BOUNDARY BEHAVIOR OF HARMONIC AND QUASIREGULAR MAPPINGS

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The 12th Romanian–Finnish Seminar
August 17–21, 2009, Turku, Finland
(Joint research with S. Ponnusamy)

Abstract

We study the connection between multiplicities of the zeros and boundary behavior of bounded harmonic and quasiregular mappings of the plane.

Sufficient conditions for the existence of angular (non-tangential) limit at a boundary point, provided that multiplicities of zeroes of the function grow fast enough on a given sequence of points approaching the boundary. We compare these results and also consider sharpness of such conditions in the planar case.

This talk is based on joint research with S. Ponnusamy.

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Introduction

Lindelöf's theorem

Suppose that γ is a curve, with parameter interval $[0, 1]$, such that $|\gamma(t)| < 1$ if $t < 1$ and $\gamma(1) = 1$. If f is a bounded analytic function of the unit disk $\mathbb{D}$ and

$$\lim_{t \to 1} f\big(\gamma(t)\big) = \alpha,$$

then f has angular limit α at 1, i.e. limit in each angular region contained in the unit disk with the vertex in 1.

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Questions and remarks

- What about generalizations of Lindelöf's theorem for other classes of functions?
  - Holds for quasiconformal mappings in any dimension $n \geq 2$.
  - Does **not** hold for non-univalent quasiregular mappings for $n \geq 3$ (Rickman).
  - Lindelöf type results have been proved by Vuorinen with various assumptions for quasiregular mappings in $\mathbb{R}^n$.
- Terminology:
  - $\mathbb{H}^n$ is the upper half space and $\mathbb{B}^n$ is the unit ball.
  - For a discrete and open mapping $f\colon G \to \mathbb{R}^n$, the local (topological) index $i(x, f)$ is the infimum of $\sup_y \mathrm{card} f^{-1}(y) \cap U$ where U runs through the neighborhoods of x.

QR mappings ($n \geq 2$)

For example, we may prove the following:

Theorem (R., 2005)

Let $f\colon \mathbb{H}^n \to \mathbb{B}^n$ be a bounded K-quasiregular mapping, let $t_k = 2^{-k}$ and $f(t_k e_n) = 0$ for all $k = 0, 1, \ldots$

(1) If $\limsup_{k\to\infty} t_k^\gamma \mu(t_k e_n, f) = \infty$, where $\mu(t_k e_n, f) = C_1 i(t_k e_n, f)^{1/(n-1)}$ and $\gamma = C_2 \log(1/\beta)$, then $f \equiv 0$.

(2) If $\mu(t_k e_n, f) \to \infty \text{ as } k \to \infty$, then f has an angular limit 0 at the origin.

- What about the case $n = 2$?

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Hyperbolic metrics (1/2)

The hyperbolic metrics in the upper half plane $\mathbb{H}$ and the unit disk $\mathbb{D}$ are defined by

$$\cosh \rho_{\mathbb{H}}(x, y) = 1 + \frac{|x-y|^2}{2\mathrm{Im}\,(x)\,\mathrm{Im}\,(y)}, \quad x, y \in \mathbb{H}, \tag{2.1}$$

and

$$\sinh^2\big(\frac{1}{2}\rho_{\mathbb{D}}(x, y)\big) = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}, \quad x, y \in \mathbb{D}, \tag{2.2}$$

respectively. If there is no danger of confusion, we denote both $\rho_{\mathbb{H}}(x, y)$ and $\rho_{\mathbb{D}}(x, y)$ simply by $\rho(x, y)$.

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Hyperbolic metrics (2/2)

- A hyperbolic disk with the center x and the radius $M > 0$ is denoted by $D(x, M)$.
- A well-known fact is that hyperbolic disks are disks in the Euclidean geometry as well.
- Both for $(\mathbb{D}, \rho_{\mathbb{D}})$ and $(\mathbb{H}, \rho_{\mathbb{H}})$ one can define the hyperbolic distance in terms of the absolute ratio.
- Since the absolute ratio is invariant under Möbius transformations the hyperbolic metric also remains invariant under these transformations.

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Analytic case

Let $f(z)$ be analytic at z_0 and with $f(z_0) = 0$, but not identically zero. Then $f(z)$ has a zero of order n at z_0 if

$$f(z_0) = f'(z_0) = \ldots = f^{(n-1)}(z_0) = 0, \text{ and } f^{(n)}(z_0) \neq 0.$$

If $f(z)$ is analytic at z_0, and has zero of order n at z_0, we have $\mu(z_0, f) = n$. Recall the following version of the Schwarz' lemma.

