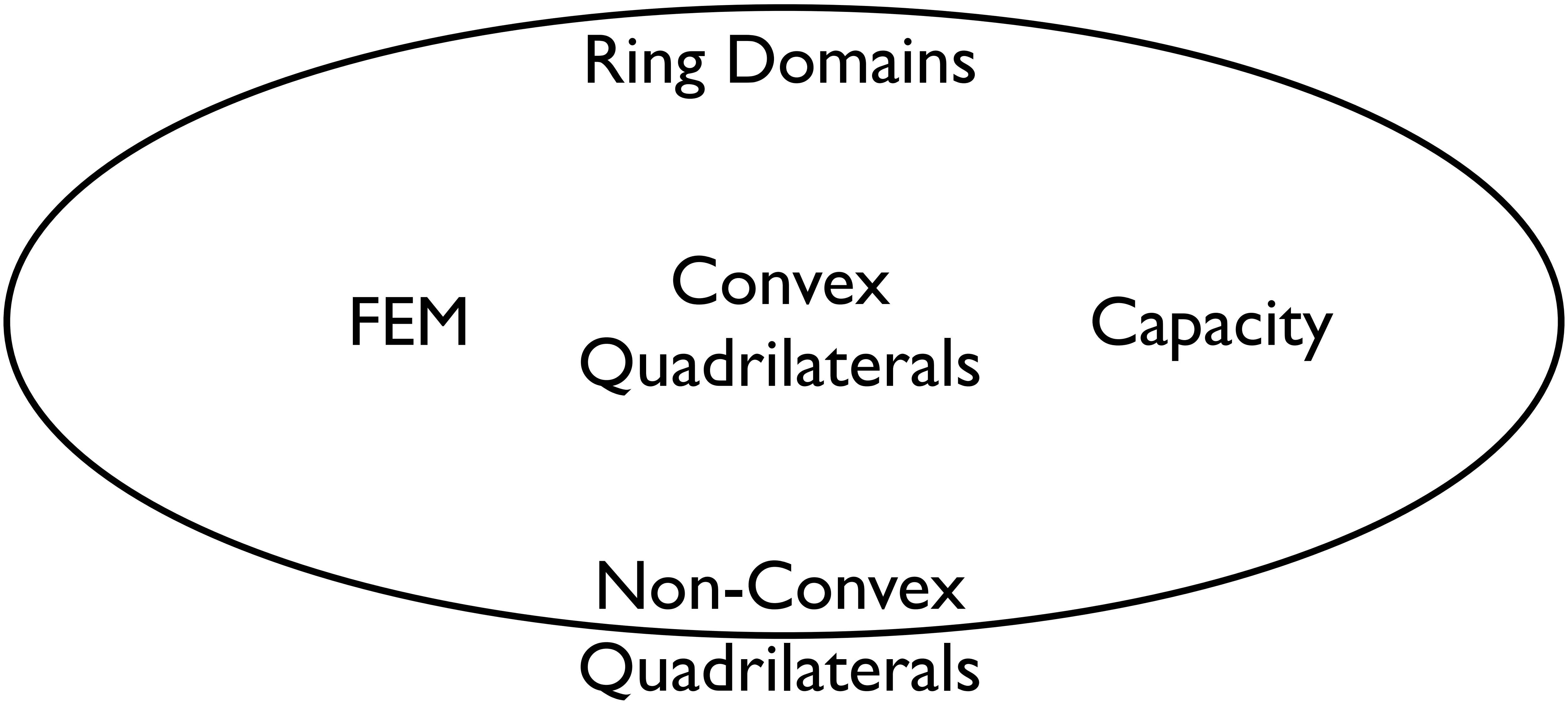

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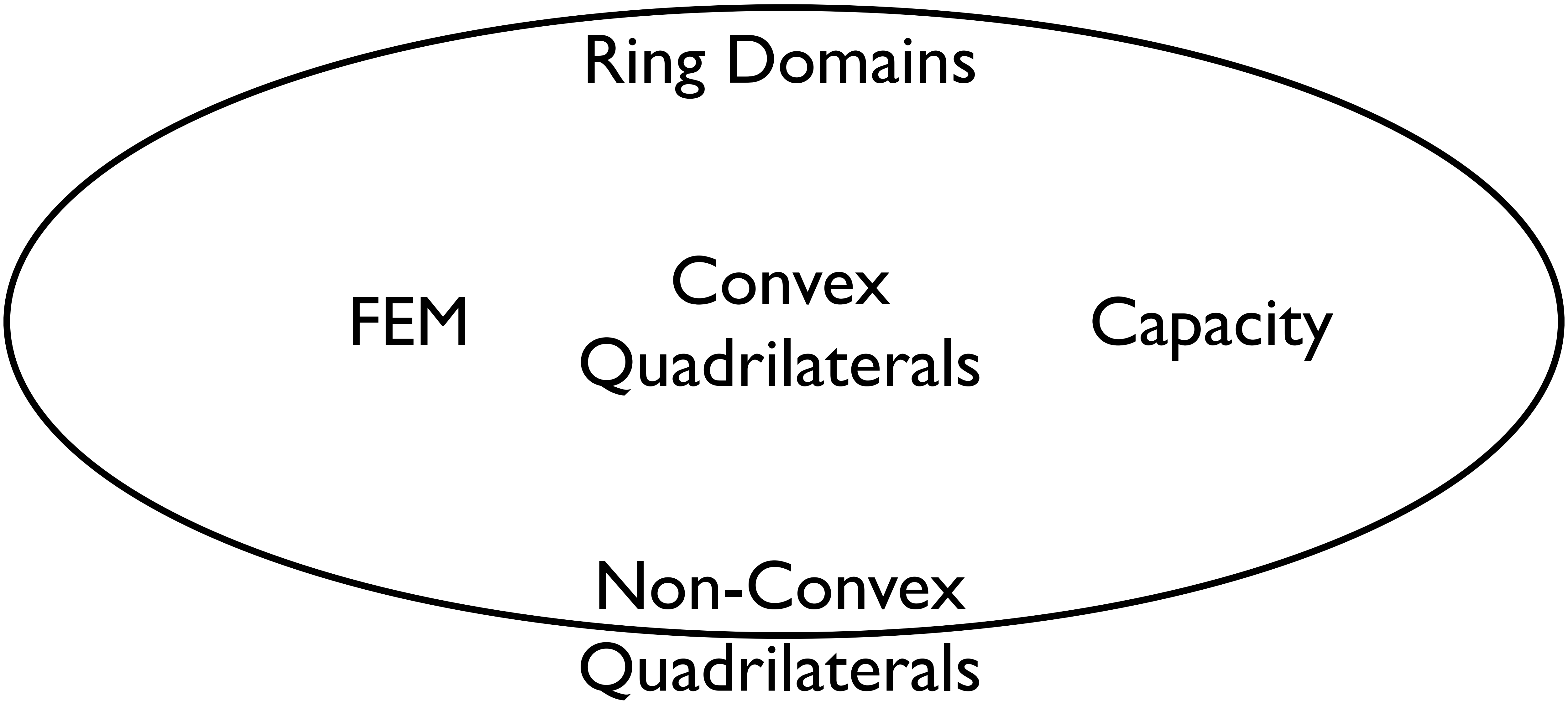


Here Be the 3D-Dragons

Objective:
Compute the quantities of interest as accurately as possible using FEM.

Dirichlet-Neumann Problem

Let D be a region in the complex plane whose boundary ∂D consists of a finite number of regular Jordan curves, so that at every point, except possibly at finitely many points, of the boundary a normal is defined. Let ψ to be a real-valued continuous function defined on ∂D. Let $\partial D = A \cup B$ where A, B both are unions of Jordan arcs. Find a function u satisfying the following conditions:

1. u is continuous and differentiable in $\overline{D}$.
2. $u(t) = \psi(t), \qquad \text{for all } t \in A.$
3. If $\partial/\partial n$ denotes differentiation in the direction of the exterior normal, then

$$\frac{\partial}{\partial n} u(t) = \psi(t), \qquad \text{for all } t \in B.$$

Modulus of a Quadrilateral

One can express the modulus of a quadrilateral $(D; z_1, z_2, z_3, z_4)$ in terms of the solution of the Dirichlét-Neumann problem as follows. Let $\gamma_j, j = 1, 2, 3, 4$ be the arcs of ∂D between (z_4, z_1) , (z_1, z_2) , (z_3, z_3) , (z_3, z_4), respectively. If u is the (unique) harmonic solution of the Dirichlét-Neumann problem with boundary values of u equal to 0 on γ_2, equal to 1 on γ_4 and with $\partial u/\partial n = 0$ on $\gamma_1 \cup \gamma_3$, then:

$$\mathrm{M}(D; z_1, z_2, z_3, z_4) = \int_D |\nabla u|^2 \, dm.$$

Reciprocal identity: $\mathrm{M}(D; z_1, z_2, z_3, z_4) \cdot \mathrm{M}(D; z_2, z_3, z_4, z_1) = 1$

Modulus of a Polygon

Suppose that w is a vector of p complex numbers such that the points $w_1, \ldots, w_q$, $q \geq 5$, are the vertices of a polygon D and that they define a positive orientation of the boundary. Choose indices $k_1, k_2 \in \{1, \ldots, p-1\}$ with $k_1 < k_2$ and set $z_1 = w_{k_1}$, $z_2 = w_{k_1+1}$, $z_3 = w_{k_2}$, $z_4 = w_{k_2+1}$. Then we define

$$\mathrm{cmodu}(w, k_1, k_2) = \mathrm{M}(D; z_1, z_2, z_3, z_4),$$

$$\mathrm{modu}(w, k_1, k_2) = \mathrm{M}(D; z_2, z_3, z_4, z_1)\,.$$

Reciprocal relation:

