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Complex linear differential equations in the unit disc at a glance – An overview of the research conducted on the 21st Century

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Part I: Growth of solutions

- **Classical plane results and their analogues in *D***
- **Direct problem & inverse problem**
- **Solutions of slow/moderate/fast growth**

Part II: Tools

Part III: Oscillation of solutions

H. Wittich (AASF 1966):

The entire coefficients of

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f'(z) + A_0(z)f = 0 \qquad (1)$$

are polynomials if and only if all solutions of (1) **are entire functions of finite order of growth.**

M. Frei (CMH 1961):

Let $A_q(z)$ **be the last transcendental entire coefficient, while** $A_{q+1}(z), \ldots, A_{k-1}(z)$ **are polynomials. Then** (1) **possesses at least** $k-q$ **linearly independent entire solutions of infinite order.**

Exact orders of growth by G. Gundersen, E. Steinbart and S. Wang (TAMS 1998).

J.H. (PhD 2000):

- **The analytic coefficients of**

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'(z)+A_0(z)f=0 \qquad (2)$$

  are in the Korenblum space $A^{-\infty}$ if and only if all solutions of (2) are analytic in D and of finite order.

- **Let $A_q(z)$ be the last analytic coefficient not in $A^{-\infty}$, while $A_{q+1}(z),\ldots,A_{k-1}(z)\in A^{-\infty}$. Then (2) possesses at least $k-q$ linearly independent analytic solutions of infinite order.**

Similar results by D. Benbourenane (PhD 2001).

Growth between coefficients and solutions:

- **Direct problem:**

  Growth of coefficients $\to$ growth of solutions

- **Inverse problem:**

  Growth of solutions $\to$ growth of coefficients

Some attempts:

- **I. Chyzhykov, G. Gundersen and J.H. (PLMS 2003)**
- **Z.-X. Chen and K.H. Shon (JMAA 2004)**

J.H., R. Korhonen and J. Rättyä (TAMS 2008):

Direct Problem		Inverse Problem	
Assumption	***Result***	***Assumption***	***Result***
$A_j \in B^{\frac{1}{k-j}}$	$f \in N$	$f \in N$	$A_j \in \bigcap_{0<p<\frac{1}{k-j}} B^p$
$A_j \in H^{\frac{1}{k-j}}_{k-j}$	$f \in F$	$f \in F$	$A_j \in \bigcap_{0<p<\frac{1}{k-j}} H^p_{\frac{1}{p}}$
$A_j \in B^{\frac{1}{k-j}}_{\alpha}$	$\rho(f) \leq \alpha$	$\rho(f) \leq \alpha$	$A_j \in \bigcap_{0<p<\frac{1}{k-j}} B^p_{\alpha}$
$A_j \in H^{\frac{1}{k-j}}_{(\alpha+1)(k-j)}$	$\rho(f) \leq \alpha$	$\rho(f) \leq \alpha$	$A_j \in \bigcap_{0<p<\frac{1}{k-j}} H^p_{\frac{\alpha+1}{p}}$

Necessary and sufficient conditions for solutions to be of finite order, plus estimates for the number of linearly independent solutions of maximal growth, in terms of

- **Nevanlinna characteristic** $T(r, g)$
  R. Korhonen and J. Rättyä (JMAA 2009)

- **maximum modulus** $M(r, g)$
  R. Korhonen and J. Rättyä (PAMS 2007)
  I. Chyzhykov, J.H. and J. Rättyä (to appear in JAMS)

- φ**-order of growth**
  I. Chyzhykov, J.H. and J. Rättyä (to appear in Jd'AM)

Recall: $\rho(g) = \limsup_{r\to 1^-} \frac{\log^+ T(r,g)}{-\log(1-r)}$, $\rho_M(g) = \limsup_{r\to 1^-} \frac{\log^+ M(r,g)}{-\log(1-r)}$

C. Pommerenke (CV 1982): Let f be a solution of

$$f'' + A(z)f = 0,$$

where $A(z)$ is analytic in D.

- $\displaystyle\int_0^1 M(r,A)^2(1-r)^3\,dr < \infty \implies f \in H^2$
- $\displaystyle\int_D |A(z)|^{\frac{1}{2}}\,dm(z) < \infty \implies f \in N$

Summary of growth "indicators" for solutions:

- **Hardy spaces, Bergman spaces, etc.**
  J.H., R. Korhonen and J. Rättyä (NMJ 2007)
  J. Rättyä (CVEE 2007)

- **The classes** N **and** F

- **The orders of growth** $\rho(g)$, $\rho_M(g)$ **and** $\rho_\varphi(g)$

- **Iterated order of growth**
  T.-B. Cao and H.-X. Yi (JMAA 2006)
  J.H., R. Korhonen and J. Rättyä (RM 2006)

Part I: Growth of solutions $\longrightarrow$ DONE!

