Geometric properties of quasihyperbolic and j-metric balls

Riku Klén
riku.klen@utu.fi
University of Turku

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Contents of the talk

- Short introduction
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Definitions

Let $G \subsetneq \mathbb{R}^n$, $n \geq 2$, be a domain and define

- the *quasihyperbolic distance* for $x, y \in G$ by
$$k_G(x,y) = \inf_{\alpha \in \Gamma_{xy}} \int_\alpha \frac{|dz|}{d(z)},$$
where $d(z) = d(z, \partial G)$ and Γ_{xy} is the collection of all rectifiable curves in G joining x and y.
- the *distance ratio metric* or *j-metric* for $x, y \in G$ by
$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x), d(y)\}}\right).$$
- for $m \in \{k_G, j_G\}$ we define the *metric ball* (*metric disk* in the case $n = 2$) for $r > 0$ and $x \in G$ by
$$B_m(x,r) = \{y \in G \colon m(x,y) < r\}.$$

Definitions

- G is *starlike with respect to* $x \in G$ if for all $y \in G$ the line segment $[x, y]$ is contained in G and G is *strictly starlike with respect to* x if each line from the point x meets ∂G at exactly one point.
- if G is starlike w.r.t. x for all $x \in G$ then it is *convex*.
- a domain $G \subset \mathbb{R}^n$ is *close-to-convex* if $\mathbb{R}^n \setminus G$ can be covered with non-intersecting half-lines ($\{z \in \mathbb{R}^n : z = x + ty,\ x, y \in \mathbb{R}^n, y \neq 0, t > 0\}$ or $\{z \in \mathbb{R}^n : z = x + ty,\ x, y \in \mathbb{R}^n, y \neq 0, t \geq 0\}$).

Clearly convex domains are starlike and starlike domains are close-to-convex. For example union of two disjoint convex domains is always close-to-convex.

The radius and the shape of the metric balls

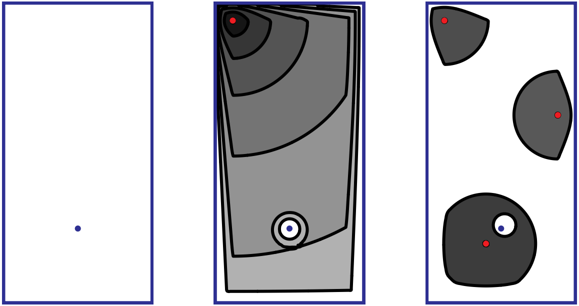

Figure: Examples of j-metric disks.

The motivation of the study

It is natural to pose the following open problem [Vuorinen, 2006, 8.1]:

Open problem

Does there exist $r_0 > 0$ such that $B_m(x, r)$ is convex (in Euclidean geometry) for all $r \in (0, r_0)$? ($G \subset \mathbb{R}^n$ domain, m metric)

Instead of convexity we can also consider other properties like starlikeness.

Punctured space

Explicit formula for the quasihyperbolic distance is known only in a few special domains. One of these special domains is $\mathbb{R}^n \setminus \{0\}$.

Theorem [Martin & Osgood, 1986]

For $x, y \in \mathbb{R}^n \setminus \{0\}$ and $n \geq 2$

$$k_{\mathbb{R}^n \setminus \{0\}}(x, y) = \sqrt{\alpha^2 + \log^2 \frac{|x|}{|y|}}, \qquad (1)$$

where $\alpha \in [0, \pi]$ is the angle between line segment $[x, 0]$ and $[0, y]$ at the origin.

Punctured space

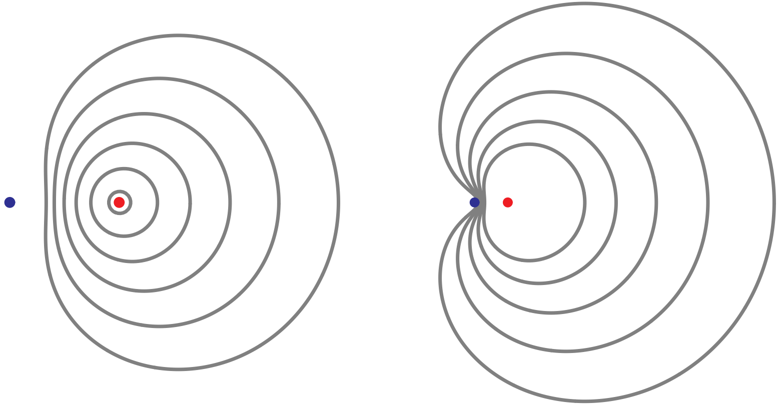

Figure: Examples of quasihyperbolic disks in $\mathbb{R}^n \setminus \{0\}$.

