Invariant Schwarzian derivative of higher order and its applications

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August 18, 2009

The 12th Romanian-Finnish Seminar, University of Turku

Introduction

Note

Joint work with
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(Classical) Schwarzian derivative

- pre-Schwarzian derivative:

$$T_f = \frac{f''}{f'}$$

- Schwarzian derivative:

$$S_f = (T_f)' - \frac{1}{2}(T_f)^2 = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

Basic properties

- $S_f = 0$ iff f is a Möbius transformation.
- $S_{g\circ f} = (S_g \circ f) \cdot (f')^2 + S_f$
- In particular,

$$S_{g\circ f} = (S_g \circ f) \cdot (f')^2$$

for a Möbius transformation f.

Norm

For $c \in \mathbb{R}$, set

$$\|\varphi\|_c = \sup_{|z|<1} (1 - |z|^2)^c |\varphi(z)|$$

for a function φ on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Univalence criteria

Theorem (Nehari)

Let f be a non-constant meromorphic function on $\mathbb{D}$. If f is univalent then $\|S_f\|_2 \leq 6$. Conversely, if $\|S_f\|_2 \leq 2$ then f is univalent. The numbers 6 and 2 are sharp.

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Theorem (Becker, Becker-Pommerenke)

Let f be a non-constant analytic function on $\mathbb{D}$. If f is univalent then $\|T_f\|_1 \leq 6$. Conversely, if $\|T_f\|_1 \leq 1$ then f is univalent. The numbers 6 and 1 are sharp.

A key property

The usefulness of the quantity $\|S_f\|_2$ comes from the invariance property

$$S_{g\circ f} = (S_g)\circ f \cdot (f')^2$$

for $f \in \mathrm{Aut}(\mathbb{D})$.

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$$S_{g\circ f} = (S_g)\circ f \cdot (f')^2$$

for $f \in \mathrm{Aut}(\mathbb{D})$.
Equivalently, setting $\sigma[f](z) = (1-|z|^2)^2 S_f(z)$, we have

$$\sigma[g\circ f] = \sigma[g]\circ f \cdot \left(\frac{f'}{|f'|}\right)^2$$

for $f \in \mathrm{Aut}(\mathbb{D})$.

Higher-order Schwarzians

Tamanoi's Schwarzian derivatives

Tamanoi (1995):

$$W = \frac{f'(z)(f(\zeta) - f(z))}{\frac{1}{2}f''(z)(f(\zeta) - f(z)) + f'(z)^2} = \sum_{n=0}^{\infty} S_n[f](z)\frac{(\zeta - z)^{n+1}}{(n+1)!}.$$

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$S_n[f]$ is called (Tamanoi's) Schwarzian derivative of virtual order n.

Recursions

By the relation

$$\partial_z W - \partial_\zeta W = -1 - \tfrac{1}{2} S_2[f](z) W^2,$$

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we have

$$S_n[f] = S_{n-1}[f]' + \tfrac{1}{2} S_2[f] \sum_{k=1}^{n-1} \binom{n}{k} S_{k-1}[f] S_{n-k-1}[f], \quad n \geq 3.$$

First several relations

$$S_0[f] = 1$$
$$S_1[f] = 0$$
$$S_2[f] = S_f$$
$$S_3[f] = S_2[f]'$$
$$S_4[f] = S_3[f]' + 4S_2[f]^2$$
$$S_5[f] = S_4[f]' + 5S_2[f]S_3[f]$$
$$S_6[f] = S_5[f]' + 6S_2[f]S_4[f] + 10S_2[f]^3.$$

Simple observations

$S_n[f] = 0$ does not necessarily imply $S_{n+1}[f] = 0$.

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Example: Let $f(z) = e^{az}$ for a constant $a \neq 0$. Then $S_2[f] = -a^2/2$. Therefore, $S_3[f] = 0$ but $S_4[f] = a^4$.

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This example also tells us that $S_3[f] = 0$ does not imply univalence of f.

