Resolvents of kernels associated with absolutely continuous semigroups

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Abstract

Let $\mathcal{P} := (P_t)_{t>0}$ and $\mathcal{P}^* := (P_t^*)_{t>0}$ be semigroups in duality, such that their associated resolvents $\mathcal{V} = (V_\alpha)_{\alpha\geqslant 0}$ and $\mathcal{V}^* = (V_\alpha^*)_{\alpha\geqslant 0}$ are of the form

$$V_\alpha f(x) = \int v_\alpha(x,y) f(y)\mu(dy); \quad V_\alpha^* f(x) = \int v_\alpha(y,x) f(y)\mu(dy)$$

and whose densities $(v_\alpha)_{\alpha\geqslant 0}$ satisfy:

$$\forall (x,y) \notin A \ \exists C(x,y) \in [0,+\infty) \text{ such that the function}$$

$$\alpha \mapsto \frac{C(x,y)}{\alpha} - v_\alpha(x,y)$$

is completely monotone on $(0,+\infty)$.

Then $\mathcal{P}$ and $\mathcal{P}^*$ are absolutely continuous.

The interest of the absolute continuity of a semigroup $\mathcal{P}$ resides in the following characterization: $\mathcal{P}$ is absolutely continuous if and only if, for any other semigroup $\mathcal{P}'$, the resolvent associated with the product semigroup $\mathcal{P} \otimes \mathcal{P}'$ is absolutely continuous. This characterization is essentially used in the proof of the theorem.

1 Preliminaries.

Let $(X, \mathcal{X}, \mu)$ be a space with measure. All measures will be positive, σ-finite (unless explicitly mentioned otherwise). $\mathcal{F}(X)$ will denote the (ordered convex cone) of numerical, positive measurable functions $X \to [0, +\infty]$. We shall adhere to the notations from [6].

Notation 1 *We consider the following operation, denoted $*$, on $\mathcal{F}(X \times X)$:*

$$(f * g)(x, y) := \int_X f(x, u) g(u, y) \mu(du)$$

Let us define f^{*n} as: $f^{*1} = f$; $f^{*(n+1)} = f * f^{*n}$.

It is clear that $f*g$ is well defined. The measurability is proved considering first characteristic functions. Thus, $*$ is a well defined operation, with the following properties:

$$f * (g_1 + g_2) = f * g_1 + f * g_2$$

$$(f_1 + f_2) * g = f_1 * g + f_2 * g$$
$$(\alpha f) * g = \alpha (f * g)$$
$$f * (\alpha g) = \alpha (f * g)$$

By Tonelli's theorem, we have:

$$(f * g) * h = f * (g * h)$$

This operation is not commutative; however, in the examples, we will work in commutative parts.

Let recall the following basic result about absolutely continuous resolvents in duality :

Theorem 1 *(Kunita–Watanabe [7]) Let $\mathcal{V} = (V_\alpha)_{\alpha \geqslant 0}$ and $\mathcal{V}^* = (V^*_\alpha)_{\alpha \geqslant 0}$ be sub–markovian resolvents of kernels , absolutely continuous and in duality with respect to μ.*

There exists a family $(v_\alpha)_{\alpha \geqslant 0}$ with the properties:

- *v_α is a numerical, positive, measurable function on $(X \times X, \mathcal{X} \otimes \mathcal{X})$, which is $\mathcal{V}^\alpha$–excessive in the first variable and is $\mathcal{V}^{*\,\alpha}$–excessive in the second variable;*
- *$V_\alpha f(x) = \int v_\alpha(x, y) f(y) \mu(dy), \forall x \in X$, $\forall f \in \mathcal{F}$*
- *$V^*_\alpha f(x) = \int v_\alpha(y, x) f(y) \mu(dy)$, $\forall x \in X$, $\forall f \in \mathcal{F}$.*

These properties determine uniquely the family (v_α). Moreover, the following relation holds:

$$(*)\ \ v_\alpha = v_\beta + (\beta - \alpha) v_\alpha * v_\beta,\ \forall 0 \leqslant \alpha < \beta$$

Notation 2 *Let us denote by*

$$A := \{(x,y) |\ v_0(x,y) = +\infty\}$$

$$A_x := \{y |\ (x,y) \in A\}\ ;\ A^y := \{x |\ (x,y) \in A\}$$

The sets A_x resp. A^x are (co–)polar, $\mathcal{V}$ and μ–negligible.

