Uniform Continuity and φ-uniform Domains

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Notation

- $\mathbb{R}^n$: the Euclidean n-space ($n \geq 2$).
- $B^n(a,r) := \{x \in \mathbb{R}^n : |x-a| < r\}$, the Euclidean ball with radius r centered at a.
- $S^{n-1}(a,r) := \{x \in \mathbb{R}^n : |x-a| = r\}$ sphere with radius r centered at a.
- We set $B^n(r) := B^n(0,r)$, $B^n := B^n(1)$ and $S^{n-1}(r) := S^{n-1}(0,r)$, $S^{n-1} := S^{n-1}(1)$.
- $H^n := \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$, the n-dimensional upper half-space.
- $G \subsetneq \mathbb{R}^n$: a domain (i.e. an open connected non-empty set).
- We denote by ∂G, $\overline{G}$ and '$\mathrm{diam}\, G$' the boundary, closure and diameter of G, respectively.

- For $x \in G$ we write $\delta(x) := d(x, \partial G)$, the Euclidean distance from x to ∂G.
- Given $x, y \in G$, $\Gamma(x, y)$ stands for the collection of all rectifiable paths $\gamma \subset G$ joining x and y.
- The w-length, $(w : G \to (0, \infty))$, of $\gamma \in \Gamma(x, y)$ is

$$\ell_w(\gamma) = \int_\gamma w(z)|dz|.$$

Problems

Problem A: Uniform continuity

(A1) Find a sharp bound for the modulus of continuity of the identity mapping

$$id:\ (G,k_G)\to(G,|\cdot|)\,.$$

(A2) Characterize domains G such that

$$id:\ (G,j_G)\to(G,k_G)\,.$$

is uniformly continuous.

Problems

Problem B: Generalization of uniform domains

(B1) Examples of φ-uniform domains.

(B2) Complement of φ-uniform domains.

Problem C: Properties of φ-uniform domains

(C1) φ-uniformity under bilipschitz mapping.

(C2) Suppose that $(z_i)_{i=1}^m$ is a finite non-empty sequence of points in a domain $G \subsetneq \mathbb{R}^n$. Is it true that G is φ_0-uniform implies $G \setminus \{z_1, z_2, \dots, z_m\}$ also is φ-uniform for some φ depending on φ_0 only.

Introduction

Uniform continuity

Let (X_j, d_j), $j = 1, 2$, be metric spaces. A function $f : X_1 \to X_2$ is said to be *uniformly continuous* if there exists a function, *modulus of continuity* $\omega : [0, r_1) \to [0, r_2)$ such that $\omega(0) = 0$ and $\omega(t) \to 0$, as $t \to 0$, and for all $x, y \in X_1$ with $d_1(x, y) < r_1$ we have $d_2(f(x), f(y)) \le \omega(d_1(x, y)) < r_2$.

Introduction

The weighted metric

The w-length minimizing property

$$\rho_G(x,y) = \inf_{\gamma \in \Gamma(x,y)} \ell_w(\gamma) = \inf_{\gamma \in \Gamma(x,y)} \int_\gamma w(z)|dz|\,,$$

defines the weighted metric in G.

The hyperbolic metrics

$$\rho_{B^n}\ : w(z) = 2/(1-|z|^2)\,, \quad \rho_{H^n}\ : w(z) = 1/z_n\,.$$

Introduction

The quasihyperbolic metric

- The *quasihyperbolic metric* [GP-76] is defined by the quasihyperbolic length minimizing property

$$k_G(x,y) = \inf_{\gamma\in\Gamma(x,y)} \ell_k(\gamma), \quad \ell_k(\gamma) = \int_\gamma \frac{|dz|}{\delta(z)},$$

  where $\ell_k(\gamma)$ is the quasihyperbolic length of γ.
- For a given pair of points $x, y \in G$, the infimum is always attained [GO-79], i.e. there always exists a quasihyperbolic geodesic $J_G[x,y]$ which minimizes the above integral, $k_G(x,y) = \ell_k(J_G[x,y])$ and furthermore with the property that the distance is additive on the geodesic: $k_G(x,y) = k_G(x,z) + k_G(z,y)$ for all $z \in J_G[x,y]$.