$$\mathrm{cmodu}(w, k_1, k_2) \cdot \mathrm{modu}(w, k_1, k_2) = 1\,.$$

Modulus of a Ring Domain

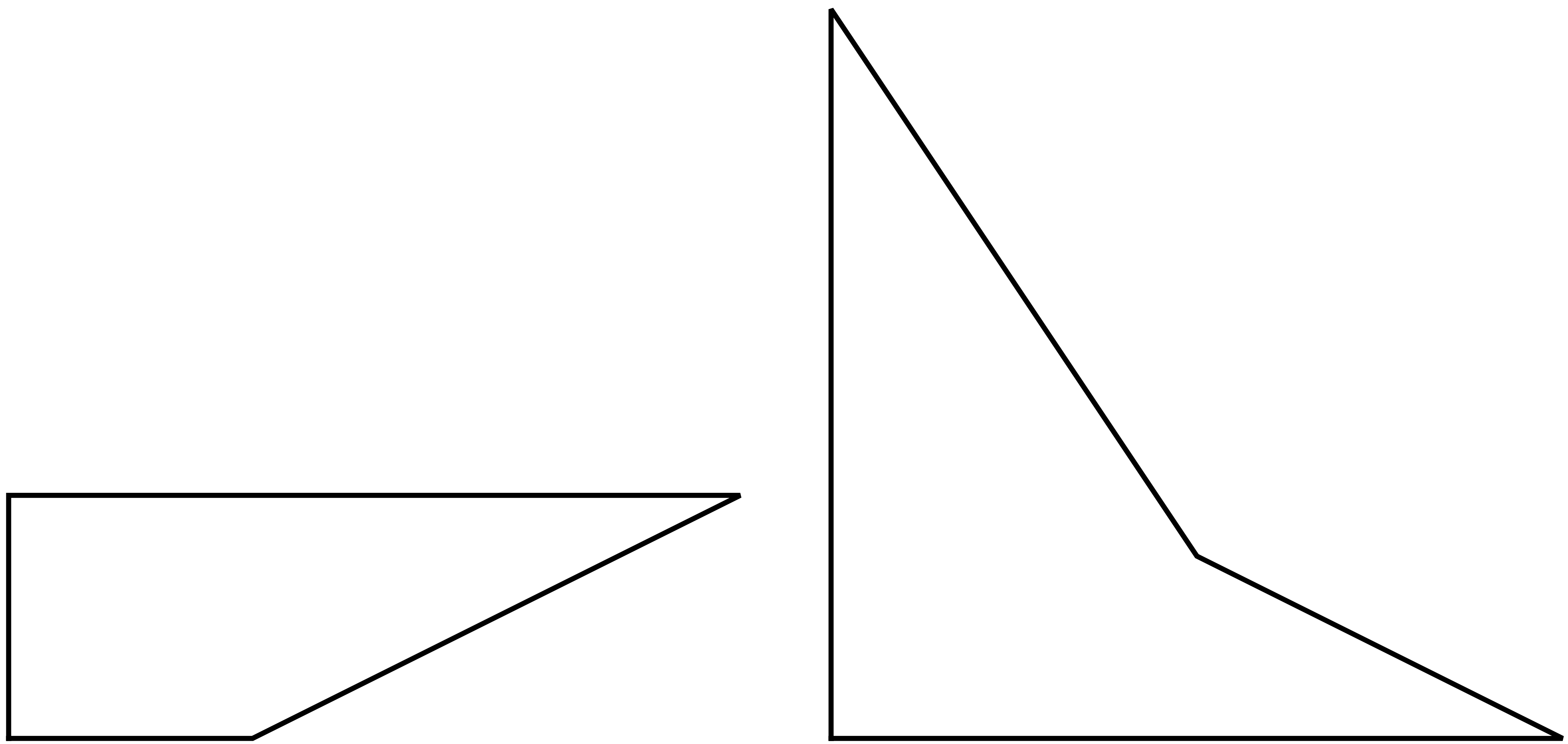
Let E and F be two disjoint compact sets in the extended complex plane . Then one of the sets E, F is bounded and without loss of generality we may assume that it is E . If both E and F are connected and the set $R = \setminus (E \cup F)$ is connected, then R is called a *ring domain*. In this case R is a doubly connected plane domain. The *capacity* of R is defined by

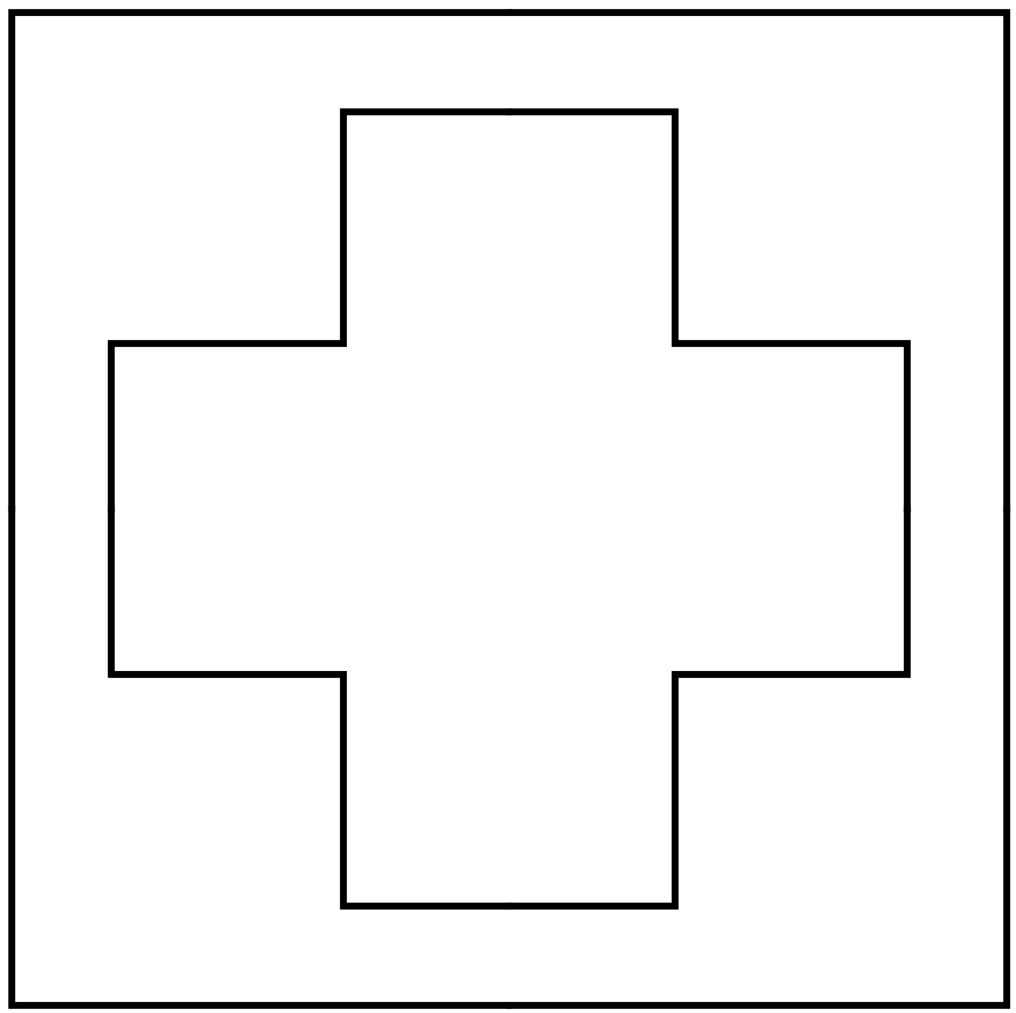
$$\operatorname{cap} R = \inf_u \int_R |\nabla u|^2 \, dm,$$

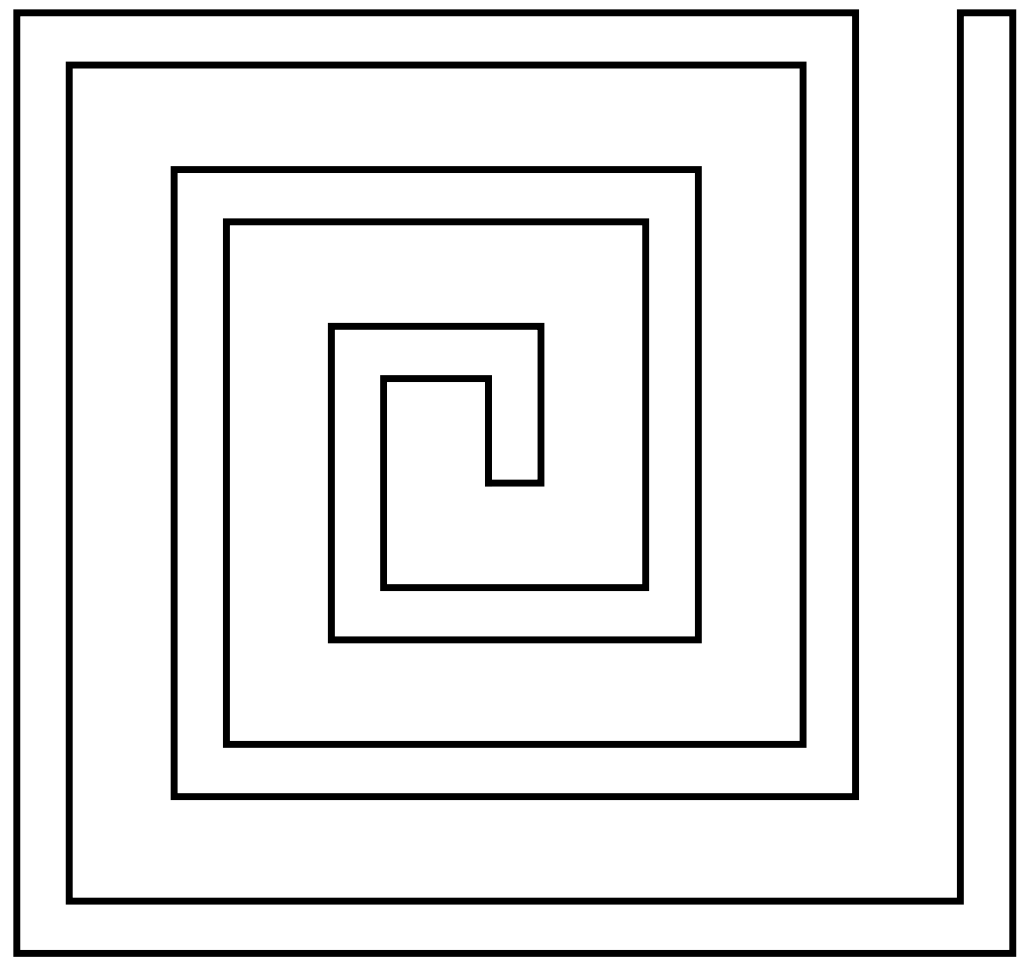
where the infimum is taken over all nonnegative, piecewise differentiable functions u with compact support in $R \cup E$ such that $u = 1$ on E.