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By definition,

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for a Möbius transformation f, in general.

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for a Möbius transformation f, in general. For instance, since $S_3[f] = S_2[f]' = (S_f)'$, we have

$$S_3[g \circ f] = S_3[g] \circ f \cdot (f')^3 + 2S_2[g] \circ f \cdot f' f''$$

for a Möbius transformation f.

Motivation

One of our motivations of the present research is to find quantities, say $\tilde{S}_n[f]$, analogous with Schwarzians, which satisfy the relation

$$\tilde{S}_n[g \circ f] = \tilde{S}_n[g] \circ f \cdot (f')^n$$

or

$$\tilde{S}_n[g \circ f] = \tilde{S}_n[g] \circ f \cdot \left(\frac{f'}{|f'|} \right)^n$$

for a Möbius transformation f or for a conformal isometry f.

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Peschl-Minda derivatives

ρ-derivative

Let $\rho = \rho(z)|dz|$ be a (smooth) conformal metric on a domain Ω.

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The ρ-derivative: for $\varphi \in C^\infty(\Omega)$,

$$\partial_\rho \varphi = \frac{\partial \varphi}{\rho}$$

Standard metrics

- Spherical metric:
  $\mathbb{C}_{+1} = \widehat{\mathbb{C}}$ with $\lambda_{+1} = \lambda_{+1}(z)|dz| = |dz|/(1+|z|^2)$.

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- hyperbolic metric:
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These metrics have constant Gaussian curvatures $+4, 0, -4$, respectively. λ_δ-derivatives are called spherical derivative, (usual) derivative and hyperbolic derivative according to the cases $\delta = +1, 0, -1$.

Definition of Peschl-Minda derivatives

For a holomorphic map $f : \Omega \to \Omega'$, we define the Peschl-Minda derivative $D^n f = D^n_{\sigma,\rho} f$ of order n with respect to ρ and σ inductively by

$$D^1 f = \frac{\sigma \circ f}{\rho} f'$$

$$D^{n+1} f = \left[\partial_\rho - n\partial_\rho(\log \rho) + (\partial_\sigma \log \sigma) \circ f \cdot D^1 f\right] D^n f \quad (n \geq 1).$$

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References:

E. Schippers, *The calculus of conformal metrics*, Ann. Acad. Sci. Fenn. Math. 32 (2007), 497–521
Kim-Sugawa, *Invariant differential operators associated with a conformal metric*, Michigan Math. J. 55 (2007), 459–479

Classical cases (due to Peschl)

Peschl (1955): for a holomorphic map $f : \mathbb{C}_\delta \to \mathbb{C}_\varepsilon$,

$$\frac{f\left(\frac{\zeta+z}{1-\delta\bar{z}\zeta}\right) - f(z)}{1 + \varepsilon\overline{f(z)}f\left(\frac{\zeta+z}{1-\delta\bar{z}\zeta}\right)} = \sum_{n=1}^{\infty} \frac{D^n f(z)}{n!} \cdot \zeta^n$$

$$D^1 f(z) = \frac{(1+\delta|z|^2)f'(z)}{1+\varepsilon|f(z)|^2},$$

$$D^2 f(z) = \frac{(1+\delta|z|^2)^2 f''(z)}{1+\varepsilon|f(z)|^2} + \frac{2\delta\bar{z}(1+\delta|z|^2)f'(z)}{1+\varepsilon|f(z)|^2} - \frac{2\varepsilon(1+\delta|z|^2)^2\overline{f(z)}f'(z)^2}{(1+\varepsilon|f(z)|^2)^2},$$