Punctured space

By (1) it is easy to find a solution to Open problem in the case $G = \mathbb{R}^n \setminus \{0\}$.

Theorem [K, 2008]

Let $G = \mathbb{R}^n \setminus \{0\}$. Then the quasihyperbolic ball $B_k(x, r)$ is

- convex for $r \in (0, 1]$ and all $x \in G$,
- starlike w.r.t. x for $r \in (0, 2.8329\ldots]$ and all $x \in G$.

Properties of quasihyperbolic balls

Theorem [Väisälä, 2007]

The quasihyperbolic ball $B_k(x, r)$ is starlike w.r.t. x for $r \in (0, \pi/2]$ and all $x \in G$.

The radius $\pi/2$ is presumably not sharp.

Theorem [Väisälä, 2009]

In the case $n = 2$ the quasihyperbolic disks $B_k(x, r)$ are convex for all $x \in G$ and $r \in (0, 1]$.

The radius 1 is sharp. The proof is based on the result for the domain $\mathbb{R}^2 \setminus \{0\}$ and Voronoi cells.

Convex and starlike domains

Theorem [Martio & Väisälä, to appear]

In a convex domain the quasihyperbolic balls $B_k(x, r)$ are convex for all $x \in G$ and $r > 0$.

Theorem [K, 2008]

In a starlike domain w.r.t. x the quasihyperbolic balls $B_k(x, r)$ are starlike w.r.t. x for all $r > 0$.

Close-to-convexity of $B_k(x,y)$ in $\mathbb{R}^n \setminus \{0\}$

Proposition 2

The function

$$f(z) = \cos\sqrt{z^2-1} + \sqrt{z^2-1}\sin\sqrt{z^2-1}$$

has exactly one zero, denoted by λ, on $(2,\pi)$.

Proof.

By a simple computation $f'(z) = z\cos\sqrt{z^2-1} < 0$, because by assumption $\sqrt{z^2-1} \in (\sqrt{3}, \sqrt{\pi^2-1}) \subset (\pi/2, \pi)$. Therefore $f(z)$ is continuous and strictly decreasing on $(2,\pi)$. Because $f(2) > 0$ and $f(\pi) < 0$, the assertion follows. □

Close-to-convexity of $B_k(x,y)$ in $\mathbb{R}^n \setminus \{0\}$

Lemma 3

Let $G = \mathbb{R}^2 \setminus \{0\}$, $r > 0$ and $x \in G$. The quasihyperbolic disk $B_k(x,r)$ is close-to-convex for $r \leq \lambda$ is not close-to-convex for $r > \lambda$.

Lemma 4

If the domain $G \subset \mathbb{R}^n$ is invariant under rotation about a line l and $G \cap L$ is close-to-convex for every plane L with $l \subset L$, then G is close-to-convex.

Close-to-convexity of $B_k(x, y)$ in $\mathbb{R}^n \setminus \{0\}$

Theorem 5

Let $G = \mathbb{R}^n \setminus \{0\}$, $r > 0$ and $x \in G$. The quasihyperbolic ball $B_k(x, r)$ is close-to-convex for $r \leq \lambda$.

The proof follows from the two previous lemmas. The radius λ is sharp in the case $n = 2$. In the case $n > 2$ the sharp radius is unknown.

Summary for quasihyperbolic balls

G	$B_k(x,r)$	r_0
any	starlike w.r.t x	$\pi/2$
convex	convex	∞
starlike w.r.t. x	starlike w.r.t. x	∞
any, $n=2$	convex	1
$\mathbb{R}^n \setminus \{0\}$	convex	1
$\mathbb{R}^n \setminus \{0\}$	starlike w.r.t. x	$\kappa \approx 2.83$
$\mathbb{R}^n \setminus \{0\}$	close-to-convex	$\lambda \approx 2.97$

Radii of convexity and starlikeness

Theorem [K, 2008]

The *j*-metric balls $B_j(x, r)$ are convex for all $x \in G$ and $r \leq \log 2$.

Theorem [K, 2008]

The *j*-metric balls $B_j(x, r)$ are starlike w.r.t. x for all $x \in G$ and $r \leq \log(1 + \sqrt{2})$.

Convex and starlike domains

Theorem [K, 2008]

In a convex domain the j-metric balls $B_j(x,r)$ are convex for all $x \in G$ and $r > 0$.

Theorem [K, 2008]

In a starlike domain w.r.t. x the j-metric balls $B_j(x,r)$ are starlike w.r.t. x for all $r > 0$.

j-metric balls in $\mathbb{R}^n \setminus \{0\}$

Lemma 6

Let $G = \mathbb{R}^n \setminus \{0\}$ *and* $x \in G$*. For* $r \in (0, \log(1+\sqrt{3})]$ *the j-metric ball* $B_j(x,r)$ *is close-to-convex.*

Proof.