Part II: Tools

- **Wiman-Valiron Theory**
- **Pointwise growth**
- **Carleson measures**
- **Logarithmic derivatives**

Part III: Oscillation of solutions

Wiman-Valiron theory in the unit disc

- **P. Fenton and M. Strumia (JLMS 2009)**

Applications to differential equations

- **P. Fenton and J. Rossi (to appear in TAMS)**

Let $A_0(z),\ldots,A_{k-1}(z)$ be analytic in D, and let f solve

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'(z)+A_0(z)f=0.$$

J.H., R. Korhonen and J. Rättyä (AASF 2004):

- $|f(re^{i\theta})|\leq C\exp\left(n_c\int_0^r \max_{0\leq j\leq k-1}|A_j(te^{i\theta})|^{\frac{1}{k-j}}\,dt\right)$
- $|f(re^{i\theta})|\leq C_1\exp\left(C_2\sum_{j=0}^{k-1}\sum_{n=0}^{j}\int_0^r |A_j^{(n)}(se^{i\theta})|(1-s)^{k-j+n-1}\,ds\right)$

Merom. case: Y.-M. Chiang and W. Hayman (CMH 2004).
Nonhomog. case: J.H., R.K. and J.R. (AASF 2009).

C. Pommerenke (CV 1982): Let $A(z)$ be analytic in D s.th.

$$\lim_{|a|\to 1^-} \int_D |A(z)|^2(1-|z|^2)^3 \frac{1-|a|^2}{|1-\bar{a}z|^2}\, dm(z) = 0.$$

Then $d\mu(z) = |A(z)|^2(1-|z|^2)^3\, dm(z)$ is a Carleson measure.
Let f solve $f'' + A(z)f = 0$. Then

$$\begin{aligned} ||f||^2_{H^2} &\leq |f(0)| + |f'(0)| + \frac{1}{\pi}\int_D |f''(z)|^2(1-|z|^2)^3\, dm(z) \\ &= |f(0)| + |f'(0)| + \frac{1}{\pi}\int_D |f(z)|^2 d\mu(z). \end{aligned}$$

Carleson's theorem then implies $f \in H^2$.

Generalizations by J.H., R.K. and J.R. (NMJ 2007).

I. Chyzhykov, G. Gundersen and J.H. (PLMS 2003):

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left(\frac{1}{1-|z|}\right)^{(\rho(f)+2+\varepsilon)(k-j)}$$

J.H., R. Korhonen and J. Rättyä (BLMS 2004):

$$\int_0^{2\pi} \left|\frac{f^{(k)}(re^{i\theta})}{f^{(j)}(re^{i\theta})}\right|^{\alpha} d\theta = O\left(\frac{T(R,f)}{1-r}\right)^{\alpha(k-j)}, \quad 0 < \alpha(k-j) < 1$$

Further estimates:

- **Blaschke products and functions in N: J.H. (KMJ 2007)**
- **The case $\alpha = \frac{1}{k-j}$: I.C., J.H. and J.R. (to appear in Jd'AM)**
- **Proximate order: I.C., J.H. and J.R. (to appear in JAMS)**

Part I: Growth of solutions $\longrightarrow$ DONE!

Part II: Tools $\longrightarrow$ DONE!

Part III: Oscillation of solutions

- **Disconjugate/nonoscillatory/oscillatory equations**
- **Blaschke-oscillatory equations**
- **Prescribed zero sequences**
- **Separated/uniformly separated/exponential zero sequences**

Definition: **Let $A(z)$ be analytic in D. The equation**

$$f'' + A(z)f = 0$$

is called

- **disconjugate (resp. non-oscillatory) if each non-trivial solution has at most one zero (resp. finitely many zeros) in D;**
- **oscillatory if there is at least one solution with infinitely many zeros in D;**
- **Blaschke-oscillatory if the zero sequence $\{z_n\}$ of each non-trivial solution satisfies**

$$\sum_n (1 - |z_n|) < \infty.$$

Z. Nehari (BAMS 1949): If

$$|A(z)| \leq \frac{1}{(1-|z|^2)^2} \tag{3}$$

for all $z \in D$, then $f'' + A(z)f = 0$ is disconjugate.

B. Schwarz (TAMS 1955): If there exists $R \in (0,1)$ such that (3) holds for all z with $R \leq |z| < 1$, then $f'' + A(z)f = 0$ is non-oscillatory.