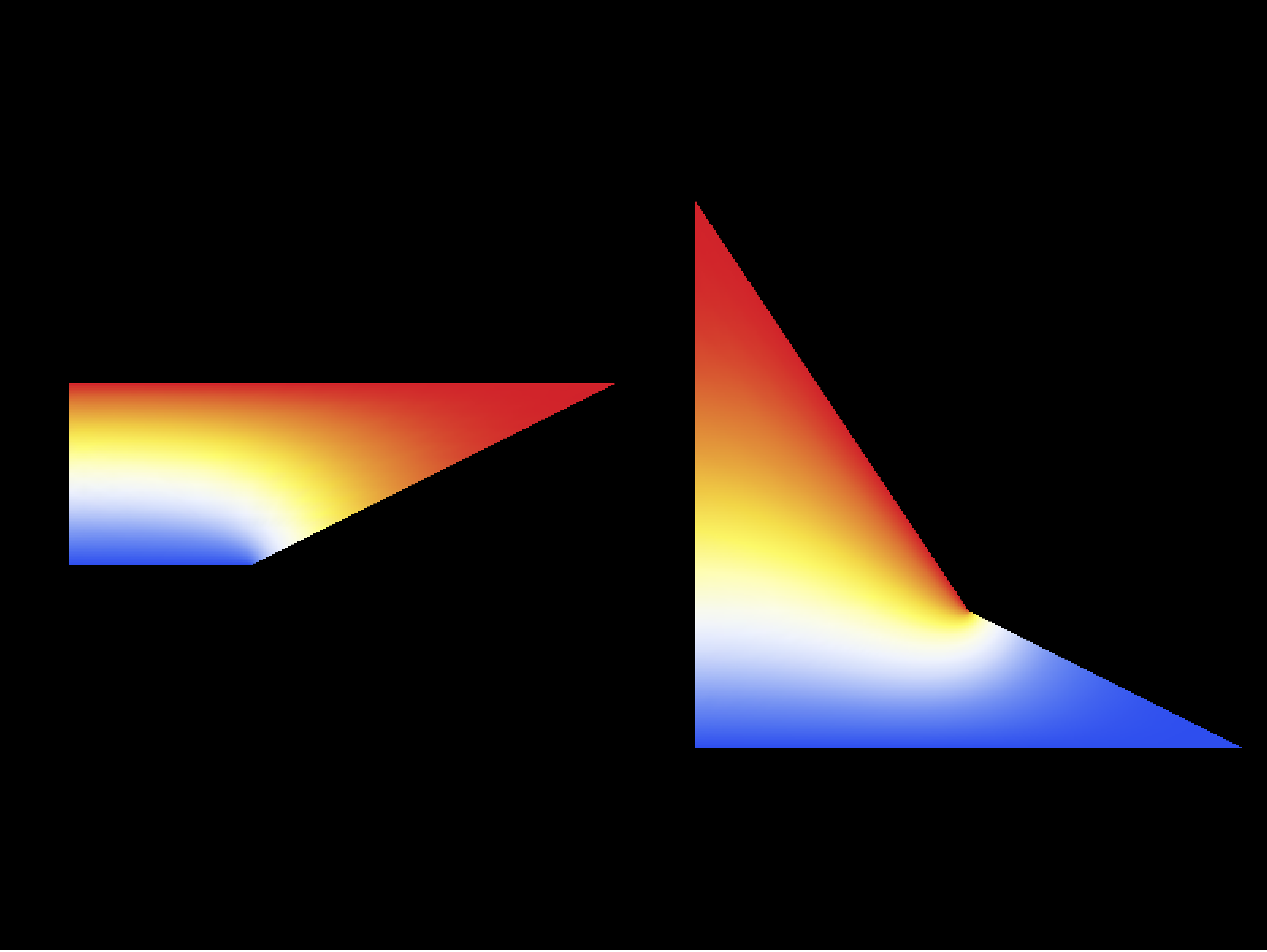
Conformal modulus: $M(R) = 2\pi / \operatorname{cap} R$

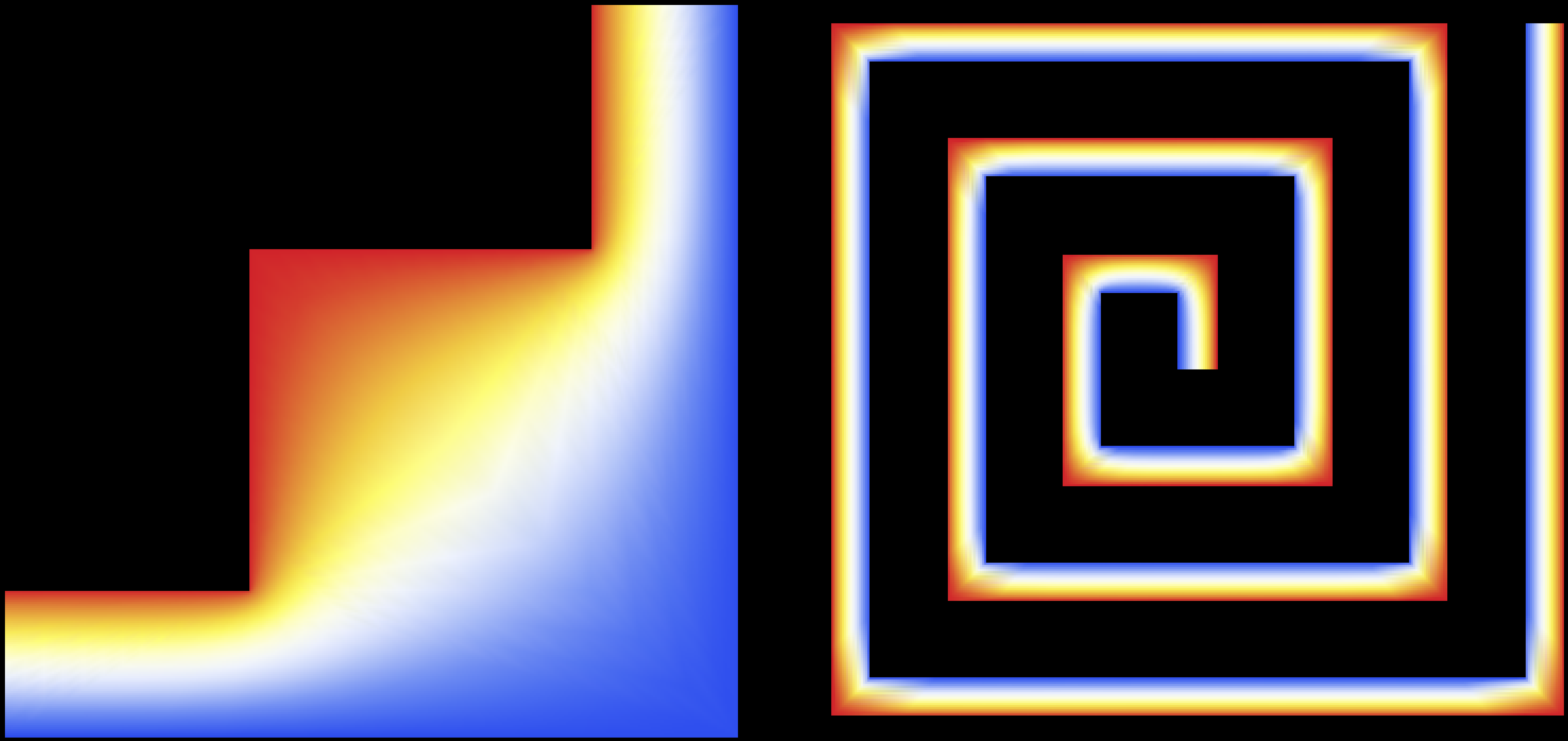
Domains of Interest



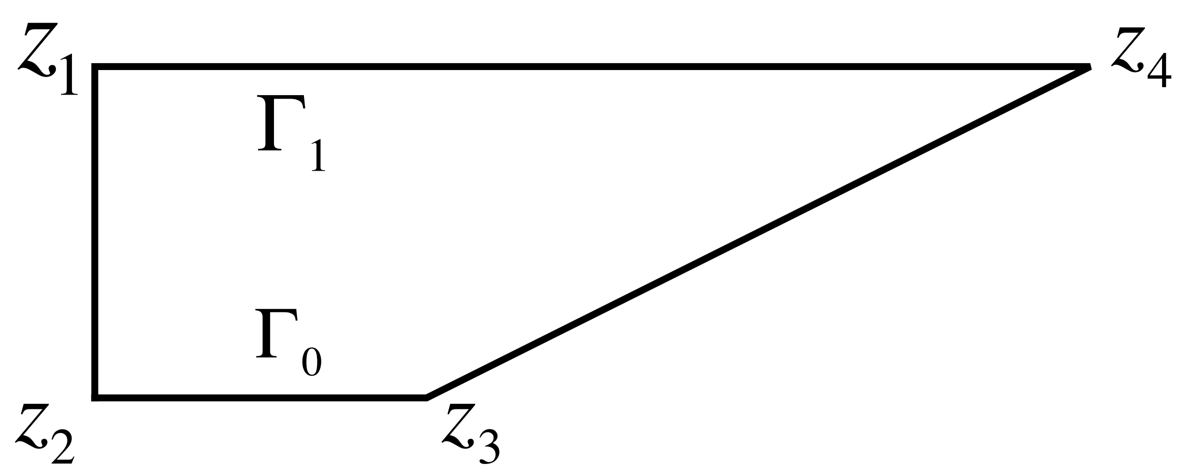








Model Problem



$$
\begin{aligned}
\Delta u &= 0, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \Gamma_0, \\
u &= 1, \quad \text{on } \Gamma_1, \\
\frac{\partial u}{\partial n} &= 0, \quad \text{on } \Gamma_2 \cup \Gamma_3,
\end{aligned}
$$

Variational problem: Find $u \in V$ such that

$$a(u, v) = 0, \quad \forall\, v \in V,$$

where

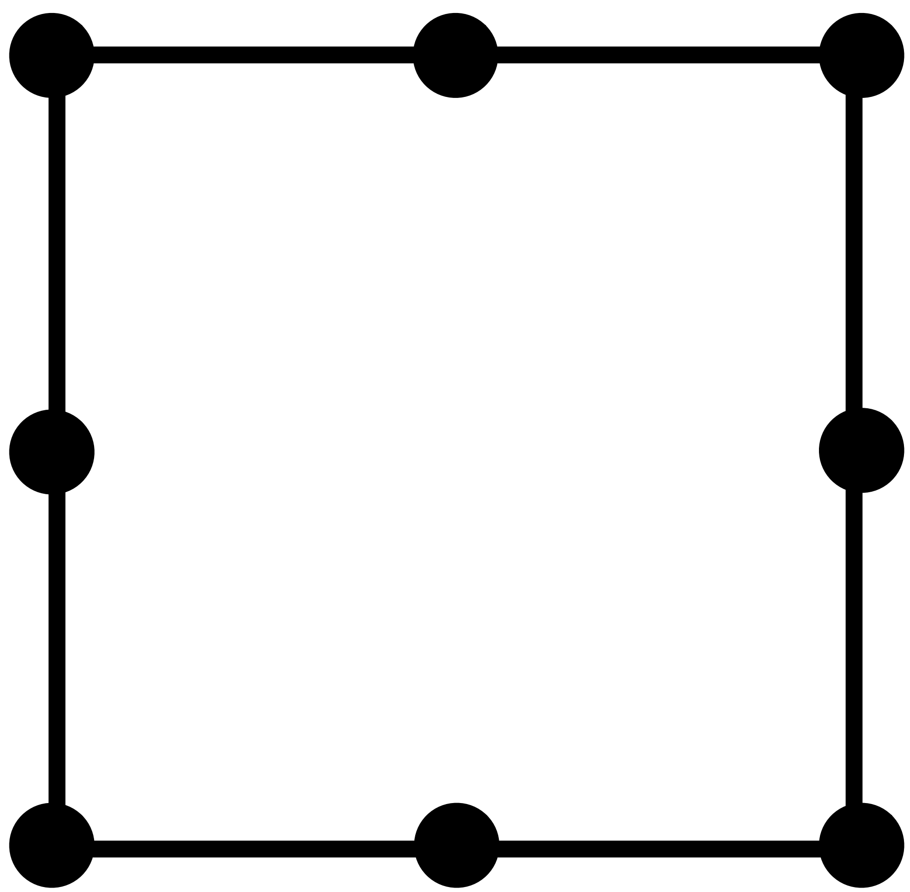
$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \; d\omega,$$