Introduction

The distance ratio metric

For $x, y \in G$ the *distance ratio metric* j_G is defined [Vu-85] by

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{\delta(x), \delta(y)\}}\right) .$$

In a slightly different form, this metric was studied in [GO-79].

Facts

- $k_G(x,y) \geq j_G(x,y), \forall x, y \in G$.
- $j_{B^n}(x,y) \leq \rho_{B^n}(x,y) \leq 2j_{B^n}(x,y)$ for all $x, y \in B^n$ with equality on the right hand side when $y = -x$ (see [AVV, Lemma 7.56]).
- j_G is not geodesic.

Motivation to Problem A

Lemma [(A1) for ρ_{B^n} [Vu-book, (2.27)]]

For $x, y \in B^n$ let $t = \sqrt{(1-|x|^2)(1-|y|^2)}$. Then

$$\tanh^2 \frac{\rho_{B^n}(x,y)}{2} = \frac{|x-y|^2}{|x-y|^2+t^2} \,,$$

$$|x-y| \leqslant 2\tanh \frac{\rho_{B^n}(x,y)}{4} = \frac{2|x-y|}{\sqrt{|x-y|^2+t^2}+t} \,,$$

where equality holds for $x=-y$. In particular, the identity map is uniformly continuous with $\omega(t) = 2\tanh(t/4)$.

Earle and Harris [EH-08] provided several applications of this inequality and extended this inequality to other metrics such as the Carathéodory metric.

Solution to (A1)

Quasihyperbolic counterpart of Lemma

(1) If x, y are on a diameter of B^n and $w = |x-y|\, e_1/2$, then we have

$$k_{B^n}(x,y) \geq k_{B^n}(-w,w) = 2\log\frac{2}{2-|x-y|} \geq |x-y|\,,$$

where the first inequality becomes equality when $y=-x$.

(2) If $x, y \in B^n$ are arbitrary and $w = |x-y|\, e_1/2$, then

$$k_{B^n}(x,y) \geq k_{B^n}(-w,w) = 2\log\frac{2}{2-|x-y|} \geq |x-y|\,,$$

where the first inequality becomes equality when $y=-x$.

Solution to (A1)

Proof.

We use

Facts

- For $x \in B^n$ we have

$$k_{B^n}(0,x) = j_{B^n}(0,x) = \log \frac{1}{1-|x|}\,.$$

- For $b \in S^{n-1}$ and $0 < r < s < 1$ we have

$$k_{B^n}(br,bs) = j_{B^n}(br,bs) = \log \frac{1-r}{1-s}\,.$$

Solution to (A1)

Proof Contd.

(1) If x and y are in the opposite sides of the origin, then we have

$$k_{B^n}(x,y) = k_{B^n}(x,0) + k_{B^n}(0,y) = \log \frac{1}{(1-|x|)(1-|y|)} .$$

Also, we have

$$k_{B^n}(-w,w) = 2k_{B^n}(0,w) = 2\log \frac{1}{1-|w|} .$$

We need to prove that

$$(1-|w|)^2 \geq (1-|x|)(1-|y|) .$$

Solution to (A1)

Proof Contd.

It suffices to show that

$$\frac{|x|+|y|}{2} = \frac{|x-y|}{2} = |w| \le 1 - \sqrt{(1-|x|)(1-|y|)}\,,$$

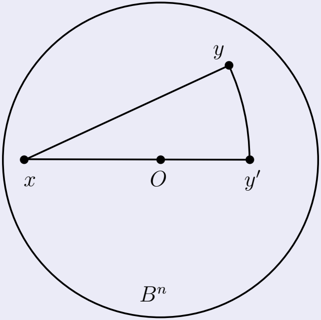
which is equivalent to $(|x|-|y|)^2 \ge 0$.
If x and y are in the same side of the origin, then proof goes in a similar way.