Schwarz' lemma

Let $f : \mathbb{D} \to \mathbb{D}$ be analytic with $f(0) = 0$ and $\mu(0, f) = p \geq 1$. Then

$$|f(z)| \leq |z|^p \text{ for all } z \in \mathbb{D}.$$

Multiplicity and boundary behavior

Theorem A (Ponnusamy, R.)

Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function with $b_k \in \mathbb{D}$ such that $f(b_k) = 0$ for all $k = 1, 2, \ldots$, where $b_k \to \beta \in \partial\mathbb{D}$, $\rho(b_k, b_{k+1}) \leq M_k < \infty$ and $\mu_k = \mu(b_k, f)$. If

$$\left(\frac{1 - e^{-M_k}}{1 + e^{-M_k}}\right)^{\mu_k} \to 0 \text{ as } k \to \infty,$$

then $f(z)$ has an angular limit 0 at β.

Multiplicity and boundary behavior

Corollary (Uniformly distributed zeros)

Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function with $b_k \in \mathbb{D}$ such that $f(b_k) = 0$ for all $k = 1, 2, \ldots$, where $b_k \to \beta \in \partial\mathbb{D}$, $\rho(b_k, b_{k+1}) \leq M < \infty$ and $\mu(b_k, f) \to \infty$. Then $f(z)$ has an angular limit 0 at β.

Multiplicity and boundary behavior

Theorem B (Ponnusamy, R.)

Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function with $b_k \in \mathbb{D}$ such that $f(b_k) = 0$ for all $k = 1, 2, \ldots$, where $b_k \to 1$ and $\mu_k = \mu(b_k, f)$. If

$$\liminf_{k\to\infty} |b_k|^{\mu_k} = 0,$$

then $f \equiv 0$ on $\mathbb{D}$.

Example 1

Example 1

Let $b_k = 1 - 2^{-k}$ and $\mu_k = k$ for $k = 1, 2, \ldots$. Then

$$\sum_{k=1}^{\infty} \mu_k (1 - b_k) = \sum_{k=1}^{\infty} k 2^{-k} = 2 < \infty,$$

and hence, one may construct an analytic function $B(z)$, a so called Blaschke product, whose zeros are precisely $\{b_k\}$ with respective multiplicities μ_k. By Corollary $B(z)$ has angular limit 0 at 1.

Example 2 (P. Lappan)

- It is possible to construct a Blaschke product $B_0\colon \mathbb{D} \to \mathbb{D}$ with infinitely many zeroes b_k on the positive real axis such that $\mu(b_k, B_0) \to \infty$ as $k \to \infty$, but B_0 does not have an angular limit at 1.
- Let $b_k = 1 - e^{-k}$. If we form the Blaschke product $B(z)$ having zeros (of order one) at the points b_k, then $B(z)$ does not have a radial limit at $z = 1$, since there is a sequence $\{c_k\} \subset (0, 1)$ such that $b_{k-1} < c_k < b_k$ and a number $\sigma > 0$ such that $|B(c_k)| \geq \sigma$ for each k.

Example 2 (P. Lappan), continued

- Let $\{m_k\}$ be any sequence of positive integers, and let $n_k = m_k + 2\sum_{j=1}^{k-1} m_j$.
- Note that $n_{k+1} - n_k = m_k + m_{k+1}$. Let $a_k = b_{n_k}$, let $p_k = n_k + m_k$, and let $d_k = c_{p_k}$.
- We now have

$$a_k = b_{n_k} < b_{n_k+m_k-1} < c_{p_k} = d_k < b_{p_k} < b_{p_k+m_{k+1}-1} < a_{k+1}.$$

- Now define

$$B_0(z) = \prod_{k=1}^{\infty} \left(\frac{|a_k|(a_k - z)}{a_k(1 - \bar{a}_k z)} \right)^{m_k} .$$

Example 2 (P. Lappan), continued

- Next we show that $|B_0(d_k)| \geq |B(d_k)|$.
- By comparing the position of factors, we have for $q < k$, $a_q = b_{n_q} < b_{n_q+1}$, and hence

$$\left(\frac{|d_k - a_q|}{|1 - \bar{a}_q d_k|}\right)^{m_q} > \prod_{j=n_q+1}^{n_q+m_q} \frac{|d_k - b_j|}{|1 - \bar{b}_j d_k|}.$$

- Also, for $q \geq k$, as $n_{q+1} - m_{q+1} = n_q + m_q$, we have

$$\left(\frac{|d_k - a_q|}{|1 - \bar{a}_q d_k|}\right)^{m_q} > \prod_{j=n_q-m_q+1}^{n_q} \frac{|d_k - b_j|}{|1 - \bar{b}_j d_k|}.$$

Example 2 (P. Lappan), remarks

- Essentially, using the points $c_{p_k} = d_k$, we have that the corresponding zeros of $B_0(z)$, counted according to multiplicity, are farther away from the points d_k than the corresponding zeros of $B(z)$.
- All of the factors of B_0 are considered, but there are some factors of $B(z)$ not counted in this scheme.