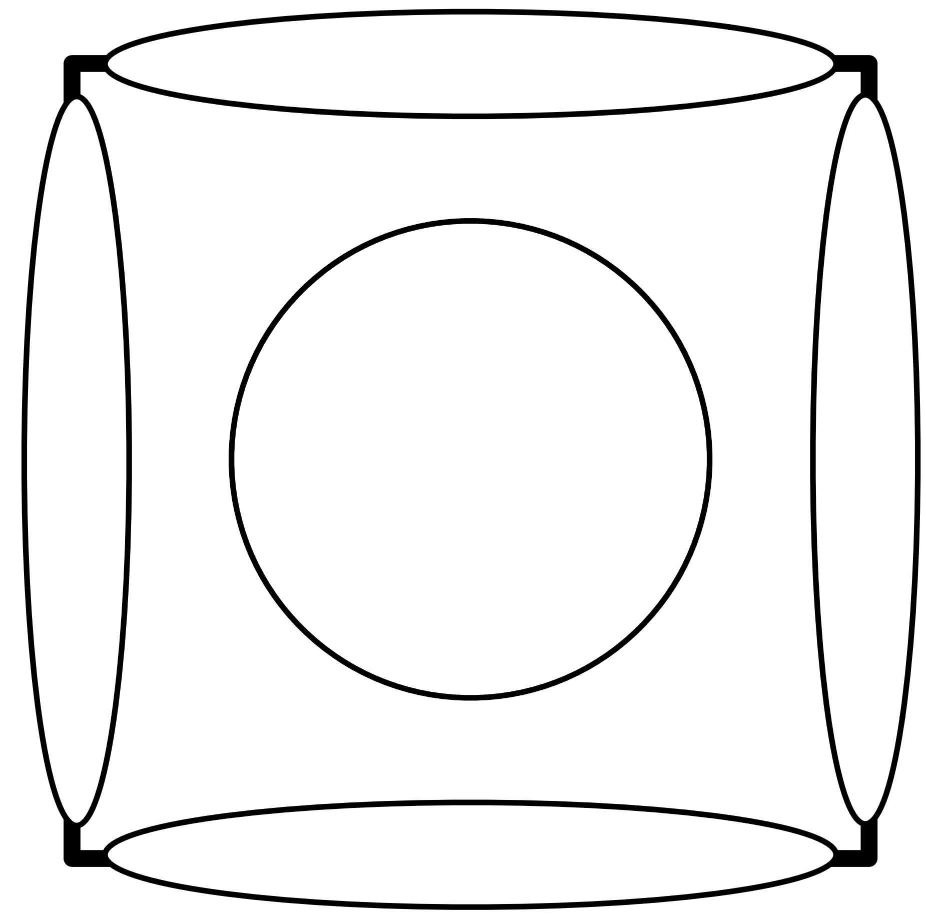
$$V = H_0^1.$$

FEM in 20s

- Choose your finite elements
- Discretize the computational domain
- Compute the local contributions of different elements
- Sum the local contributions to form a global system; set up the right hand side from the boundary conditions
- Solve the global linear system
- Solution minimizes the energy!