Solution to (A1)

Proof Contd.

(2) Choose $y' \in B^n$ such that $|x-y| = |x-y'| = 2|w|$ with x and y' on a diameter of B^n (see the figure below). Then

$$k_{B^n}(x,y) \geq k_{B^n}(x,y') \geq k_{B^n}(-w,w)\,.$$



Solution to (A1)

Generalization to bounded domains in $\mathbb{R}^n$

(3) Let $G \subsetneq \mathbb{R}^n$ be a domain with $\operatorname{diam} G < \infty$ and $r = \sqrt{n/(2n+2)}\,\operatorname{diam} G$. Then we have

$$k_G(x,y) \geq 2\log\left(\frac{2}{2-t}\right) \geq t = |x-y|/r\,,$$

for all distinct $x, y \in G$ with equality in the first step when $G = B^n(z,r)$ and $z = (x+y)/2$.

For the proof of (3) we use Jung's Theorem

Let $G \subset \mathbb{R}^n$ be a domain with $\operatorname{diam} G < \infty$. Then there exists $z \in \mathbb{R}^n$ such that $G \subset B^n(z,r)$, where $r \leq \sqrt{n/(2n+2)}\,\operatorname{diam} G$.

Definitions

Uniform domains

- In 1979, the class of uniform domains was introduced by Martio and Sarvas [MS-79]: a domain G is said to be uniform if, for some $C > 0$, any pair of points $x, y \in G$ can be joined by a rectifiable path $\gamma \subset G$ such that
  1. $\ell(\gamma) \leq C\,|x - y|$; and
  2. $\min\{\ell(\gamma[x,z]), \ell(\gamma[z,y])\} \leq C\,\delta(z)$ for all $z \in \gamma$.
- Gehring and Osgood [GO-79]: a domain G is *uniform* if and only if there exists a constant $c \geq 1$ such that

$$k_G(x,y) \leq c\, j_G(x,y) + d \qquad (1)$$

  for all $x, y \in G$.
- Vuorinen [Vu-85, 2.50]: the additive constant d can be chosen to be 0.

Definitions

φ-uniform domains

- Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous strictly increasing function with $\varphi(0) = 0$. A domain $G \subsetneq \mathbb{R}^n$ is said to be *φ-uniform* if [Vu-85]

$$k_G(x,y) \le \varphi(|x-y|/\min\{\delta(x),\delta(y)\})$$

  for all $x, y \in G$.
- Clearly, uniform domains are φ-uniform with $\varphi(t) = C\log(1+t)$. Here the constant C is the constant of uniformity.

Solution to (A2)

Theorem (Characterization of φ-uniform domains)

The identity mapping id : $(G, j_G) \to (G, k_G)$ is uniformly continuous if and only if G is φ-uniform.

Proof.

Sufficiency part is trivial. Indeed, for $x, y \in G$ we have

$$k_G(x, y) \le \varphi(\exp(j_G(x, y)) - 1) = \omega(j_G(x, y))$$

where $\omega(t) = \varphi(e^t - 1)$.
For the necessary part, we define $\varphi : (0, \infty) \to (0, \infty)$ by

$$\varphi(t) = \sup\{k_G(x, y) : j_G(x, y) \le t\} \quad t > 0\,.$$

(B1): Examples of φ-uniform domains

- Uniform domains are φ-uniform as indicated before.
- Consider domains G satisfying the following property [Vu-85, 2.50]: there exists a constant $C \geq 1$ such that each pair of points $x, y \in G$ can be joined by a rectifiable path $\gamma \in G$ with

$$\ell(\gamma) \leq C\,|x-y| \text{ and } d(\gamma, \partial G) \geq \min\{\delta(x), \delta(y)\}/C\,.$$

  Then G is φ-uniform with $\varphi(t) = C^2 t$.
- In particular, every convex domain is φ-uniform with $\varphi(t) = t$.
- However, in general, convex domains need not be uniform (for example parallel strips).
- There exist φ-uniform domains with "arbitrary" $\varphi(t)$.

