

On Generalized Gauss and Martinelli-Bochner Means

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Decentralized Gauss mean

Pseudospherical harmonics

Pseudospherical function

Generalized Gauss-Helmholtz formula

Martinelli-Bochner mean

- ▶ Gauss' mean-value formula: $G \in \mathbb{R}^N$; $u : G \rightarrow \mathbb{R}$: *harmonic*; $B_{[a]}(\rho) \Subset G$: open ball with centered a and radius $\rho \Rightarrow u(a) = \text{mean-value of } u \text{ on } \partial B_{[a]}(\rho)$.
- ▶ Gauss mean-value formula is a centralized formula.
- ▶ **Bochner-Martinelli formula:** $G \in \mathbb{C}^m$, ∂G : smooth boundary; $\psi \in \mathcal{O}(\overline{G}) \Rightarrow$

$$\psi(a) = \int_G \psi(\zeta) \mathcal{K}_a(\zeta), \quad \forall a \in G.$$

- ▶ The Bochner-Martinelli kernel \mathcal{K}_a is universal: independent on the shape of the domain.
- ▶ **Aronov-Kytmanov Theorem:** *Any C^1 -function on the closure of a domain $G \in \mathbb{C}^m$ with piecewise smooth boundary is holomorphic in G if and only if it admits the Bochner-Martinelli representation property.*

- ▶ Question: When does a (locally Lipschitz) function $\psi \in C^\lambda(\overline{D})$ admit the (centralized) Gauss mean-value property" at every point of D ?
- ▶ Question: Can the Gauss mean-value formula be decentralized, and when does a function $\psi \in C^\lambda(\overline{D})$ admit the "decentralized Gauss mean-value property" on \overline{D} ?

The **Gauss' lemma** for a double layer potential with unit density is a decentralized (mean-value) formula: In \mathbb{R}^2 ,

$$\frac{-1}{2\pi} \int_{\partial D} \partial_\nu \left(\log \frac{1}{r_a} \right) d\sigma = \begin{cases} = 1, & a \in D, \\ \frac{1}{2}, & a \in \partial D, \\ 0, & a \notin \overline{D}, \end{cases}$$

In \mathbb{R}^3 : the left-hand side integral is: $\int_{\partial D} \partial_\nu \frac{1}{r_a^{2m-2}} d\sigma$.

- ▶ V. G. Maz'ya ([4, p. 178]): "the needs of practical applications as well as the intrinsic logic of the theory itself dictate the consideration of more general curves and surfaces".
- ▶ Cialdea ([1, p. 376]): "to investigate how the classical algorithms of analysis ... can be generalized in such a way as to have a general range of applicability".
- ▶ Objective: To study integral means with respect to a general singular form defined in a neighborhood of a domain (in an ambient space) that is not necessarily smooth or lying in \mathbb{C}^m , and investigate how the Gauss' lemma and the Helmholtz representation formula can be coalesced and generalized to such a domain.

Definition

semi-Riemann domain (X, p) : X : a reduced complex space of dimension $m > 0$; $p : X \rightarrow W$: holomorphic; W : domain in \mathbb{C}^N with thin inverse image; Σ : thin analytic subset of W ; all fibers of p over $W \setminus \Sigma$ are discrete.

- ▶ $d^c := (1/4\pi i)(\partial - \bar{\partial})$; $a \in X$, $p^{[a]} := p - p(a)$, $r_a := \|p^{[a]}\|$.
- ▶ $v_p := dd^c r_a^2 = \frac{i}{2\pi} \partial \bar{\partial} r_a^2$.
- ▶ To each *pre-kernel* $q_a \in C^2(\bar{D} \setminus p^{-1}(a'))$ are associated the *kernel forms*

$$\mathfrak{K}_a := \frac{i}{2\pi} \partial q_a \wedge v_p^{m-1};$$

$$\Sigma_a := -d^c q_a \wedge v_p^{m-1};$$

$$\Xi_a := \frac{1}{4\pi} dq_a \wedge v_p^{m-1}.$$

- ▶ q_a : real-valued $\Rightarrow \Sigma_a = \Re(\mathfrak{K}_a)$, $\Xi_a = \Im(\mathfrak{K}_a)$.
- ▶ $X^0 := X \setminus p^{-1}(\Sigma)$; $D^0 := X^0 \cap D$; $\widehat{\partial S} := p^{-1}(p(\partial S))$ for $S \subseteq X$; $a \in X$: *properly contained in D* iff $a \in D \setminus \widehat{\partial D}$.

Definition

Assume: D : a weak Stokes domain with $dD \neq \emptyset$, and $\{q_z\}_{z \in X}$: an *admissible potential* in a neighborhood of \overline{D} .