h- vs p-FEM Elements





Nodal degrees of freedom

Lagrange

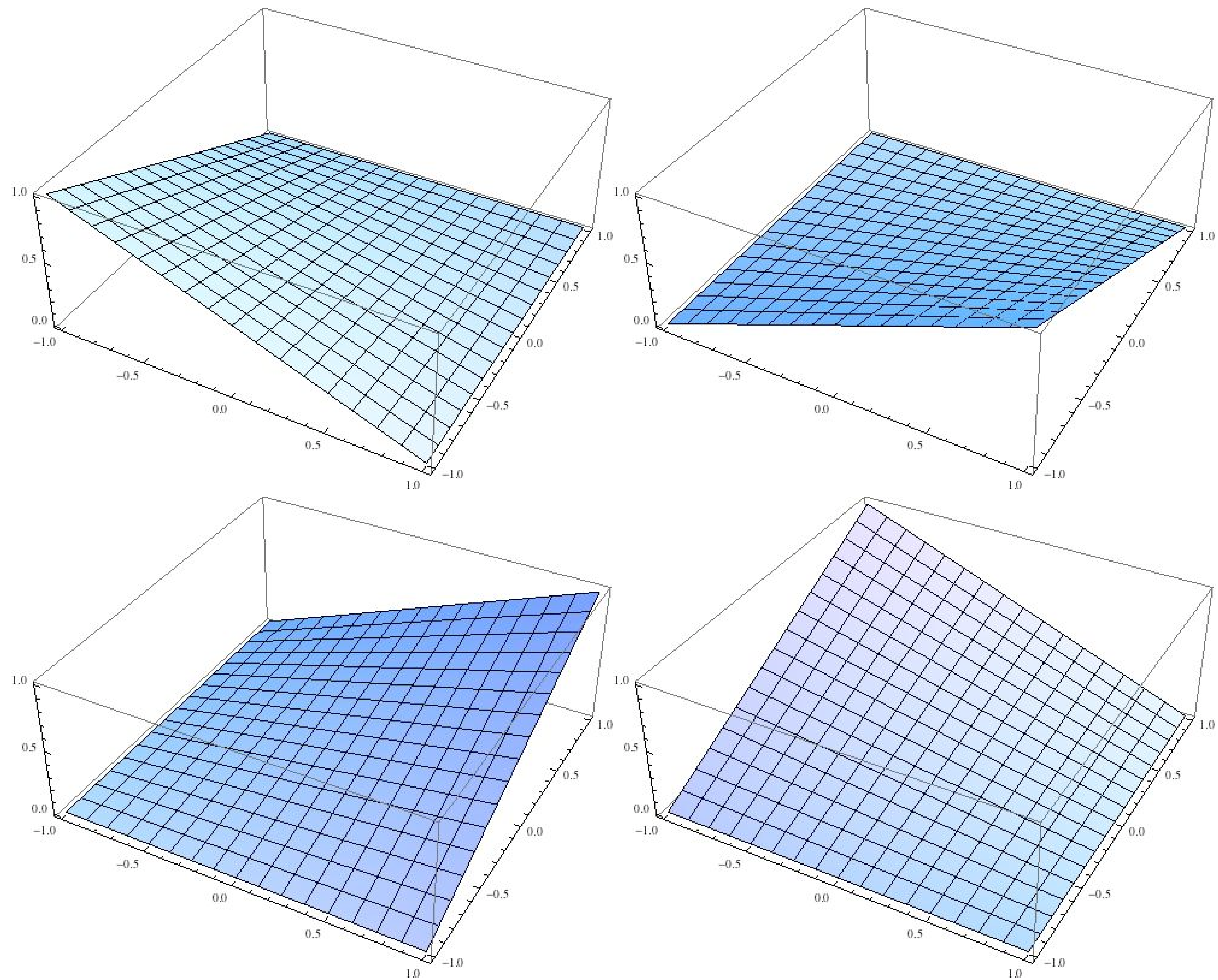
Modal degrees of freedom

Legendre

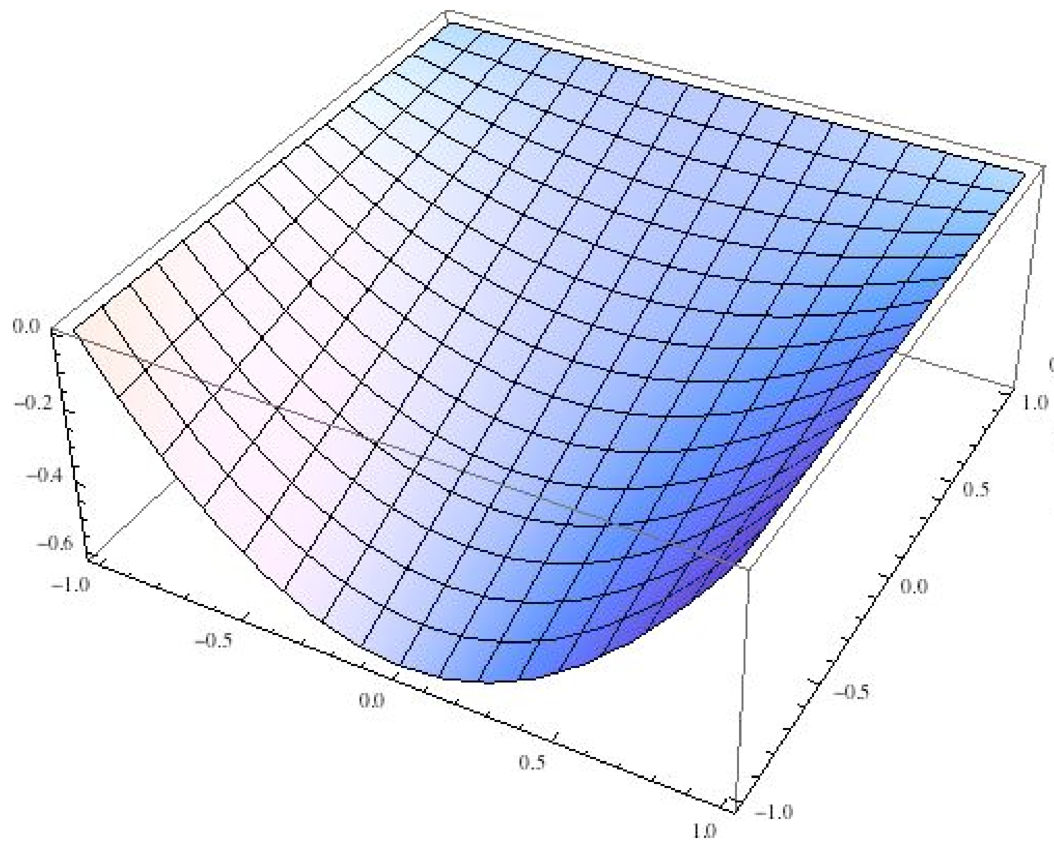
hp-Method

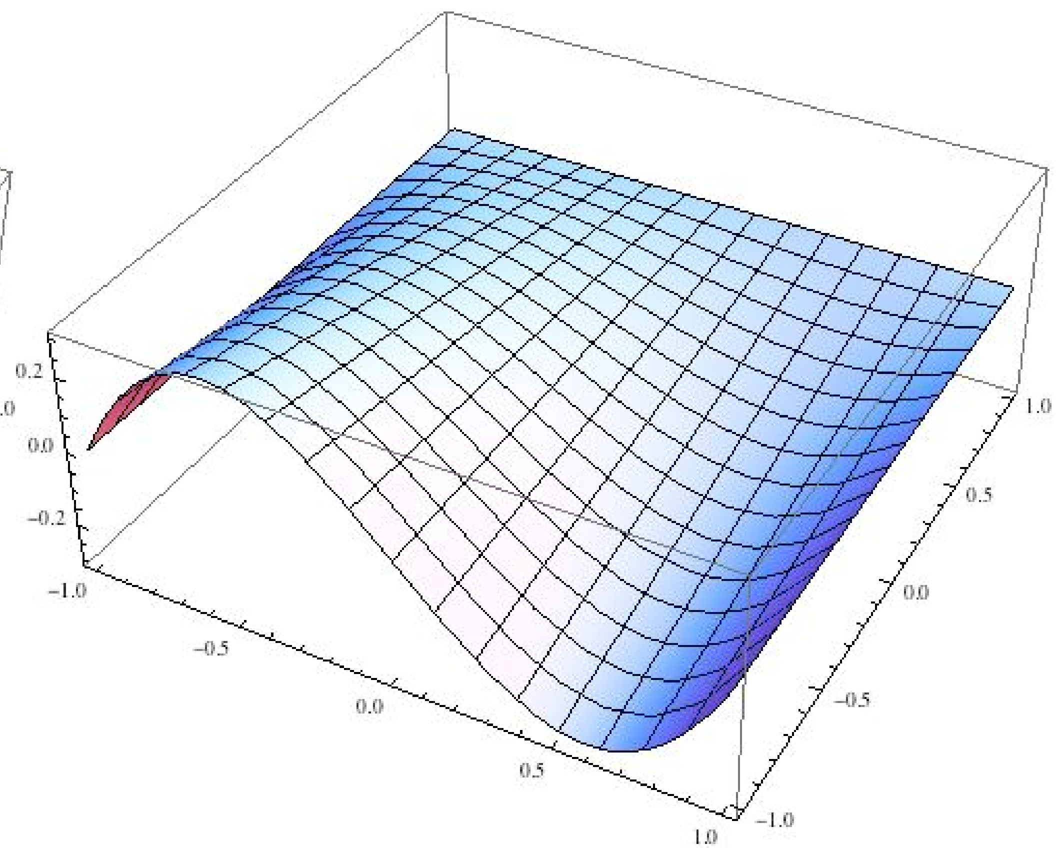
- h-method or classical FEM
  - Refine the mesh
- p-method
  - Keep the mesh fixed, increse the order of basis polynomials
- hp-method: Combine h and p!
- Number of unknows ~ #elements p^2