(B2): Complement of φ-uniform domains

- Since simply connected uniform domains in plane are quasidisks [MS-79], it follows that the complement of such a uniform domain also is uniform.
- Since the half-strip defined by $S = \{(x, y) \in \mathbb{R}^2 : \; x > 0, \; -1 < y < 1\}$ is convex, by the above discussion we observe that it is φ-uniform with $\varphi(t) = t$. On the other hand, by considering the points $z_n = (n, -2)$ and $w_n = (n, 2)$ we see that $G := \mathbb{R}^2 \setminus \overline{S}$ is not a φ-uniform domain. Indeed, we have $j_G(z_n, w_n) = \log 5$ and for some $m \in \mathbb{R} \cap J_G[z_n, w_n]$

$$\begin{aligned} k_G(z_n, w_n) \geq k_G(m, w_n) &\geq \log\left(1 + \frac{|m - w_n|}{\delta(w_n)}\right) \\ &\geq \log(1 + n) \to \infty \quad \text{as } n \to \infty. \end{aligned}$$

(B2): Complement of φ-uniform domains

- Consider the domain

$$D = \left\{(x, y) \in \mathbb{R}^2 : -\exp(-1 - x) < y < \exp(-1 - x),\ x > 0\right\}.$$

It is clear by previous investigation that D is φ-uniform with $\varphi(t) = 4t$. By a similar investigation to the previous example we see that its complementary domain $G := \mathbb{R}^2 \setminus \overline{D}$ is not φ-uniform.

(B2): Complement of φ-uniform domains

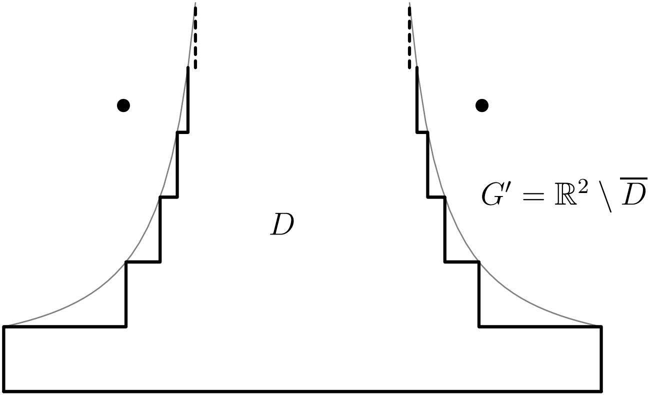


Figure: An unbounded φ-uniform domain $D \subset \mathbb{R}^2$ whose complement $G' = \mathbb{R}^2 \setminus \overline{D}$ is not φ-uniform.

(B2): Complement of φ-uniform domains

Question!

Are there any bounded φ-uniform domains whose complements are not φ-uniform ?

Let T be the triangle with vertices $(1, -1)$, $(0, 0)$ and $(1, 1)$. Consider the domain D bounded by the surface of revolution generated by revolving T about the vertical axis. (see Figure 2).

Then we see that D is φ-uniform and its complementary domain $G = \mathbb{R}^3 \setminus \overline{D}$ is not φ-uniform.

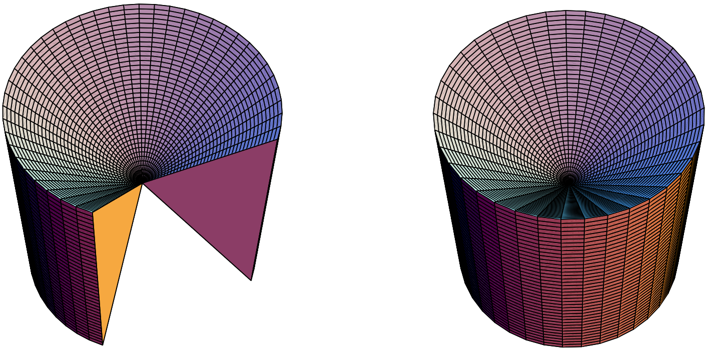


Figure: A bounded φ-uniform domain in $\mathbb{R}^3$ whose complement is not φ-uniform.