1.1 Assumption (C)

For each $(x,y) \notin A$, the function $f(\alpha) := v_\alpha(x,y)$ is completely monotone on $[0,+\infty]$, as $f^{(k)}(\alpha) = (-1)^k k!\ v_\alpha^{*(k+1)}(x,y)$. Using Bernstein's theorem [12], there exists, for each $(x,y) \notin A$, a (positive Radon) measure on $[0,+\infty]$, denoted by λ_{xy}, for which

$$v_\alpha(x,y) = \int_0^{+\infty} e^{-\alpha t} \lambda_{xy}(dt)$$

$\forall \alpha \geqslant 0$.

We formulate now the main assumption on the resolvent $\mathcal{V}$.

$$(C)\ \ \forall (x,y) \notin A\ \exists C(x,y) \in [0,+\infty) \text{ such that } \forall k \geqslant 1, \forall \alpha > 0 :$$

$$(\alpha v_\alpha)^{*\,k}(x,y) \leqslant C(x,y)$$

The condition (C) is equivalent with:

$$\forall (x,y) \notin A\ \exists C(x,y) \in [0,+\infty) \text{ such that the function}$$

$$\alpha \mapsto \frac{C(x,y)}{\alpha} - v_\alpha(x,y)$$

is completely monotone on $(0,+\infty)$.

Indeed, the k–th order derivative of this function is:

$$(-1)^k k! \left[\frac{C(x,y)}{\alpha^{k+1}} - v_\alpha^{*(k+1)} \right]$$

Let us remark that the condition (C) is "almost" necessary (see [12]) in order to have an absolutely continuous semigroup. Indeed, let us suppose that

$$P_t f(x) = \int p_t(x,y) f(y) \mu(dy)$$

where $t \mapsto p_t(x,y)$ is defined and bounded: for all $(x,y) \notin A$ there exists $C(x,y) \in [0,+\infty)$ such that $p_t(x,y) \leqslant C(x,y)$, for all $t > 0$. As we are allowed to take the derivative under the integral sign in

$$v_\alpha(x,y) = \int_0^\infty e^{-\alpha t} p_t(x,y) dt$$

we get:

$$(\alpha v_\alpha)^{*\,k}(x,y) = \frac{\alpha^k}{(k-1)!} \int_0^\infty e^{-\alpha t} t^{k-1} p_t(x,y) dt \leqslant$$

$$\leqslant C(x,y) \frac{\alpha^k}{(k-1)!} \int_0^\infty e^{-\alpha t} t^{k-1} dt = C(x,y)$$

2 Main result.

Theorem 2 *Let $\mathcal{P} := (P_t)_{t>0}$ and $\mathcal{P}^* := (P_t^*)_{t>0}$ be semigroups in duality, such that their associated resolvents $\mathcal{V}$ and $\mathcal{V}^*$ are absolutely continuous and satisfy (C).*

Then $\mathcal{P}$ and $\mathcal{P}^*$ are absolutely continuous.

2.1 Overview

We carry out the proof in several steps. First, using a Doob's type reasoning, we construct a density $q_0(t,x,y)$; this is a measurable function, but it is defined only a.e. Another way to obtain a density is through an inversion formula for the Laplace transform, namely using Philips' operator. It is defined except a polar set; however, in order to prove the measurability, some (right) continuity in t seems unavoidable. This was obtained only under an additional condition [10].

In both cases, the semigroup property (for the corresponding family of kernels) cannot be guaranteed.

A combination of the two kernel–functions allow the construction of a *family* $(Q_t)_{t>0}$ of absolutely continuous kernels, associated with the resolvent.