$$D^3 f(z) = \frac{(1+\delta|z|^2)^3 f'''(z)}{1+\varepsilon|f(z)|^2} - \frac{6\varepsilon(1+\delta|z|^2)^3\overline{f(z)}f'(z)f''(z)}{(1+\varepsilon|f(z)|^2)^2} + \frac{6\delta\bar{z}(1+\delta|z|^2)^2 f''(z)}{1+\varepsilon|f(z)|^2} + \frac{6\delta^2\bar{z}^2(1+\delta|z|^2)f'(z)}{1+\varepsilon|f(z)|^2} - \frac{12\delta\varepsilon\bar{z}(1+\delta|z|^2)^2\overline{f(z)}f'(z)^2}{(1+\varepsilon|f(z)|^2)^2} + \frac{6\varepsilon^2(1+\delta|z|^2)^3\overline{f(z)}^2 f'(z)^3}{(1+\varepsilon|f(z)|^2)^3}.$$

Invariant Schwarzian derivatives

Tamanoi's Schwarzian derivatives revisited

Define a sequence of polynomials $P_n = P_n(x_1, \ldots, x_n)$ of n indeterminates $x_1, \ldots, x_n$ inductively by

$$P_0 = 1, P_1 = 0, P_2 = x_2 - 3x_1^2/2,$$

and

$$P_n = \sum_{k=1}^{n-1} (x_{k+1} - x_1 x_k) \frac{\partial P_{n-1}}{\partial x_k} + \frac{1}{2} P_2 \sum_{k=1}^{n-1} \binom{n}{k} P_{k-1} P_{n-k-1}, \quad n \geq 3.$$

$$
\begin{aligned}
P_3 &= x_3 - 4x_1x_2 + 3x_1^3, \\
P_4 &= x_4 - 5x_1x_3 + 5x_1^2x_2, \\
P_5 &= x_5 - 6x_1x_4 + \frac{15}{2}x_1^2x_3 - 10x_1x_2^2 + 30x_1^3x_2 - \frac{45}{2}x_1^5, \\
P_6 &= x_6 - 7x_1x_5 + \frac{21}{2}x_1^2x_4 - 35x_1x_2x_3 + \frac{105}{2}x_1^3x_3 \\
&\quad + 105x_1^2x_2^2 - 210x_1^4x_2 - \frac{315}{4}x_1^6, \\
P_7 &= x_7 - 8x_1x_6 + 14x_1^2x_5 - 56x_1x_2x_4 + 84x_1^3x_4 - 35x_1x_3^2 \\
&\quad + 420x_1^2x_2x_3 - 420x_1^4x_3 - 420x_1^3x_2^2 + 420x_1^5x_2.
\end{aligned}
$$

$$\begin{aligned} P_3 &= x_3 - 4x_1x_2 + 3x_1^3, \\ P_4 &= x_4 - 5x_1x_3 + 5x_1^2x_2, \\ P_5 &= x_5 - 6x_1x_4 + \frac{15}{2}x_1^2x_3 - 10x_1x_2^2 + 30x_1^3x_2 - \frac{45}{2}x_1^5, \\ P_6 &= x_6 - 7x_1x_5 + \frac{21}{2}x_1^2x_4 - 35x_1x_2x_3 + \frac{105}{2}x_1^3x_3 \\ &\quad + 105x_1^2x_2^2 - 210x_1^4x_2 - \frac{315}{4}x_1^6, \\ P_7 &= x_7 - 8x_1x_6 + 14x_1^2x_5 - 56x_1x_2x_4 + 84x_1^3x_4 - 35x_1x_3^2 \\ &\quad + 420x_1^2x_2x_3 - 420x_1^4x_3 - 420x_1^3x_2^2 + 420x_1^5x_2. \end{aligned}$$

By letting $q_n[f] = f^{(n+1)}/f'$, we have

$$S_n[f] = P_n(q_1[f], q_2[f], \ldots, q_n[f]), \quad n \geq 0.$$

Invariant Schwarzian derivatives

Let Ω and Ω' be domains with conformal metrics ρ and σ respectively. Define for a non-constant holomorphic map $f : \Omega \to \Omega'$,

$$\Sigma^n f = P_n(Q^1 f, \dots, Q^n f), \quad n \geq 0,$$

where

$$Q^n f = \frac{D^{n+1} f}{D^1 f}.$$

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where

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$\Sigma^n f$ will be called the invariant Schwarzian derivative of virtual order n. To indicate the metrics involved, we sometimes write $\Sigma^n f = \Sigma^n_{\sigma,\rho} f$ and $Q^n f = Q^n_{\sigma,\rho} f$.