By symmetry of G we may assume $x = e_1$. By definition of the *j*-metric

$$B_j(x,r) = B^n(e_1, e^r - 1) \setminus \overline{B^n}(c,s),$$

where

$$c = \frac{e_1}{e^r(2-e^r)} \qquad s = \frac{e^r - 1}{e^r(e^r-2)}.$$

j-metric balls in $\mathbb{R}^n \setminus \{0\}$

Proof.

By geometry $B_j(x,r)$ is close-to-convex if $r \le r_0$, where r_0 is such that

$$s_0^2 + |e_1 - c_0|^2 = (e^{r_0} - 1)^2$$

or equally

$$\left(\frac{e^{r_0} - 1}{e^{r_0}(e^{r_0} - 2)}\right)^2 + \left(1 - \frac{1}{e^{r_0}(2 - e^{r_0})}\right)^2 = (e^{r_0} - 1)^2. \quad (7)$$

Equality (7) is equivalent to $r_0 = \log(1 + \sqrt{3})$ and the assertion follows. $\square$

j-metric balls in $\mathbb{R}^n \setminus \{0\}$

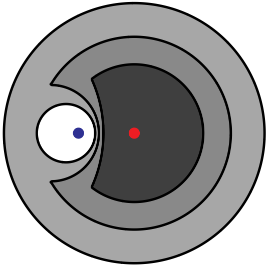

Figure: Examples of j-metric disks in $\mathbb{R}^2 \setminus \{0\}$.

j-metric balls in $\mathbb{R}^n \setminus \{0\}$

Lemma 8

Let $r > 0$ and $B_i = B^n(x_i, r_i)$ such that $|x_i| \geq r/\sqrt{2}$ and $r_i < |x_i|$ and if $|x_i| < r$ then $\sqrt{r_i^2 + |x_i|^2} \geq r$. Then

$$B = B^n(0, r) \setminus \bigcup_{i=1}^{\infty} \overline{B_i}$$

is close-to-convex.

Radius of close-to-convexity

Theorem 9

Let $G \subset \mathbb{R}^n$ a domain and $x \in G$. For $r \in (0, \log(1+\sqrt{3})]$ the j-metric ball $B_j(x,r)$ is close-to-convex.

The proof is based on the two previous lemmas and the following fact [K, 2008, (4.1)]

$$B_{j_G}(x,r) = \bigcap_{z \in \partial G} B_{j_{\mathbb{R}^n \setminus \{z\}}}(x,r). \tag{10}$$

Connected *j*-metric balls

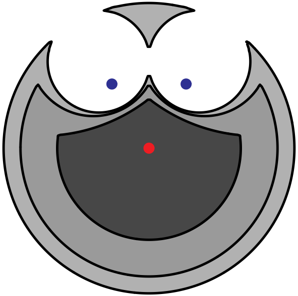

Figure: Examples of disconnected *j*-metric disks.

Summary for j-metric balls

G	$B_j(x,r)$	r_0
any	convex	$\log 2 \approx 0.69$
any	starlike w.r.t. x	$\log(1+\sqrt{2}) \approx 0.88$
any	close-to-convex	$\log(1+\sqrt{3}) \approx 1.01$
convex	convex	∞
starlike w.r.t. x	starlike w.r.t x	∞

References

- R. Klén: *Local Convexity Properties of j-metric Balls.* Ann. Acad. Sci. Fenn. Math. 33 (2008), 281-293.
- R. Klén: *Local Convexity Properties of Quasihyperbolic Balls in Punctured Space.* J. Math. Anal. Appl. 342 (2008) 192-201.
- G.J. Martin, B.G. Osgood: *The quasihyperbolic metric and the associated estimates on the hyperbolic metric.* J. Anal. Math. 47 (1986), 37–53.
- O. Martio, J. Väisälä: *Quasihyperbolic geodesics in convex domains II.* Pure Appl. Math. Q., to appear.
- J. Väisälä: *Quasihyperbolic geometry of domains in Hilbert spaces.* Ann. Acad. Sci. Fenn. Math. 32 (2007), 559-578.
- J. Väisälä: *Quasihyperbolic geometry of planar domains.* Ann. Acad. Sci. Fenn. Math. 34 (2009), 447-473.
- M. Vuorinen: *Metrics and quasiregular mappings.* Proc. Int. Workshop on Quasiconformal Mappings and their Applications, IIT Madras, Dec 27, 2005 - Jan 1, 2006, ed. by S. Ponnusamy, T. Sugawa and M. Vuorinen, *Quasiconformal Mappings and their Applications*, Narosa Publishing House, 291–325, New Delhi, India, 2007.