Planar harmonic functions

- A real-valued function $u(x, y)$ is harmonic if it satisfies the Laplace equation

$$\Delta u = \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = 0.$$

- A complex-valued function $f(x + iy) = u(x, y) + iv(x, y)$ from a domain $f : D \to \mathbb{C}$ is a harmonic if the two coordinate functions are harmonic.
- A complex-valued harmonic function is a harmonic mapping if it is univalent (one-to-one).
- Harmonic mappings in the plane are univalent complex-valued mappings whose real and imaginary parts are not necessarily conjugate i.e. do not need to satisfy the Cauchy–Riemann equations.

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Canonical representation

Lemma

In a simply connected domain $D \subset \mathbb{C}$ a complex valued harmonic function f has the representation $f = h + \bar{g}$, where h and g are analytic in D. The representation is unique up to an additive constant.

- For a harmonic mapping f of the unit disk $\mathbb{D}$, it is convenient to choose the additive constant so that $g(0) = 0$.
- The representation $f = h + \bar{g}$ is then unique and is called the canonical representation of f.

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Lindelöf's theorem: harmonic case

- The harmonic mapping $f : \mathbb{D} \to \mathbb{C}$

$$f(x, y) = \mathrm{Arg}\,(1 - z) + i\mathrm{Re}\,(1 - z)$$

  has infinitely many values at 1.
- Hence it is clear that Lindelöf's theorem does not generalize to the harmonic functions
- But having the same limit when approaching 1 radially from two separate directions is sufficient an angular limit.

Sense-preserving harmonic functions

- A complex-valued harmonic function f is sense-preserving in Ω if it satisfies a Beltrami equation $\overline{f_{\bar{z}}} = \omega f_z$, where ω is an analytic function in Ω with $|\omega(z)| < 1$.
- Since Jacobian is $J_f = |f_z|^2 - |f_{\bar{z}}|^2$, this implies that $J_f(z) > 0$ whenever $f_z(z) \neq 0$.
- If $f(z_0) = 0$ for some $z_0 \in \Omega$, the order of the zero can be defined in terms of the canonical decomposition $f = h + \bar{g}$,

$$h(z) = a_0 + \sum_{k=n}^{\infty} a_k (z - z_0)^k, \qquad g(z) = b_0 + \sum_{k=m}^{\infty} b_k (z - z_0)^k,$$

  where $n, m \geq 1$ and $a_n, b_m \neq 0$.
- Then $m > n$ or $m = n$ and $|b_n| < |a_n|$. In either case we say that f has zero of order n at z_0 and write $\mu(z_0, f) = n$.

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Schwarz' lemma (harmonic case)

Lemma (Mateljevic, Vuorinen)

Let f be a sense-preserving complex-valued function harmonic in $\mathbb{D}$, with $f(0) = 0$ and $|f(z)| < 1$.

Then

$$|f(z)| \leq \frac{4}{\pi} \arctan |z|^{\mu(0,f)} \leq \frac{4}{\pi} |z|^{\mu(0,f)},$$

for each $z \in \mathbb{D}$.

Angular regions in hyperbolic geometry

Lemma

Suppose that $b_1, b_2, \ldots$ is a sequence of points on the positive imaginary axis with $\lim_{k\to\infty} b_k = 0$ and $0 < m < \rho_{\mathbb{H}}(b_k, b_{k+1}) < M_k$. Then there exists $\varphi = \varphi(m)$ such that the angular region

$$C_\varphi = \{z \in \mathbb{H} : |\arg z - \pi/2| < \varphi \text{ and } |z| < |b_1|\}$$

is contained in the set $D = \bigcup_{k=1}^{\infty} D(b_k, M_k)$.

Lindelöf type result for harmonic functions

Theorem (Ponnusamy, R.)

Fix $m > 0$. Let $f\colon \mathbb{H} \to \mathbb{D}$ be a harmonic function, $\mu_k = \mu(b_k, f)$, and let $\{b_k\}$, $b_k \to 0$ as $k \to \infty$, be a sequence of points on the positive imaginary axis such that $0 < m \le \rho_{\mathbb{H}}(b_k, b_{k+1}) = M_k$ and $f(b_k) = 0$ for all $k = 1, 2, \ldots$ with $\lim_{k\to\infty} b_k = 0$. If

$$\left(\frac{1 - e^{-M_k}}{1 + e^{-M_k}}\right)^{\mu_k} \to 0 \text{ as } k \to \infty,$$

then f has an angular limit 0 at 1.

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