The (decentralized) *Gauss mean* of $\psi \in \mathcal{L}^2(\partial D)$ is defined by:

$$\mathcal{G}(\psi, z) := \int_{dD} \psi d^c(-q_z) \wedge v_p^{m-1}, \quad \forall z \in X.$$

- **Theorem:** The linear mapping $\mathcal{G} : \psi \mapsto \mathcal{G}\psi$, where $(\mathcal{G}\psi)(z) := \mathcal{G}(\psi, z)$, is a compact operator on $\mathcal{L}^2(\partial D)$.

Definition

$\psi \in C^\lambda(D \setminus \{a\})$ is a *pseudospherical harmonic* ($\psi \in \hat{\mathcal{H}}_{a,l}(D)$) iff.

(i) ψ is semiharmonic; (ii) ψ is *a-homogeneous of order l* , namely, ψ satisfies the equation $(E_{p,a} + \bar{E}_{p,a})(\psi) = l\psi$ a.e. in D^* .

- **Proposition:** $\psi \in C^\lambda(D)$ belongs to $\hat{\mathcal{H}}_{a,l}(D) \Leftrightarrow$ for some thin analytic set A in D and $\forall a \in D \setminus A$, the restriction of

$$\Psi_a(z) := (r_a(z))^{-l} \psi(z), \quad z \in D \setminus \rho^{-1}(a'),$$

to each (suff. small) $\partial U_{[a]}(\rho_0)$ satisfies the equation

$$(\Delta_S)_{a,\rho_0} \Psi_a + \frac{l(l+2m-2)}{\rho_0^2} \Psi_a = 0.$$

$(\Delta_S)_{a,\rho_0}$ = the (induced) spherical Laplacian, (thus Ψ_a is a "surface spherical harmonic at $a \in D$ ").

Definition

$\psi \in C^\lambda(D \setminus \{a\})$ is *a-radially symmetric*, respectively, *a-spherically symmetric*, in D , iff ψ is *a-homogeneous of order 0*, respectively, $E_{p,a}(\psi) = \bar{E}_{p,a}(\psi)$, a.e. in D^* .

Example

Set $\tilde{x}_j := \operatorname{Re}(p_j)$, $\tilde{y}_j := \operatorname{Im}(p_j)$, $1 \leq j \leq m$. Choose $a \in D$, any real-valued $U \in C^1(\mathbb{R})$ and arbitrary real constants c_k , $1 \leq k \leq m$. Let $\phi := U(\sum_{k=1}^m c_k \|\mathfrak{z}_k\|^2)$ in \overline{D} . It is easily verified that the equation $\Im(\bar{E}_{p,a}(\phi)) = 0$ in D^* . Therefore ϕ is a -spherically symmetric in D .

Example

Set $\tilde{x}_j := \operatorname{Re}(p_j)$, $\tilde{y}_j := \operatorname{Im}(p_j)$, $1 \leq j \leq m$. Choose $a \in D$, any real-valued $U \in C^1(\mathbb{R})$ and arbitrary real constants c_k , $1 \leq k \leq m$. Set $\theta_k(z) := \arctan\left(\frac{\tilde{y}_k - \tilde{y}_k(a)}{\tilde{x}_k - \tilde{x}_k(a)}\right)$, or $\theta_k(z) := \operatorname{arc cot}\left(\frac{\tilde{x}_k - \tilde{x}_k(a)}{\tilde{y}_k - \tilde{y}_k(a)}\right)$, whichever is defined. Let $\psi := U(\sum_{k=1}^m c_k \theta_k(z))$ in $\overline{D} \setminus \mathcal{S}$, where \mathcal{S} is thin analytic in D . It is easily verified that the equation $\Re(\bar{E}_{p,a}(\psi)) = 0$ a.e. in D^* . Therefore ψ is a -radially symmetric in D .

- **Decentralized Gauss Mean-value Formula:** q_z : real quasi-Newtonian; $\psi \in C^\lambda(\overline{D} \setminus \mathcal{A})$, $\mathcal{A} \subset D$: finite subset. \Rightarrow

$$\mathcal{G}(\psi, z) = \langle\langle \psi \rangle\rangle_{p,D}(z') + \int_D \frac{1}{2^m r_z^{2m-1}} \mathfrak{R}_{p,z}(\psi) v_p^m, \quad \forall z \notin \widehat{\partial D}.$$

where the first term on the right-side is the *push-forward* of ψ :

$$\langle\langle \psi \rangle\rangle_{p,D}(z') := \sum_{\xi \in D} \nu_f^y(\xi) \psi(\xi)$$

- For almost every $a \in D$, ψ : a -radially symmetric near a . \Rightarrow

$$\mathcal{G}(\psi, a) = \langle\langle \psi \rangle\rangle_{p,D}(a'). \quad (1)$$

Definition

A : a thin analytic set in D ; $\{T_A(r_n)\}$: any nested sequence of tubular neighborhoods of A ; $\psi_n := \psi|_{D \setminus T_A(r_n)}$. $\psi \in C^\lambda(\overline{D} \setminus A)$ is a -pseudospherical (**wrt** a real admissible q_a) iff:

$$\lim_{n \rightarrow \infty} \mathcal{G}(\psi_n, a) = \langle\langle \psi \rangle\rangle_{p,D \setminus A}(a').$$