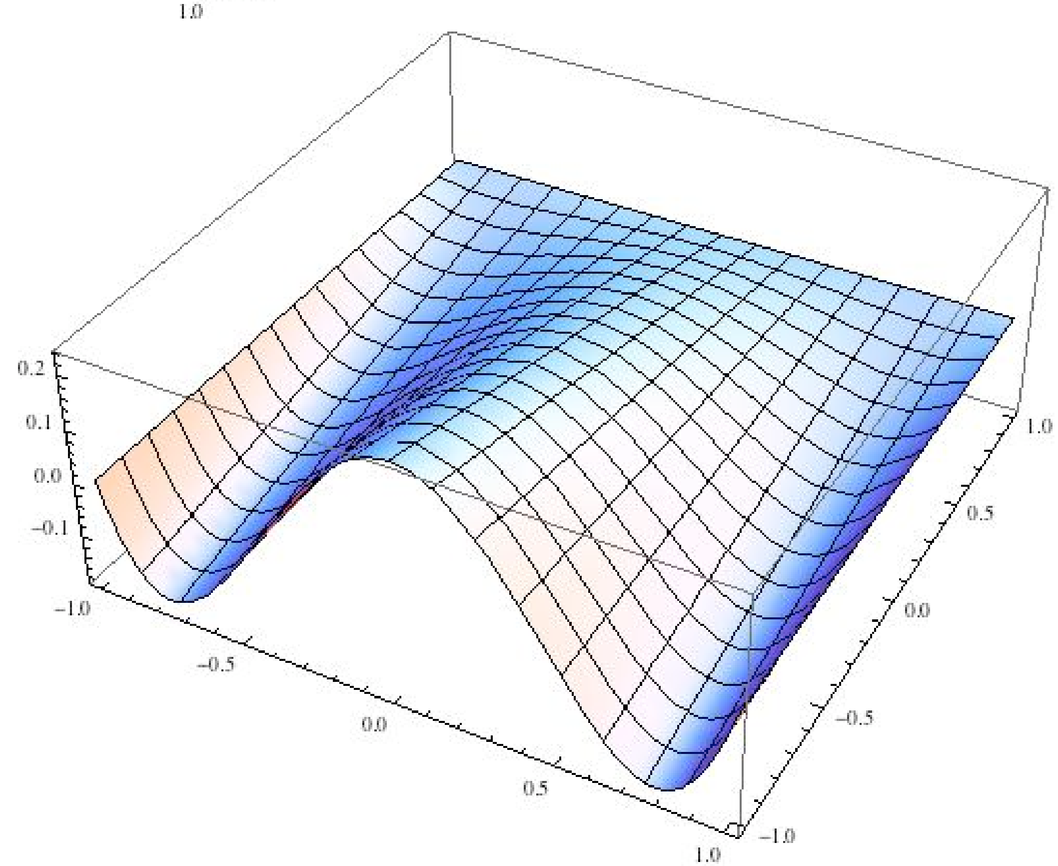
Linear Modes



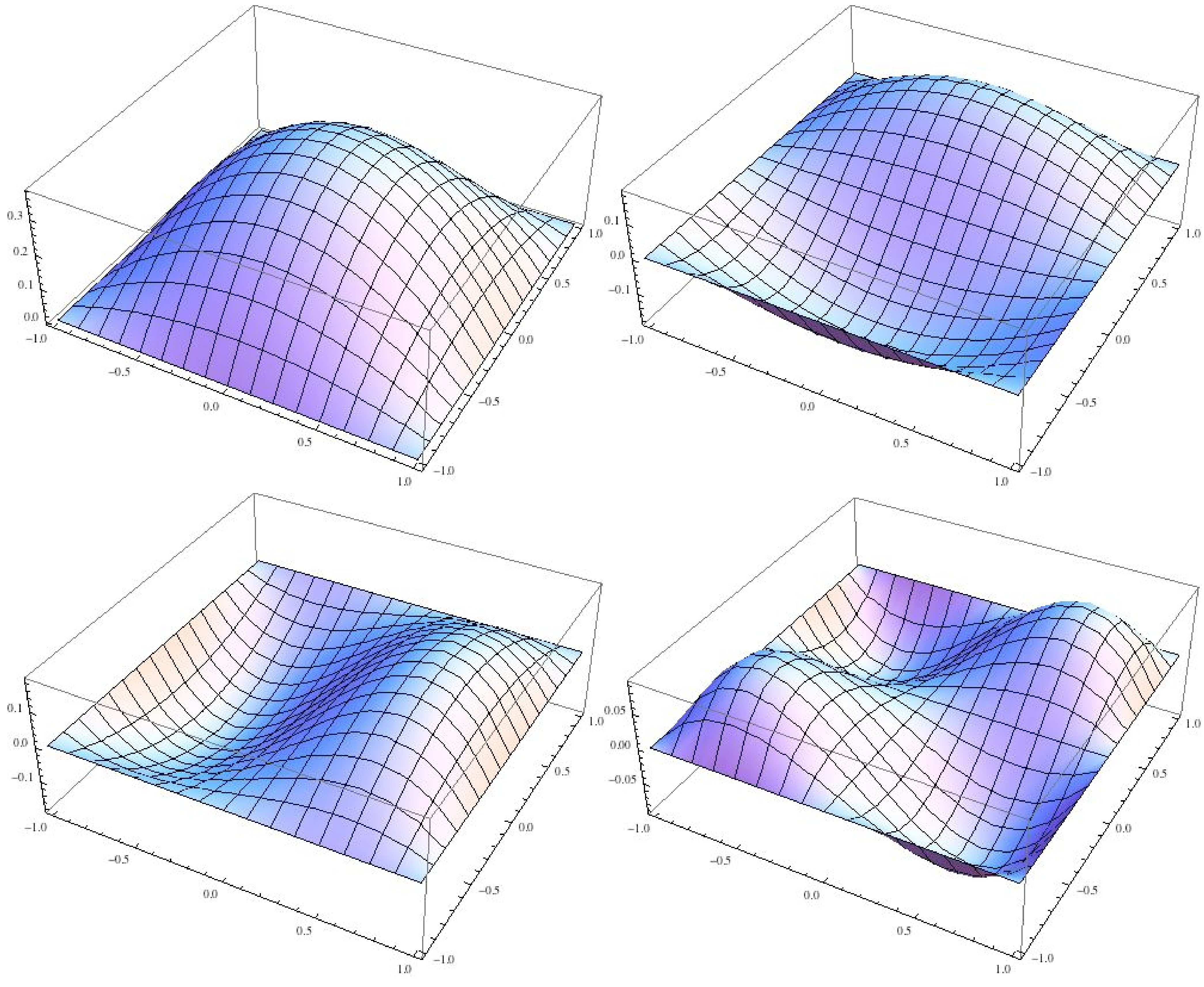
Side Modes


0.0
−0.2
−0.4
−0.6
−1.0
−0.5
0.0
0.5
1.0
−1.0
−0.5
0.0
0.5
1.0


0.2
0.0
−0.2
−1.0
−0.5
0.0
0.5
1.0
−1.0
−0.5
0.0
0.5
1.0


0.2
0.1
0.0
−0.1
−1.0
−0.5
0.0
0.5
1.0
−1.0
−0.5
0.0
0.5
1.0

Inner Modes



Exponential Convergence

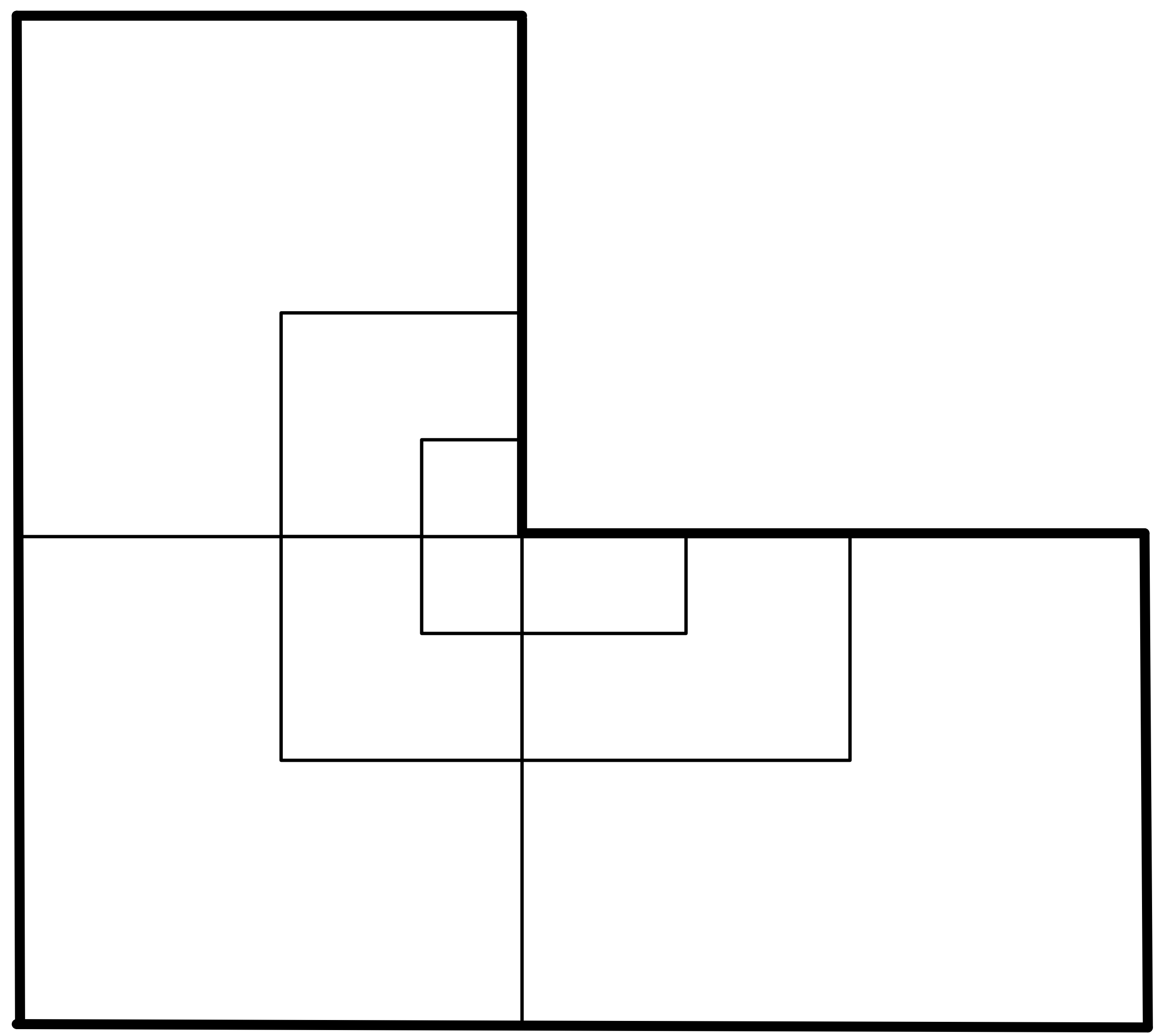
Theorem [Babuska & Guo]

Let $\Omega \subset \mathbb{R}^2$ be a polygon, v the FEM-solution, and the weak solution u_0 sufficiently smooth. Then

$$\inf_{v} \|u_o - v\|_{H^1(\Omega)} \leq C\, exp(-b\sqrt[3]{N}),$$

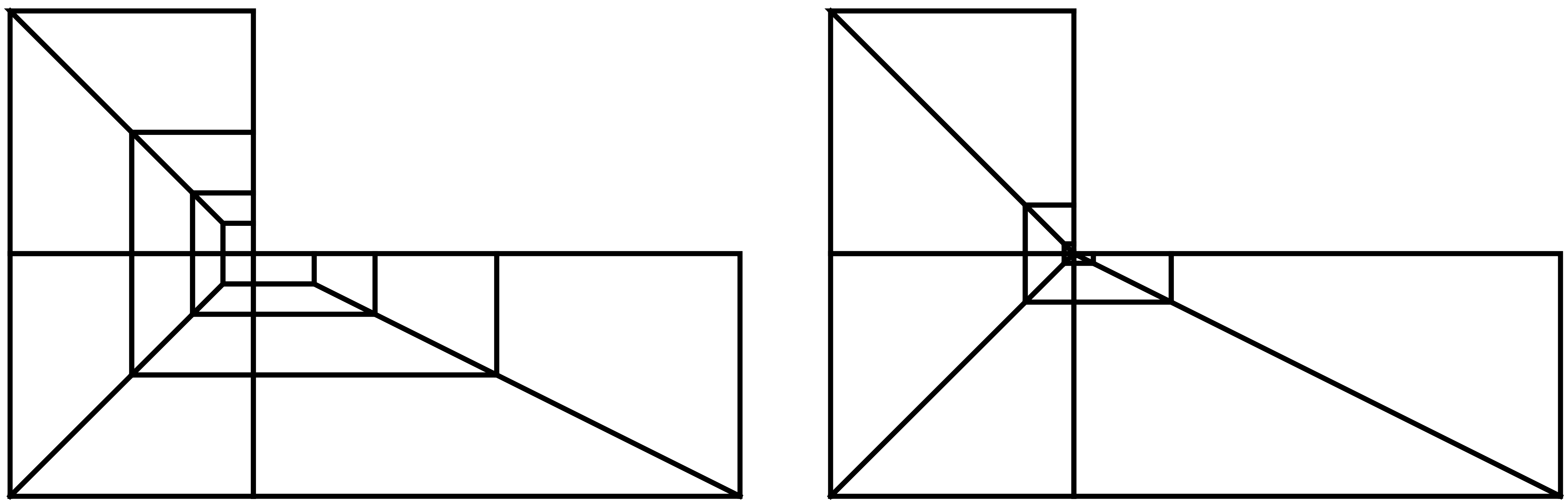
where C and b are independent of N, the number of degrees of freedom. Here v is computed on a proper geometric mesh, where the orders of individual elements depend on their originating layer, such that highest layers have smallest orders.

Geometric Mesh



Scaling and Nesting

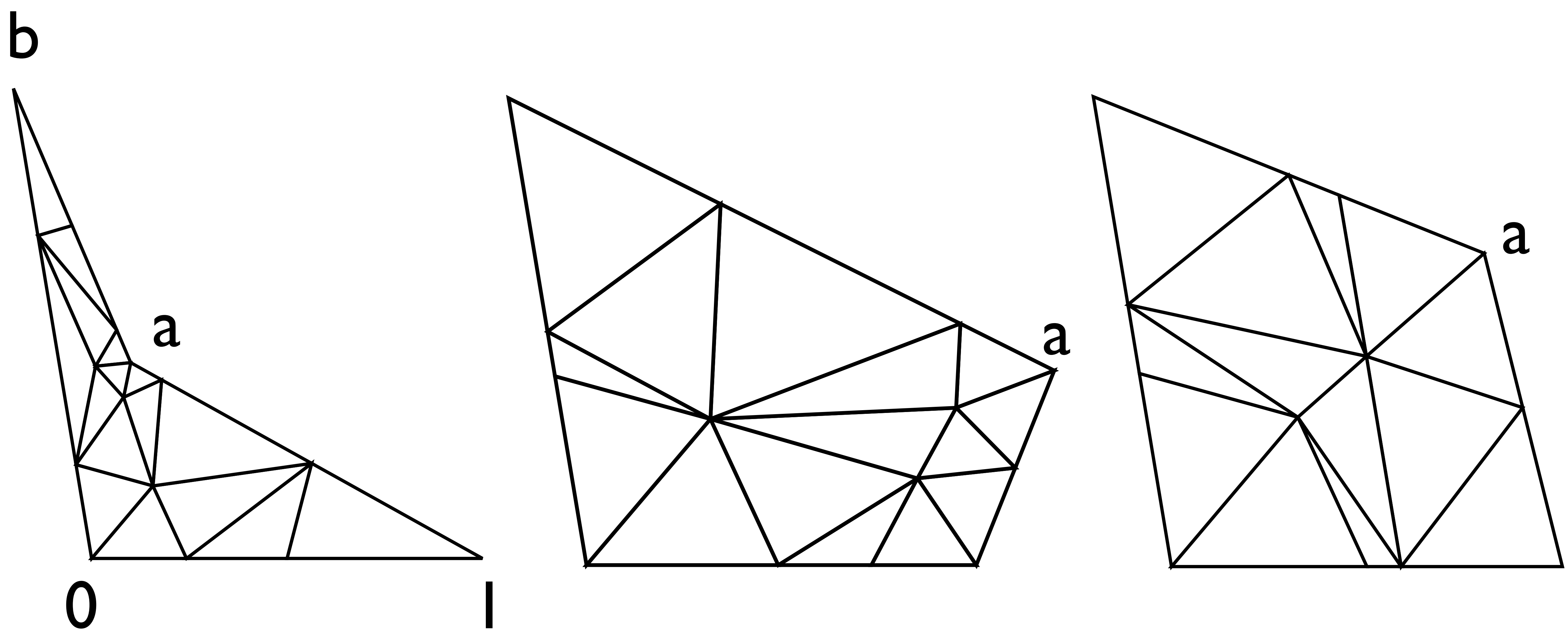
(α, ν) -meshes



α	Scaling Factor	0.5	0.2
ν	Nesting Level	3	3

Case Study:
Family of Quadrilaterals

Sample Meshes



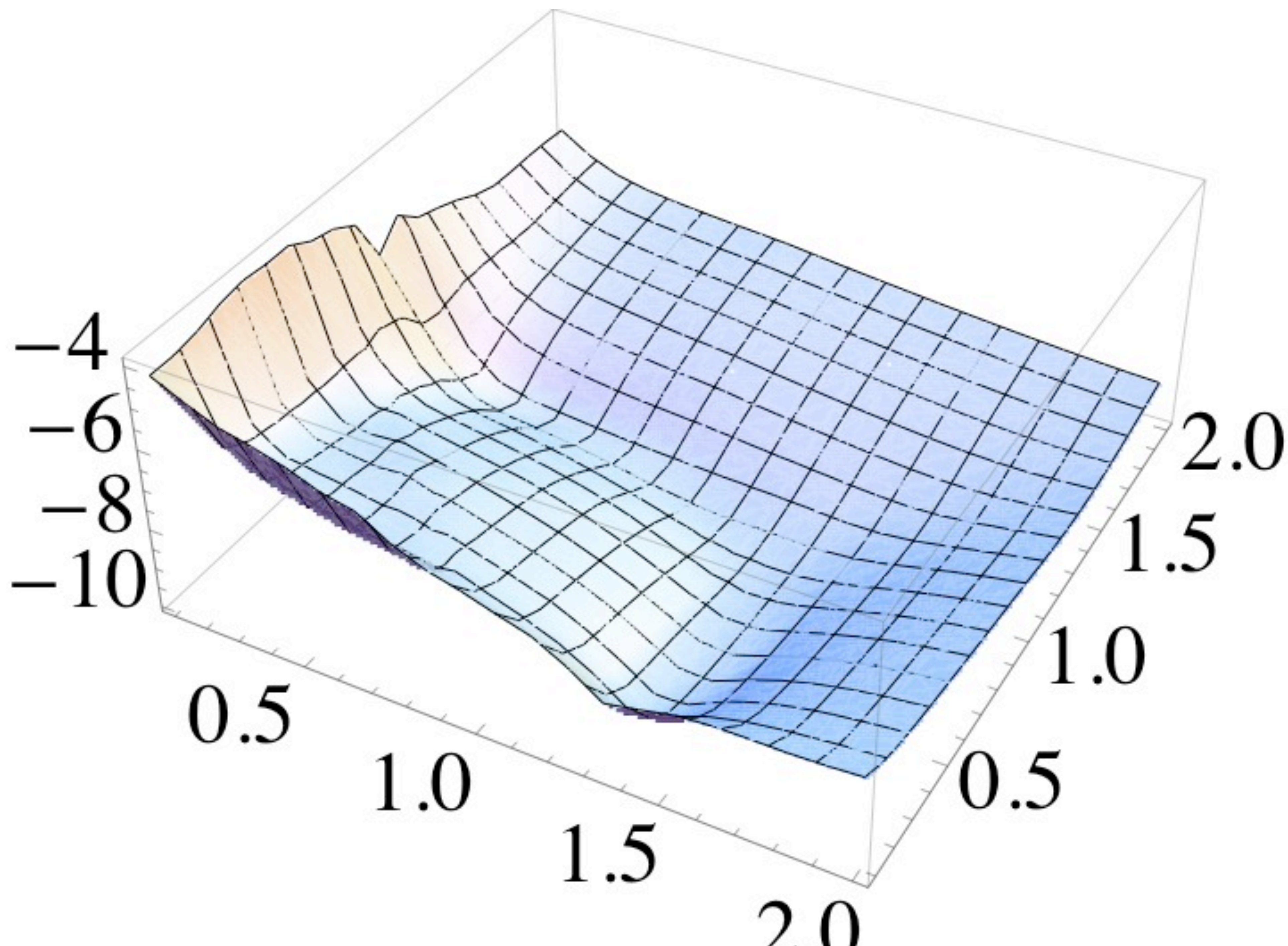
Three corners are fixed: {0,1,-0.2+1.2i}