Now, we use a characterization of absolutely continuous semigroups [8]: $\mathcal{P}$ is absolutely continuous if and only if the resolvent associated with $\mathcal{P} \otimes \mathcal{P}'$ is absolutely continuous, where $\mathcal{P}'$ is any semigroup with absolutely continuous associated resolvent (or just $\mathcal{P}'$ is the translation semigroup on $\mathbb{R}$). The key remark is that the "resolvent" associated with the family $\mathcal{Q} \otimes \mathcal{P}'$ is absolutely continuous *and* is the same as the true resolvent associated with the semigroup $\mathcal{P} \otimes \mathcal{P}'$, hence the conclusion.

2.2 A Lemma

Lemma 1 *There exists a $\mathcal{B}(\mathbb{R})\otimes\mathcal{X}^\mu\otimes\mathcal{X}^\mu$–measurable function $q : (0, +\infty) \times X \times X \longrightarrow [0, +\infty]$ such that*

$$Q_t f(x) := \int q(t, x, y) f(y) \mu(dy)$$

$$Q_t^* f(x) := \int q(t, y, x) f(y) \mu(dy)$$

are absolutely continuous kernels in duality and:

$$V_\alpha f(x) = \int_0^\infty e^{-\alpha t} Q_t f(x) dt$$

$$V_\alpha^* f(x) = \int_0^\infty e^{-\alpha t} Q_t^* f(x) dt$$

$\forall \alpha \geqslant 0$*,* $\forall f \in \mathcal{F}(X)$*,* $\forall x \in X$*.*

2.3 Proof.

1. Using the kernel

$$V f(x, y) := \int_0^\infty f(t) \lambda_{xy}(dt)$$

we define a measure, denoted μ_0, on $[0, +\infty] \times X \times X$ as follows. Let F be a positive, measurable function on $[0, +\infty] \times X \times X$ and denote by F_{xy} the function defined as $F_{xy}(t) := F(t, x, y)$. Then

$$F \mapsto \int V(F_{xy})(x, y) \mu(dx) \mu(dy)$$

is well defined, since $(x,y) \mapsto V(F_{xy})(x,y)$ is measurable. Indeed, we consider first F of the form $f(t)g(x,y)$; the map is clearly measurable; then linear combinations and monotone sequences of such functions give the conclusion. Hence, this is the measure μ_0.

We show now that λ_{xy} is absolutely continuous with respect to Lebesgue measure on $[0,+\infty)$. From condition (C) we conclude that, for each $(x,y) \notin A$ there exists a positive measure ν_{xy} pe $[0,+\infty]$ such that

$$\frac{C(x,y)}{\alpha} - v_\alpha(x,y) = \int_0^\infty e^{-\alpha t} d\nu_{xy}(t)$$

Using this fact, we prove that, for $(x,y) \notin A$ and for any positive, measurable function g on $[0,+\infty]$:

$$\int_0^\infty g(t)\mu_{xy}(dt) \leqslant \int_0^\infty C(x,y)g(t)dt$$

Indeed, from the positivity of the function $\dfrac{C(x,y)}{\alpha} - v_\alpha(x,y)$ we obtain that

$$\int_0^\infty e^{-\alpha t}.\lambda_{xy}(dt) \leqslant \int_0^\infty e^{-\alpha t}.C(x,y).dt$$

Let now consider a function

$$f(t) := \sum_{i=1}^n c_i.e^{-\alpha_i t} - \sum_{j=1}^m d_j.e^{-\beta_j t}, \forall t \in [0,+\infty]$$

with $c_i, d_j \geqslant 0$, $\alpha_i, \beta_j > 0$ and suppose that $f(t) \geqslant 0$, $\forall t \in [0,+\infty]$. We have:

$$0 \leqslant \int_0^\infty f(t)\nu_{xy}(dt) = \sum_{i=1}^n c_i \int_0^\infty e^{-\alpha_i t}\nu_{xy}(dt) - \sum_{j=1}^m d_j \int_0^\infty e^{-\beta_j t}\nu_{xy}(dt) =$$

$$= \sum_{i=1}^n c_i \left[\frac{C(x,y)}{\alpha_i} - v_{\alpha_i}(x,y)\right] \sum_{j=1}^m d_j \left[\frac{C(x,y)}{\beta_j} - v_{\beta_j}(x,y)\right] =$$