Invariance property

Lemma

Let $\Omega, \hat{\Omega}, \Omega', \hat{\Omega}'$ be plane domains with smooth conformal metrics $\rho, \hat{\rho}, \sigma, \hat{\sigma}$, respectively. Suppose that locally isometric holomorphic maps $g : \hat{\Omega} \to \Omega$ and $h : \Omega' \to \hat{\Omega}'$ are given. Then, for a non-constant holomorphic map $f : \Omega \to \Omega'$, the formulae

$$Q^n_{\hat{\sigma},\hat{\rho}}(h \circ f \circ g) = (Q^n_{\sigma,\rho} f) \circ g \cdot \left(\frac{g'}{|g'|}\right)^n$$

$$\Sigma^n_{\hat{\sigma},\hat{\rho}}(h \circ f \circ g) = (\Sigma^n_{\sigma,\rho} f) \circ g \cdot \left(\frac{g'}{|g'|}\right)^n$$

are valid on $\hat{\Omega}$.

Recursions

Compare with the recurrence formula for $S_n[f]$:

$$\Sigma^n f = \big(\partial_\rho - (n-1)\partial_\rho \log \rho\big)\Sigma^{n-1} f$$

$$+\frac{1}{2}\Sigma^2 f \sum_{k=1}^{n-1} \binom{n}{k} \Sigma^{k-1} f \Sigma^{n-k-1} f, \quad n \geq 3.$$

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Note that the classical Schwarzian derivatives cannot be extended unless one assigns projective structures on Riemann surfaces.
Furthermore, Tamanoi's Schwarzian derivatives cannot be defined for a nonconstant holomorphic map between Riemann surfaces, in general, even when projective structures are assigned.

Projective Schwarzian derivatives

Setting

Let $f : \Omega \to \Omega'$ be a nonconstant holomorphic map between Riemann surfaces with projective structures. If the source domain Ω is equipped with conformal metric ρ, we can define another kind of invariant Schwarzian derivatives of higher order, called projective Schwarzian derivatives.

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For simplicity, we will consider only plane domains (with standard projective structures) in the sequel.

Covariant derivatives

Let $\varphi = \varphi(z)dz^n$ be an n-differential on Ω. Then its covariant derivative in z-direction w.r.t. the Levi-Civita connection of ρ is defined by

$$\Lambda_\rho(\varphi) = \left[\partial\varphi - 2n(\partial \log \rho)\varphi\right] dz^{n+1}.$$

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We define $\mathfrak{D}_\rho^n f$ by

$$\mathfrak{D}_\rho^n f dz^n = \Lambda_\rho^{n-2}(S_f(z)dz^2), \quad n \geq 2.$$

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We define $\mathfrak{D}^n_\rho f$ by

$$\mathfrak{D}^n_\rho f dz^n = \Lambda^{n-2}_\rho(S_f(z)dz^2), \quad n \geq 2.$$

By naturality, for a Möbius transformation h, we have

$$\mathfrak{D}^n_{h^*\rho}(f \circ h) = (\mathfrak{D}^n_\rho f) \circ h \cdot (h')^n.$$

Another expression of Tamanoi's ones

Define a sequence of polynomials $T_n = T_n(x_2, \ldots, x_n)$ of $n-1$ indeterminates with integer coefficients, inductively, by $T_2 = x_2$ and

$$T_n = \sum_{k=2}^{n-1} \frac{\partial T_{n-1}}{\partial x_k} \cdot x_{k+1} + \frac{x_2}{2} \sum_{k=1}^{n-1} \binom{n}{k} T_{k-1} T_{n-k-1}, \quad n \geq 3.$$

Here, we also set $T_0 = 1$ and $T_1 = 0$.