Let the fourth corner get values inside [0.1,2]x[0.1,2].

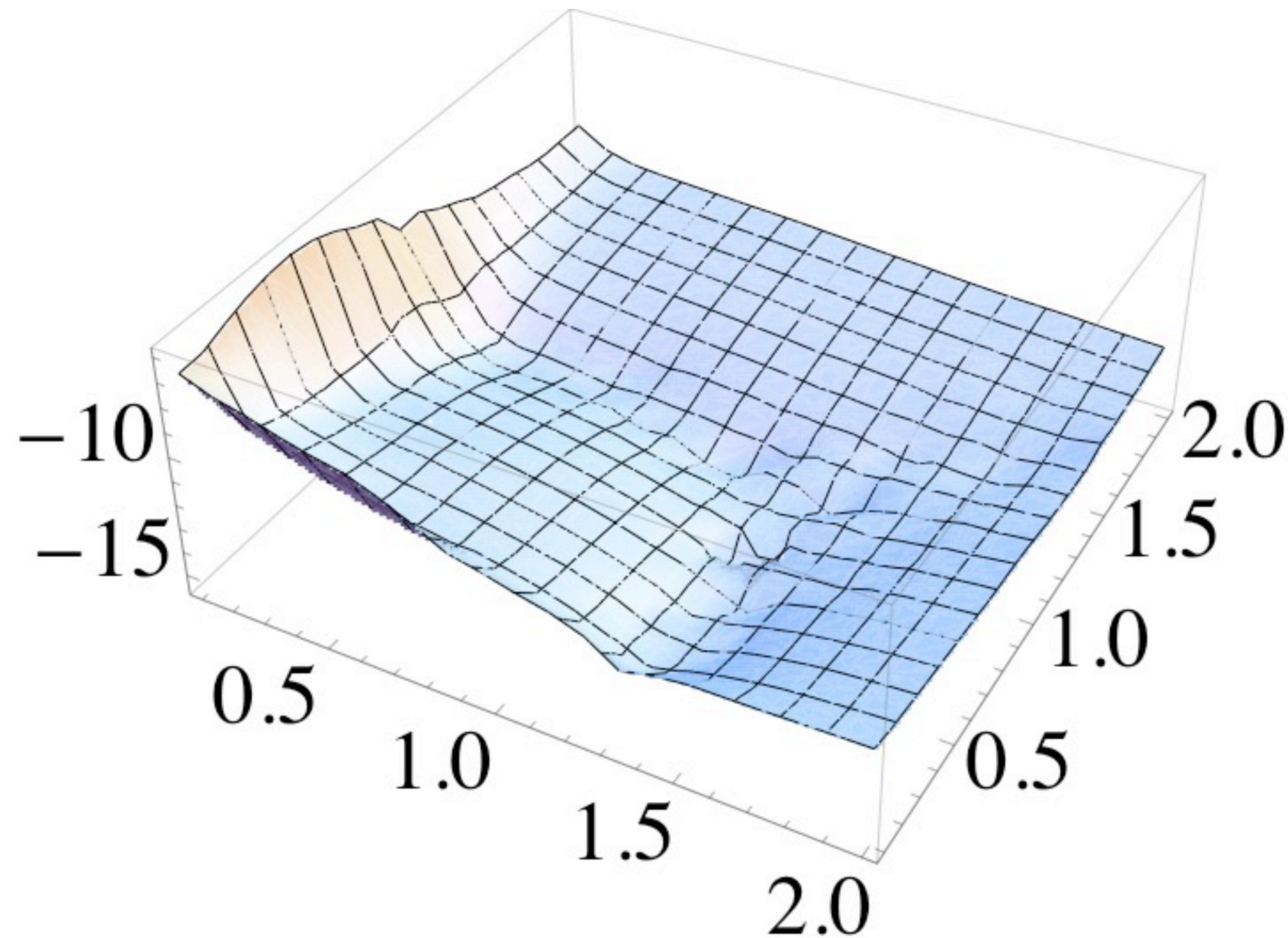
Numerical Experiments

- Given 0,1,b, choose the fourth corner; a
- Compute the domains corresponding to
  $$\mathrm{M}(D;a,b,0,1)\cdot\mathrm{M}(D;(b-1)/(a-1),1/(1-a),0,1)=1$$
- Create the (α,ν) -meshes: (0.15,12)
- Solve the systems for p=1,...,20
- Compute the reciprocal identity – 1
- Log10 indicates the number of correct decimals