$$= \sum_{i=1}^n c_i \left[\int_0^\infty e^{-\alpha_i t}C(x,y)dt - \int_0^\infty e^{-\alpha_i t}\lambda_{xy}(dt)\right] -$$

$$- \sum_{j=1}^m d_j \left[\int_0^\infty e^{-\beta_j t}C(x,y)dt - \int_0^\infty e^{-\beta_j t}\lambda_{xy}(dt)\right] =$$

$$= \int_0^\infty f(t)C(x,y)dt - \int_0^\infty f(t)\lambda_{xy}(dt)$$

Now the inequality extends to all positive, measurable functions f

Let now suppose that $(m \otimes \mu \otimes \mu)(F) = 0$. Particularly, for $\mu \otimes \mu$–almost every (x, y), we have $m(F_{xy}) = 0$, hence $V(F_{xy}) \equiv 0$, hence $\mu_0(F) = 0$. Thus, μ_0 is absolutely continuous with respect to $m \otimes \mu \otimes \mu$.

Moreover μ_0 is σ–finite. Indeed, since V_0 was supposed proper, there exist $D_n \subseteq X$ such that $X = \bigcup_n D_n$ and each $V_0(\chi_{D_n})$ is bounded. As μ is σ–finite, there exist $E_n \in \mathcal{X}$ such that $X = \bigcup\limits_n E_n$ and $\mu(E_n) < +\infty$. Now, if we denote $B_n := [0, +\infty] \times E_n \times D_n$, then

$$\mu_0(B_n) = \int_{D_n \times E_n} v_0(x,y)\mu(dx)\mu(dy) < +\infty$$

We obtain a density function, denoted as $q_0(t, x, y)$ which is positive, defined and measurable on $D \subseteq [0, +\infty] \times (X \times X \setminus A)$ where D is a set with a negligible complement.

If $\mathcal{X}$ is separable, then q_0 is already everywhere defined.

We have the equality:

$$\int F(t,x,y)\mu_0(dt,dx,dy) = \int V(F_{xy})(x,y)\mu(dx)\mu(dy) =$$

$$= \int F(t,x,y)q_0(t,x,y)dt\mu(dx)\mu(dy)$$

Writing for $F(t, x, y) = f(t)g(x, y)$ we obtain:

$$\int g(x,y)\int_0^\infty f(t)\lambda_{x,y}(dt)\mu(dx)\mu(dy) = \int g(x,y)f(t)q_0(t,x,y)dt\mu(dx)\mu(dy)$$

Specializing to $f_\alpha(t) := e^{-\alpha t}$, for $\alpha \geqslant 0$, we obtain further:

$$\int g(x,y)v_\alpha(x,y)\mu(dx)\mu(dy) = \int g(x,y)\left[e^{-\alpha t}q_0(t,x,y)dt\right]\mu(dx)\mu(dy)$$

It follows that for each $\alpha \geqslant 0$ there exists a $\mu \otimes \mu$–negligible set $A^{1\alpha} \subset X \times X$ such that

$$v_\alpha(x,y) = \int_0^\infty e^{-\alpha t}q_0(t,x,y)dt$$

for any $(x,y) \in A^{1\alpha}$.

Considering the union of the $\mu \otimes \mu$–negligible sets $A^{1\alpha}$, for rational α and using the continuity in α of both members, it follows the existence of a $\mu \otimes \mu$–negligible set $A^1 \subset X \times X$, such that

$$(1) \quad v_\alpha(x,y) = \int_0^\infty e^{-\alpha t} q_0(t,x,y) dt$$

for any $\alpha \geqslant 0$ and any $(x,y) \notin A^1$.

We have thus constructed a map q_0, which is everywhere defined if $\mathcal{X}$ is separable; but only $m \otimes \mu \otimes \mu$–a.e. in general. Anyway, (1) holds only $\mu \otimes \mu$–a.e.