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$$S_n[f] = T_n(Sf, (Sf)', \ldots, (Sf)^{(n-2)}), \quad n \geq 3.$$

Note that T_n is of weight n, in other words,

$$T_n(a^2x_2, a^3x_3, \ldots, a^nx_n) = a^nT_n(x_2, x_3, \ldots, x_n), \quad a \in \mathbb{C}.$$

Definition of projective Schwarzians

Define $V_\rho^n f$ $(n \geq 2)$ by

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Note that $V_\rho^n f = S_n[f]$ when $\rho = |dz|$.

Lemma

For Möbius transformations g and h,

$$V_{h^*\rho}^n(g \circ f \circ h) = (V_\rho^n f) \circ h \cdot (h')^n, \quad n \geq 2.$$

Corollary

Let f be a nonconstant meromorphic map on $\mathbb{D}$. For an analytic automorphism T of $\mathbb{D}$ and a Möbius transformation M,

$$V^n(M \circ f \circ T) = V^n f \circ T \cdot (T')^n.$$

In particular, $\|V^n(M \circ f \circ T)\|_n = \|V^n f\|_n, \ n \geq 2$. Here $V^n = V^n_\rho$ for $\rho = |dz|/(1-|z|^2)$.

Relation between invariant and projective Schwarzians

Schwarzian for the metric

For a conformal metric ρ on Ω, let

$$\Theta_\rho = 2\partial^2 \log \rho - 2(\partial \log \rho)^2 = 2\frac{\partial^2 \rho}{\rho} - 4\left(\frac{\partial \rho}{\rho}\right)^2.$$

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Note:

- Θ_ρ becomes holomorphic iff ρ has constant Gaussian curvature.

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- Θ_ρ is NOT a conformal invariant, but a projective invariant.

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Note:

- Θ_ρ becomes holomorphic iff ρ has constant Gaussian curvature.
- Θ_ρ is NOT a conformal invariant, but a projective invariant.

Example: $\Theta_{\lambda_\delta} = 0$ for $\delta = +1, 0, -1.$

Theorem

Let Ω, Ω' be domains with smooth conformal metrics ρ, σ, respectively, and let $f : \Omega \to \Omega'$ be a non-constant holomorphic map. Then

$$\Sigma^2 f = \rho^{-2} \left[S_f + f^* \Theta_\sigma - \Theta_\rho \right],$$

where $f^* \Theta_\sigma$ is the pull-back $(\Theta_\sigma \circ f)(f')^2$ as a quadratic differential.

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where $f^* \Theta_\sigma$ is the pull-back $(\Theta_\sigma \circ f)(f')^2$ as a quadratic differential.

Corollary

For a non-constant holomorphic map $f : \mathbb{C}_\delta \to \mathbb{C}_\varepsilon$,

$$\Sigma^2 f = \lambda_\delta^{-2} S_f.$$

Further relations

Define Θ_ρ^n $(n \geq 2)$ by

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It is a basic problem to write down a relation between $\Sigma_{\sigma,\rho}^n f$ and $V_\rho^n f$ in terms of $\Theta_\rho, \Theta_\sigma$ and their higher derivatives defined as above. We, however, only consider the case when $n = 3$ here.

Relation between $\Sigma^3 f$ and V_f

Theorem

Let Ω, Ω' be domains with smooth conformal metrics ρ, σ, respectively, and let $f : \Omega \to \Omega'$ be a nonconstant holomorphic map. Then

$$\Sigma^3_{\sigma,\rho} f = \rho^{-3} \left[V^3_\rho f + f^* \Theta^3_\sigma - \Theta^3_\rho \right] + 2\rho^{-2} f^* \Theta^2_\sigma \cdot Q^1_{\sigma,\rho} f,$$

where $f^* \Theta^n_\sigma$ is the pull-back $(\Theta^n_\sigma \circ f)(f')^n$ as an n-differential.