Problem C: Properties of φ-uniform domains

(C1): φ-uniformity under bilipschitz mapping

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an L-bilipschitz mapping, that is

$$|x - y|/L \leq |f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in \mathbb{R}^n$. If $G \subsetneq \mathbb{R}^n$ is φ-uniform, then $f(G)$ is φ_1-uniform with $\varphi_1(t) = L^2\varphi(L^2 t)$.

Problem C: Properties of φ-uniform domains

Proof.

We denote $\delta(z) := d(z, \partial G)$ and $\delta'(w) := d(w, \partial f(G))$. Since f is L-bilipschitz, it follows that $\delta(z)/L \le \delta'(f(z)) \le L\delta(z)$ for all $z \in G$.

We have the following well-known relation (see for instance [Vu-book])

$$k_G(x,y)/L^2 \le k_{f(G)}(f(x), f(y)) \le L^2 k_G(x,y)$$

for all $x, y \in G$.

Problem C: Properties of φ-uniform domains

Proof Contd.

Hence, φ-uniformity of G yields

$$\begin{aligned} k_{f(G)}(f(x), f(y)) &\leq L^2 k_G(x, y) \\ &\leq L^2 \varphi(|x - y| / \min\{\delta(x), \delta(y)\}) \\ &\leq L^2 \varphi(L^2 |f(x) - f(y)| / \min\{\delta'(f(x)), \delta'(f(y))\}) . \end{aligned}$$

This concludes our claim.

Problem C: Properties of φ-uniform domains

Problem C2: Recall

Suppose that $(z_i)_{i=1}^m$ is a finite non-empty sequence of points in a domain $G \subsetneq \mathbb{R}^n$. Is it true that G is φ_0-uniform implies $G \setminus \{z_1, z_2, \ldots, z_m\}$ also is φ-uniform for some φ depending on φ_0 only.

(C2): It is enough to discuss

Let $G \subsetneq \mathbb{R}^n$ be a φ_1-uniform domain and $z_0 \in G$. Then $G \setminus \{z_0\}$ is φ-uniform for some φ depending on φ_1 only.

Problem C: Properties of φ-uniform domains

For the proof, we use the following lemma from [Vu-85]

Lemma

[Vu-85, Lemma 2.53] If $\theta \in (0,1)$, then there exists a positive number $a(\theta)$ such that the following holds. If $x, y, z \in G$ with $x, y \in G \setminus B^n(z, \theta d(z, \partial G))$ then

$$k_{G\setminus\{z\}}(x,y) \le a(\theta)\, k_G(x,y)\,,$$

where $a(\theta) = 1 + (2/\theta) + \pi/(2\log \frac{2+2\theta}{2+\theta})$.

Problem C: Properties of φ-uniform domains

We consider three cases

(I) $x, y \in B^n(z_0, \theta\delta_1(z_0)) \setminus \{z_0\}$.

(II) $x, y \in G \setminus B^n(z_0, \theta\delta_1(z_0)/2)$.

(III) $x \in B^n(z_0, \theta\delta_1(z_0)/2) \setminus \{z_0\}$ and $y \in G \setminus B^n(z_0, \theta\delta_1(z_0))$.

Problem C: Properties of φ-uniform domains

A generalization

Let $\theta \in (0,1)$. Assume that $G \subsetneq \mathbb{R}^n$ is φ_1-uniform and $z_0 \in G$. If $E \subset B^n(z_0, \theta d(z_0, \partial G)/5)$ is a non-empty closed set such that $\mathbb{R}^n \setminus E$ is φ_2-uniform, then $G \setminus E$ is φ-uniform for φ depending on φ_1 and φ_2.

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THANK YOU