2. In the non–separable case, we need to extend conveniently q_0 everywhere. We present two ways to do this.

a) In this part, we obtain, as a density, a version for q; it is defined only $m \otimes \mu \otimes \mu$–a.e.

Using (C) for each $(x,y) \notin A$ there exists a positive measure λ_{xy} on $[0,+\infty]$ such that

$$\frac{C(x,y)}{\alpha} - v_\alpha(x,y) = \int_0^\infty e^{-\alpha t} d\lambda_{xy}(t)$$

We showed [10] that λ_{xy} is absolutely continuous with respect to Lebesgue measure on $[0,+\infty)$. More precisely, we obtained that $\lambda_{xy} \leqslant C(x,y)m$, $\forall (x,y) \notin A$. For each $(x,y) \notin A$, we have thus the density $p_t(x,y)$ (measurable, positive function on $[0,+\infty]$), such that $\lambda_{xy}(dt) = p_t(x,y).m(dt)$. Hence we have:

$$v_\alpha(x,y) = \int_0^\infty e^{-\alpha t} p_t(x,y).dt$$

$\forall \alpha \geqslant 0$, $\forall (x,y) \notin A$. However, the measurability of the function $(x,y) \mapsto p_t(x,y)$ is not proved.

b) We use now [9] the inversion of the Laplace transform (Philip's operator), in order to get a density function $p_t(x,y)$.

Let us denote:

$$f_n(t,x,y) := e^{-nt} \sum_{k=0}^\infty \frac{(nt)^k}{k!} (nv_n)^{*k}(x,y)$$

The sequence f_n is norm bounded in $L^\infty((0,+\infty))$, hence weak* compact. For each $(x,y) \notin A$, there exists a weak* convergent subsequence f_{n_k} to $p_t(x,y)$, a bounded measurable function, defined a.e. on $(0,+\infty)$. We have

$$\int_0^\infty f_{n_k}(t)g(t)dt \longrightarrow \int_0^\infty p_t(x,y)g(t)dt$$

for each $g \in L^1((0,+\infty))$. Choosing $g(t) := e^{-\alpha t}$ (with $\alpha > 0$), we get

$$v_\alpha(x,y) = \int_0^\infty e^{-\alpha t} p_t(x,y)dt$$

3. Let us define finally a function $q(t,x,y)$ as: $q_0(t,x,y)$ if $(x,y) \notin A^1$; equal to $p_t(x,y)$ if $(x,y) \in A^1 \setminus A$ and arbitrary elsewhere (for all t). Thus, we have a positive and $\mathcal{B} \otimes (\mathcal{X} \otimes \mathcal{X})^{\mu \otimes \mu}$–measurable function $q(t,x,y)$ defined everywhere on $[0,+\infty] \times X \times X$.

Under the assumption that $\mathcal{X}$ is separable, the above construction is not necessary and q is $\mathcal{B} \otimes \mathcal{X} \otimes \mathcal{X}$–measurable.

This function has the property that:

$$v_\alpha(x,y) = \int_0^\infty e^{-\alpha t} q(t,x,y)dt$$

$\forall \alpha \geqslant 0$, $\forall (x,y) \notin A$.

Thus, we define, for each $t \in (0,+\infty)$ two kernels on $(X, \mathcal{X}^\mu)$

$$Q_t f(x) := \int q(t,x,y) f(y) \mu(dy)$$

$$Q_t^* g(x) := \int q(t,y,x) g(y) \mu(dy)$$

These kernels are clearly absolutely continuous with respect to μ. They are in duality:

$$\int Q_t f(x).g(x)\mu(dx) = \int \left(\int q(t,x,y) f(y) \mu(dy) \right) g(x)\mu(dx) =$$

$$= \int \left(\int q(t,x,y) g(x) \mu(dx) \right) f(y)\mu(dy) = \int f(x).Q_t^* g(x)\mu(dx)$$