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Theorem

Let Ω, Ω' be domains with smooth conformal metrics ρ, σ, respectively, and let $f : \Omega \to \Omega'$ be a nonconstant holomorphic map. Then

$$\Sigma^3_{\sigma,\rho} f = \rho^{-3} \left[V^3_\rho f + f^*\Theta^3_\sigma - \Theta^3_\rho \right] + 2\rho^{-2} f^*\Theta^2_\sigma \cdot Q^1_{\sigma,\rho} f,$$

where $f^*\Theta^n_\sigma$ is the pull-back $(\Theta^n_\sigma \circ f)(f')^n$ as an n-differential.

Corollary

For a nonconstant holomorphic map $f : \mathbb{C}_\delta \to \mathbb{C}_\varepsilon$,

$$\Sigma^3 f = \lambda_\delta^{-3} V_f.$$

Univalence criteria

Meromorphic functions on $\mathbb{D}$

From now on, we suppose that $\Omega = \mathbb{D} = \mathbb{C}_{-1},\ \rho = \lambda_{-1}$ and $\Omega' = \mathbb{C}_{+1},\ \sigma = \lambda_{+1}$.

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For simplicity, we write $V_f = V^3_{\lambda_{-1}} f$ for a nonconstant meromorphic function $f : \mathbb{D} \to \widehat{\mathbb{C}}$. Recall

$$V_f(z) = (S_f)'(z) - \frac{4\bar{z}}{1 - |z|^2} S_f(z).$$

Univalence criteria with V_f

Theorem

Let f be a non-constant meromorphic function on the unit disk $\mathbb{D}$. If f is univalent in $\mathbb{D}$, then $\|V_f\|_3 \leq 16$. The number 16 is sharp. Conversely, if $\|V_f\|_3 \leq 3/2$, then f is univalent in $\mathbb{D}$.

A lemma

Lemma

For a locally univalent meromorphic function f on the unit disk, the inequalities

$$\frac{16}{25\sqrt{5}}\|V_f\|_3 \le \|S_f\|_2 \le \frac{4}{3}\|V_f\|_3$$

hold. Here, the constant $16/25\sqrt{5}$ is sharp.

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We now apply Nehari's univalence criterion: "$\|S_f\|_2 \leq 2 \Rightarrow \ f$ is univalent", to obtain the second part of the theorem.

A representation formula

For the proof of the last lemma, the following representation formula is needed:

$$S_f(z) = \frac{1}{\pi} \iint_{|\zeta|<1} \frac{(1-|\zeta|^2)^4 V_f(\zeta)}{(1-|z|^2)^4(\bar{z}-\bar{\zeta})} d\xi d\eta \quad (\zeta = \xi + i\eta)$$

for a locally univalent meromorphic function f on $\mathbb{D}$ with $\|V_f\|_3 < \infty$.

An example

Suppose that $S_f(z) = a(1 - z^2)^{-2}$ for a real constant a. Then $\|S_f\|_2 = |a|$ and f is univalent in $\mathbb{D}$ iff $-6 \le a \le 2$.

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Note that the functions $k(z) = z/(1-z)^2$ and $l(z) = \frac{1}{2}\log\frac{1+z}{1-z}$ satisfy $S_k(z) = -6(1-z^2)^{-2}$ and $S_l(z) = 2(1-z^2)^{-2}$.

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Lemma

Suppose that $S_f(z) = a(1-z^2)^{-2}$ in $z \in \mathbb{D}$ for a complex constant a. Then

$$\|Vf\|_3 = \frac{8\sqrt{3}}{9}|a|.$$

Open problems

- Find a (sharp) constant $c > 3/2$ such that $\|V_f\|_3 \leq c$ implies univalence of f on $\mathbb{D}$.

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- Find univalence criteria in terms of $V^n f$ for higher n.