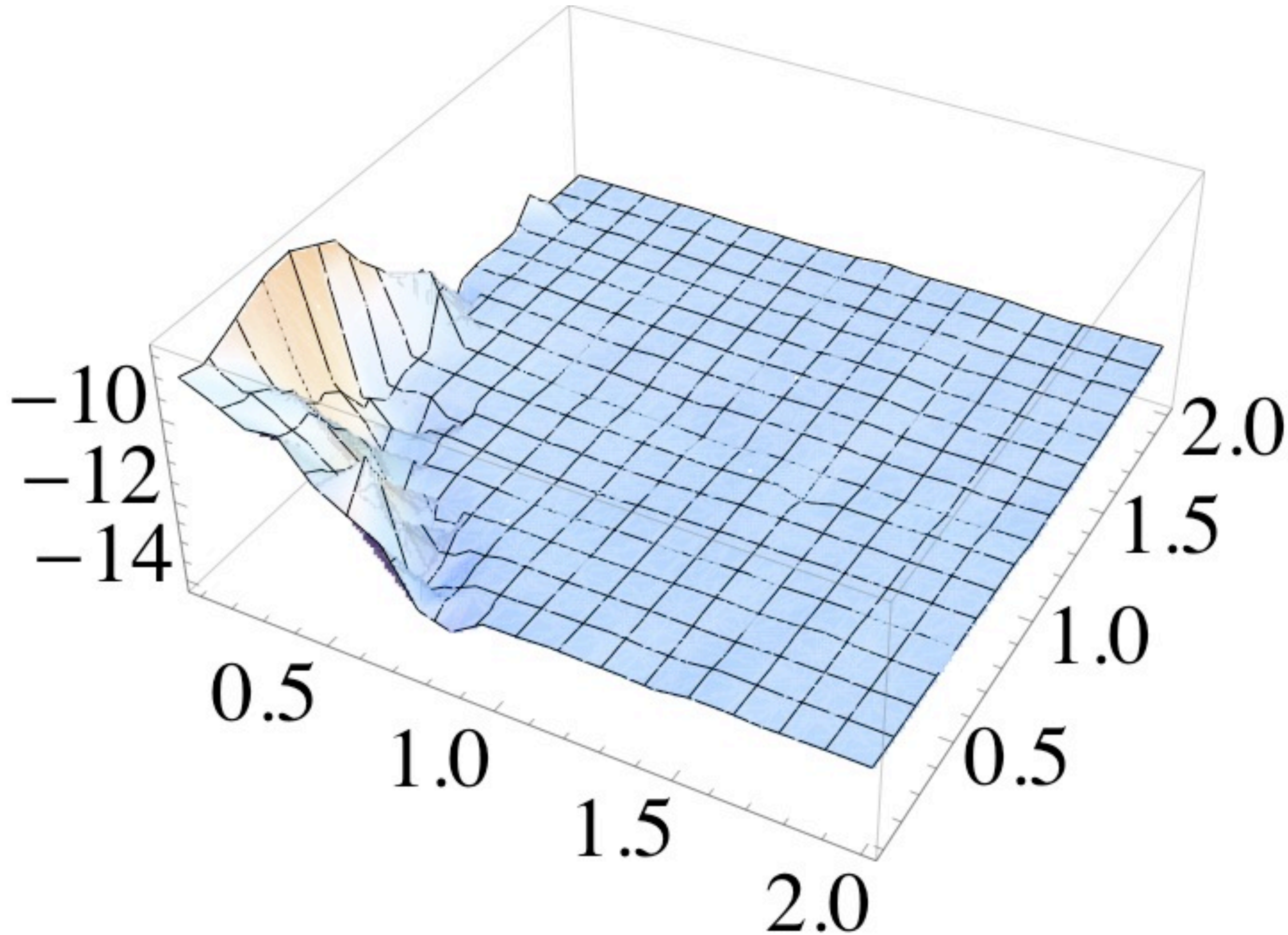
p=8, Log10 Error



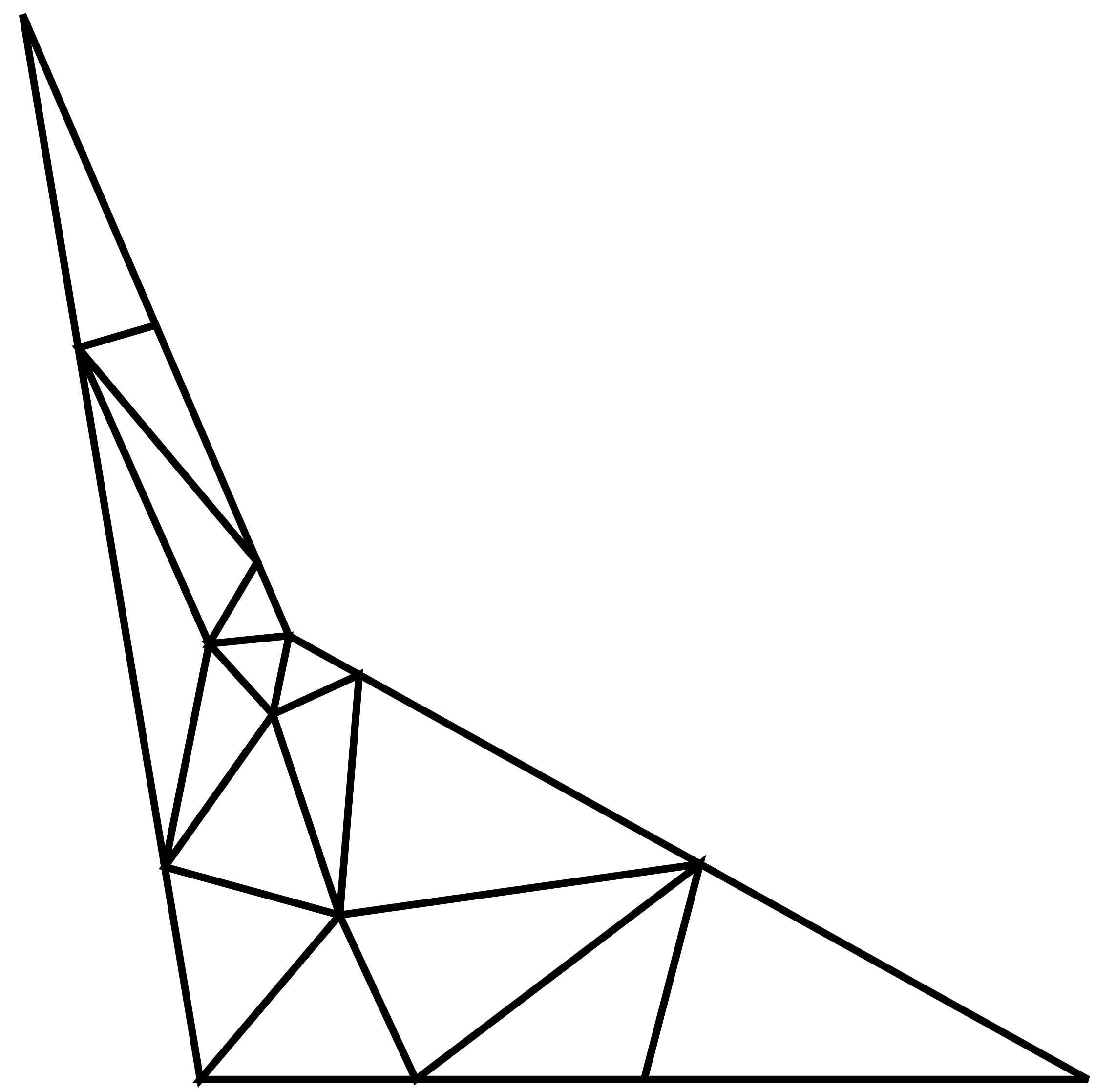
p=13, Log10 Error



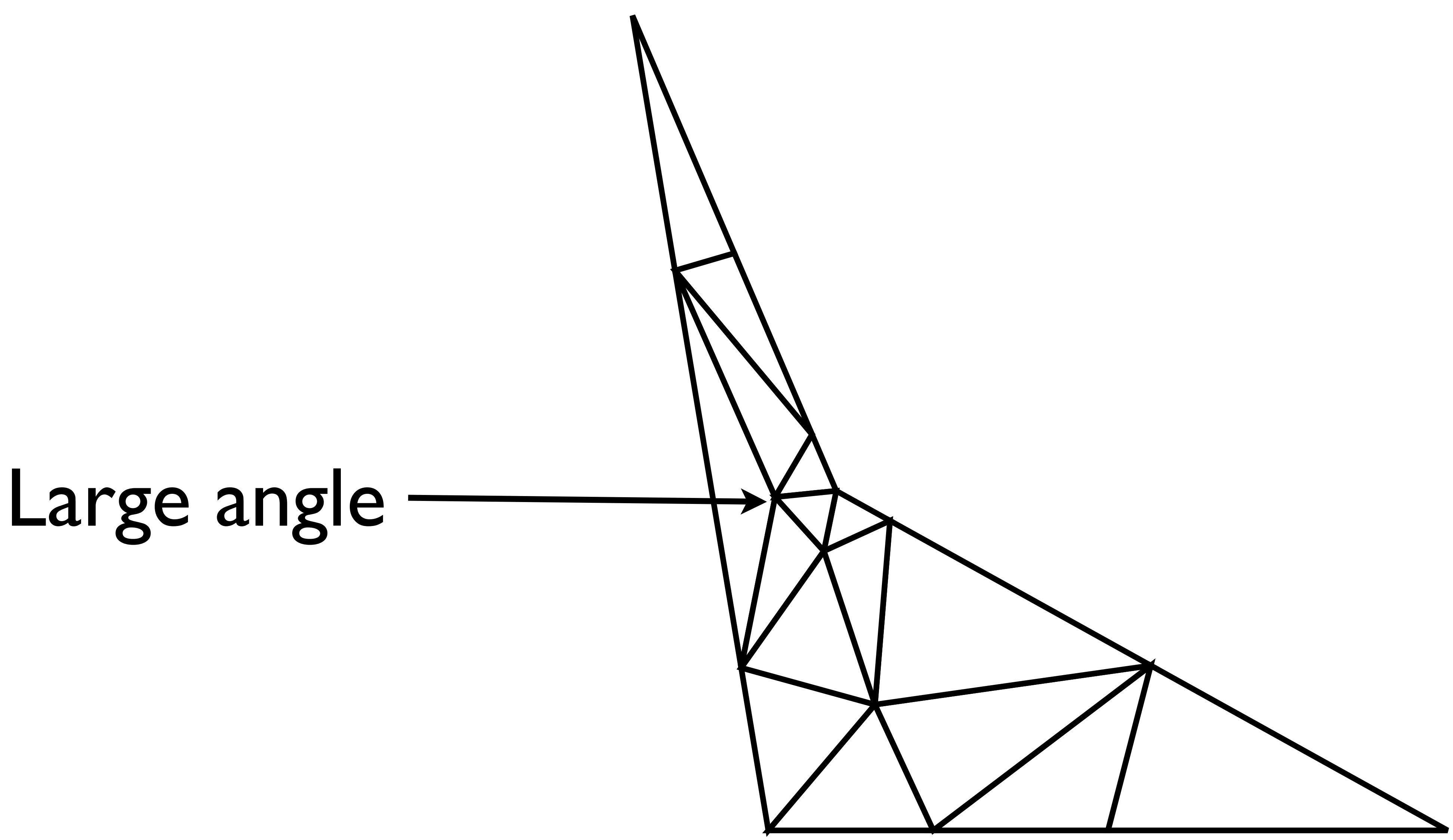
p=20, Log10 Error



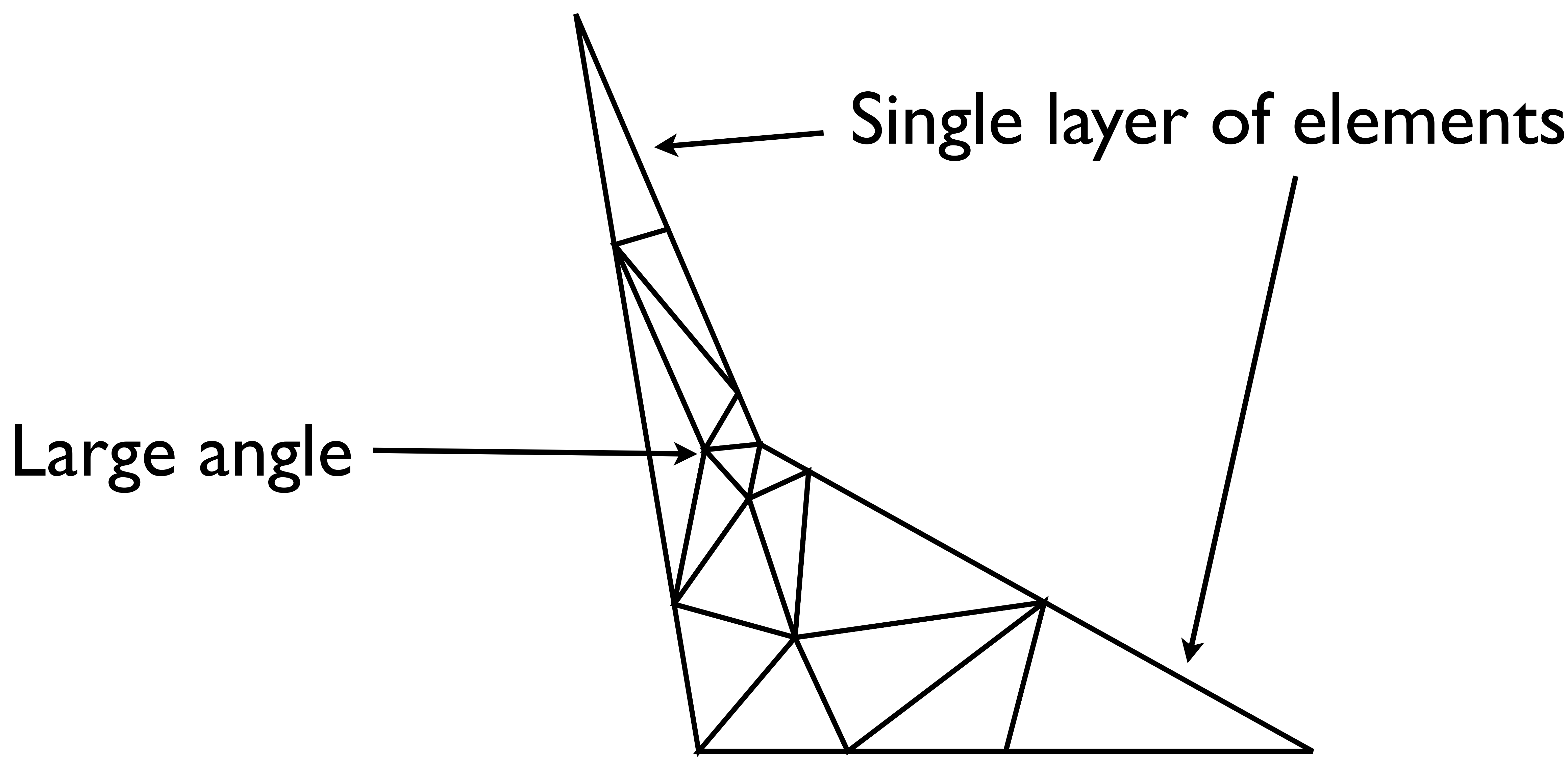
Maximal Error Case



Maximal Error Case



Maximal Error Case



Comments

- For certain geometries computation of moduli is akin to evaluation of special functions.
- Code will be available if you ask kindly. Documentation is somewhat lacking.
- Implemented in Mathematica.

Challenges

- What is the best possible initial mesh?
  - Large angles vs small angles
- Is a posteriori error analysis feasible?
- What about moduli in higher dimensions?
  - p-Laplace, non-linear

Reference

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Harri Hakula, Antti Rasila, Matti Vuorinen

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