Since for each fixed x, the section A_x is polar, hence μ-negligible, we can write:

$$\int_0^\infty e^{-\alpha t} Q_t f(x) dt = \int_0^\infty e^{-\alpha t} \left[\int q(t,x,y) f(y) \mu(dy) \right] dt =$$

$$= \int f(y)\mu(dy) \left[\int_0^\infty e^{-\alpha t} q(t,x,y) dt \right] = \int v_\alpha(x,y) f(y) \mu(dy) = V_\alpha f(x)$$

$$\int_0^\infty e^{-\alpha t} Q_t^* f(x) dt = \int_0^\infty e^{-\alpha t} \left[\int q(t,y,x) f(y) \mu(dy) \right] dt =$$

$$= \int f(y)\mu(dy) \left[\int_0^\infty e^{-\alpha t} q(t,y,x) dt \right] = \int v_\alpha(y,x) f(y) \mu(dy) = V_\alpha^* f(x)$$

for all x and α.

Thus, the proof of lemma 1 is complete.

2.4

We come now to the proof of the theorem. Let $\mathcal{V}' = (V'_\alpha)_{\alpha \geqslant 0}$ be an absolutely continuous resolvent, associated with a semigroup $\mathcal{P}' = (P'_t)_{t>0}$ (it suffices to consider the left translation semigroup on $\mathbb{R}$: $\mathcal{T}^* = (T_t^*)_{t>0}$ by $T_t^* f(x) := f(x-t)$). We can form the tensor product $(P_t \otimes P'_t)$: this is a semigroup and we denote $\mathcal{W} := (W_\alpha)_{\alpha \geqslant 0}$ the associated resolvent. We want to prove that it is absolutely continuous.

We can consider also the family of kernels $(Q_t \otimes P'_t)$ (see [5], [8]). Since we have:

$$V_\alpha f(x) = \int_0^\infty e^{-\alpha t} Q_t f(x) dt = \int_0^\infty e^{-\alpha t} P_t f(x) dt$$

it follows that, for each f and each x there exists a negligible set $N_{f,x} \subset [0, +\infty]$ such that for all $t \notin N_{f,x}$ we have $Q_t f(x) = P_t f(x)$. Now, we can write:

$$\int_0^\infty e^{-\alpha t} Q_t f(x) P'_t g(y) dt = \int_0^\infty e^{-\alpha t} P_t f(x) P'_t g(y) dt = W_\alpha (f \otimes g)(x,y)$$

The equality extends to all measurable functions F of two variables:

$$\int_0^\infty e^{-\alpha t} (Q_t \otimes P'_t) F(x,y) dt = \int_0^\infty e^{-\alpha t} (P_t \otimes P'_t) F(x,y) dt =$$

$$= W_\alpha F(x,y)$$

Now, each Q_t is μ–absolutely continuous and each V'_α is m–absolutely continuous. We prove that W_α is $\mu \otimes m$–absolutely continuous. Let $(\mu \otimes m)(M) = 0$. Hence, there exists a m–negligible set M' such that for all $t \notin M'$, the section $M_t := \{x \in X | (x,t) \in M\}$ is μ–negligible. For any such t, we have $Q_\tau(\chi_{M_t}) \equiv 0$, $\forall \tau > 0$. This can be used as:

$$Q_\tau(\chi_{M_t})(x) \leqslant \chi_{M'}(t)$$

Now:

$$W_0(\chi_M)(x_0,t_0) = \int_0^\infty (Q_\tau \otimes P'_\tau)(\chi_M)(x_0,t_0)d\tau =$$

$$= \int_0^\infty P'_\tau \left[Q_\tau(\chi_{M_\cdot})\right](x_0)(t_0)d\tau \leqslant \int_0^\infty P'_\tau(\chi_{M'})(t_0)d\tau =$$

$$= V'_0(\chi_{M'})(t_0) = 0$$

Let θ be a measure, with respect to which the resolvent $\mathcal{W}$, associated with the product semigroup $\mathcal{P} \bigotimes \mathcal{T}^*$ is absolutely continuous. We define a measure μ' on $(X, \mathcal{X})$ by:

$$\mu'(f') := \int_{X\times\mathbb{R}} P_t f(x)\theta(dx,dt)$$

Now $\mu'(f) = 0$ means that $P_t f(x) = 0$, θ–a.e. Hence, the $\mathcal{W}$–excessive function

$$F(x,t) := \begin{cases} 0 & \text{if} \quad t \leqslant 0 \\ P_t f(x) & \text{if} \quad t > 0 \end{cases}$$

is negligible with respect to θ. As $\mathcal{W}$ is absolutely continuous with respect to θ, it follows that $P_t f(x) = 0$, $\forall x \in X$, $\forall t > 0$.