M. Chuaqui and D. Stowe (AASF 2008): An estimate for the number of zeros in the previous case.

E. Hille (BAMS 1949) & B. Schwarz (TAMS 1955): If

$$A(z) = \frac{1+4\gamma^2}{(1-z^2)^2}, \quad \gamma > 0,$$

then $f'' + A(z)f = 0$ is oscillatory.

D. London (1962): If

$$\int_D |A(z)|dm(z) \leq \pi,$$

then $f'' + A(z)f = 0$ **is disconjugate.**

J.H. (2005): Let $z_1 \in D \setminus \{0\}$ **and** $z_2 = -z_1$**. Then** $f'' + A(z)f = 0$ **with**

$$A(z) = \frac{10z_1^2 - z^2}{4z_1^4}$$

has a solution with zeros at the points z_1, z_2**. Moreover,**

$$\int_D |A(z)|dm(z) \leq \frac{11}{4|z_1|^4} \int_D dm(z) \leq \frac{11\pi}{4|z_1|^4}.$$

D. London (1962): If

$$\int_D |A(z)|\,dm(z) < \infty,$$

then $f'' + A(z)f = 0$ **is non-oscillatory.**

Example: If

$$A(z) = \frac{1}{(1-z^2)^2},$$

then $f'' + A(z)f = 0$ **is non-oscillatory (in fact disconjugate). Further,**

$$\int_{D(0,r)} |A(z)|\,dm(z) \geq \frac{\pi}{4} \log \frac{1}{1-r^2}.$$

C. Pommerenke (CV 1982): If

$$\int_D |A(z)|^{\frac{1}{2}}\, dm(z) < \infty,$$

then $f'' + A(z)f = 0$ **is Blaschke-oscillatory.**

J.H. (JMAA 2006): If $f'' + A(z)f = 0$ **is Blaschke-oscillatory, then**

$$\int_D |A(z)|^{\alpha}\, dm(z) < \infty$$

for every $\alpha \in \left(0, \frac{1}{2}\right)$**.**

Example: **The functions**

$$f_1(z) = (1-z)\exp\left(\frac{1+z}{1-z}\right) \quad \textbf{and} \quad f_2(z) = (1-z)\exp\left(-\frac{1+z}{1-z}\right)$$

are linearly independent solutions of

$$f'' + A(z)f = 0, \quad A(z) = -\frac{4}{(1-z)^4}. \tag{4}$$

We see that $f_1 \in N$ **and** $f_2 \in H^\infty$**, and so** (4) **is Blaschke-oscillatory. Further,**

$$\int_{D(0,r)} |A(z)|^{\frac{1}{2}}\, dm(z) = 2\pi \log \frac{1}{1-r^2}.$$

J.H. (CMFT 2005, JMAA 2006):

- **If** $\{z_n\}$ **is a Blaschke sequence of distinct points in** D**, then find** $A(z)$ **s.th.** $f'' + A(z)f = 0$ **possesses a solution having zeros precisely at the points** z_n**.**

- **If** $\{a_n\}$ **and** $\{b_n\}$ **are two given Blaschke sequences of distinct points in** D **with no points in common, then find** $A(z)$ **s.th.** $f'' + A(z)f = 0$ **possesses two linearly independent solutions** f_1 **and** f_2 **having zeros precisely at the points** a_n **and** b_n**, respectively.**

Definition: A sequence $\{z_n\}$ **of points in** D **is called**

- **separated, if**

$$\inf_{n \neq k} \left| \frac{z_n - z_k}{1 - \bar{z}_n z_k} \right| > 0;$$

- **uniformly separated, if**

$$\inf_{k} \prod_{n \neq k} \left| \frac{z_n - z_k}{1 - \bar{z}_n z_k} \right| > 0;$$

- **exponential, if there is a constant** $c \in (0,1)$ **such that**

$$1 - |z_{n+1}| \leq c(1 - |z_n|), \quad n = 1, 2, \ldots$$

B. Schwarz (TAMS 1955):

Let $A(z)$ **be analytic in** D**, and let** $a > 1$**.**

- **If**

$$|A(z)| \leq \frac{a}{(1-|z|^2)^2}, \quad z \in D,$$

  then the zero sequence of every solution of $f'' + A(z)f = 0$ **is separated by** $\frac{2a^{1/2}}{a+1}$**.**

- **If the zero sequence of every solution of** $f'' + A(z)f = 0$ **is separated by** $\frac{2a^{1/2}}{a+1}$**, then**

$$|A(z)| \leq \frac{3a}{(1-|z|^2)^2}, \quad z \in D.$$

<u>Open problem:</u>

- **Necessary / sufficient conditions on** $A(z)$ **(analytic in** D**) such that the zero sequence of every solution of** $f'' + A(z)f = 0$ **is uniformly separated / exponential.**

Partial results by J. Gröhn, J.H. and J. Rättyä.