Let us remark that, since:

$$\mu'(f) = \int_X V_0 f(x)\mu(dx) = \int_X \int_X v_0(x,y)f(y)\mu(dy)\mu(dx) =$$

$$= \int_X f(y)\left[\int_X v_0(x,y)\mu(dx)\right]\mu(dy)$$

it follows that μ' is absolutely continuous with respect to μ and the density is $V_0^* 1$.

Remark 1 . *If we suppose that $\mathcal{X}$ is separable, the proof becomes considerably shorter; we do not need the assumption "$\mathcal{X} = \mathcal{X}^{\mu}$". Moreover, the conclusion is strengthened to:*

"There exists a $m \otimes \mu \otimes \mu$-measurable map $p : (0, +\infty) \times X \times X$, such that:

$$P_t f(x) = \int p(t, x, y) f(y) \mu(dy) \text{ and } P_t^* f(x) = \int p(t, y, x) f(y) \mu(dy)$$

for any $f \in \mathcal{F}$."

2. The condition (C) already appeared in [9], [10] [11]. However, as (Q_t) do not form a semigroup, additional assumptions were supposed in order to reach the conclusion. The main novelty of this paper is that no additional assumption is needed in order to guarantee that the semigroups are absolutely continuous.

3. Under special conditions, we do not need to suppose the existence of the dual resolvent ; also the semigroups can be associated in certain situations.

a) Let us start with an absolutely continuous, sub–markovian resolvent $\mathcal{V}$, with proper V_0. The cone of excessive functions $\mathcal{E}_{\mathcal{V}}$ is a standard H-cone of functions only if*:*

$1 \in \mathcal{E}_{\mathcal{V}}$

$s, t \in \mathcal{E}_{\mathcal{V}} \Rightarrow \min(s, t) \in \mathcal{E}_{\mathcal{V}}$

$\mathcal{E}_{\mathcal{V}}$ separates X.

See [2] for all notions about H-cones.

In this situation, one can construct a sub–markovian resolvent $\mathcal{V}^*$, with proper V_0^* in duality, if we accept to change the base measure. We arrive thus to the hypotheses of Kunita–Watanabe theorem.

b) Let us start with a standard H-cone S. If we choose a weak unit $u \in S$, we represent S as a standard H-cone of functions on the saturated space X_u.

Moreover for each standard H-cone of functions on the saturated space X and for each nearly continuous generator $p \in S$, there exists an absolutely continuous sub–markovian resolvent, such that $V_0 1 = p$ while the cone of excessive functions coincide with $\overline{S}$.

If the generator p is of the form $\sum p_n$ where $p_n \in S_0$ and $p_n \leqslant 2^{-n}$, then one can associate with the resolvent a strongly Ray semigroup on the (naturally metrisable) compact space $\overline{X_1}$. The saturated space coincide in this case with the set of non-ramification points for the semigroup.

Of course, the same construction can be done for the dual resolvent; however, the base space may be different.

Let us suppose that S satisfies the axioms of polarity and of nearly continuity. There exists then a Green space Ω such that both $X_u \setminus \Omega$ and $X_{u^*} \setminus \Omega$ are polar; and there exists [3] a finite measure $\mu = \sum \mu_n$ on Ω such that the Green potentials $G^{\mu_n} \in S_0$ and $G^{*\mu_n} \in S_0^*$ while $\sum G^{\mu_n}$, $\sum G^{*\mu_n}$ are generators. Hence, one can construct the semigroups associated with the resolvents in duality, on the common space Ω.

Open problem: *find out a condition on V_0, on the infinitesimal generator, next on $p := V_0 1$, next on $S := \mathcal{E}_{\mathcal{V}}$, such that (there exists a) resolvent which satisfies (C).*

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