Definition

Direct push-forward of ψ at $a' := p(a)$:

$$[\psi]_{p,D}(a') := (\langle\langle\psi\rangle\rangle_{p,D} + \frac{1}{2} \langle\langle\psi\rangle\rangle_{p,\partial D})(a').$$

- **Generalized Gauss-Helmholtz formula:** $\{q_a\}_{a \in X}$: an admissible potential on \bar{D} for $\{g_a\}_{a \in X}$;
 $\psi \in C^0(\bar{D}) \cap C^{1,1}(\bar{D} \setminus \mathcal{A})$, $\mathcal{A} \subset D$: finite subset. $\Rightarrow \forall a \in X$,

$$\begin{aligned} [\psi]_{p,D}(a') &= \mathcal{G}(\psi, a) + \int_{dD} q_a d^c \psi \wedge v_p^{m-1} \\ &\quad - (q_a, \Delta_p \psi)_D + (g_a, \psi)_D. \end{aligned}$$

$g_a := \Delta_p q_a$, $\Delta_p =$ the induced Laplace operator on D^* .

(Proof depends on an expression relating the Laplacian operator to the Neumann derivatives).

Definition

$D \in X$, (relative to $p^{[\xi]}$ and \mathfrak{K}_ξ) *Martinelli-Bochner mean* of $\psi \in C^0(\partial D)$:

$$\llbracket \psi \rrbracket_{\partial D}(\xi) := \int_{dD} \psi(\zeta) \mathfrak{K}_\xi(\zeta), \quad \xi \in X \setminus \widehat{\partial D}.$$

$$\llbracket \psi \rrbracket_{a,r}(\xi) := \llbracket \psi \rrbracket_{\partial D_{[a]}(r)}(\xi), \quad \forall \xi \notin \widehat{\partial D_{[a]}(r)}, \quad \forall a \in D.$$

Let $\psi \in C^\lambda(\overline{D} \setminus A)$, and $\{T_A(r_n)\}$: any nested sequence of neighborhoods of A ($T_A(r_n) = \emptyset$ if $A = \emptyset$). $D_n := D \setminus \overline{T_A(r_n)}$. Then:

$$\int_{dD} \psi \Xi_a - \int_{D \cap dT_A(r_n)} \psi \Xi_a = \frac{1}{2m} \int_{D_n} \frac{h}{r_a^{2m-1}} \mathfrak{S}_{p,a}(\psi) v_p^m.$$

provided q_a is a real, basic, admissible prekernel.

Definition

(**Wrt** q_a with admissible singularities) $\psi \in C^\lambda(\overline{D} \setminus A)$ is *a-pseudoradial* iff

$$\lim_{n \rightarrow \infty} \int_{dD_n} \psi \Xi_a = 0 \quad (2)$$

for some nested sequence $\{T_A(r_n)\}$.

► If $A = \emptyset$, then the condition (2) \Rightarrow

$$\mathcal{G}(\psi, a) = \llbracket \psi \rrbracket_{\partial D}(a).$$

Definition

$\psi \in C^\lambda(\overline{D})$ is of *GM* (Gauss-Martinelli) *type* on \overline{D} . iff ψ is both *a-pseudoradial* and *a-pseudospherical* for all $a \in D$ (**wrt** the same admissible $\{q_a\}$). For a given $a \in D$, ψ is of *a-GM type*, if ψ is both *a-pseudoradial* and *a-pseudospherical* (**wrt** some q_a).

- ▶ $\{\text{Functions of GM type on } \overline{D}\} \subset \{\text{locally Lipschitz functions on } \overline{D} \setminus A \text{ admitting both the decentralized Gauss mean-value property and the generalized Bochner-Martinelli representation in } D\}$
- ▶ Questions: When is a locally Lipschitz function of GM type ? When does a function $\psi \in C^\lambda(\overline{D})$ admit the generalized BM representation property ?

D : a weak Stokes domain in a normal Riemann domain; $dD \neq \emptyset$.

- ▶ $\psi \in C^\lambda(\overline{D})$ is *nearly holomorphic* in D iff, **wrt** an admissible $\{q_a\}$ for $\{g_a\}$, $g_a \in C^0(\overline{D} \setminus p^{-1}(a'))$, the equation $\langle \psi, \overline{q}_a \rangle_D = (\psi, -g_a)_D$ holds $\forall a \hat{\in} D$.
- ▶ **Theorem:** If $\{q_a\}$ is a basic admissible potential for $\{g_a\}$, $g_a \in C^0(\overline{D} \setminus p^{-1}(a'))$, then (with respect to $\{q_a\}$) a function $\psi \in C^\lambda(\overline{D})$ is nearly holomorphic if and only if ψ admits the generalized BM representation

$$\llbracket \psi \rrbracket_{\partial D}(a) = \langle \langle \psi \rangle \rangle_{p,D}(a'), \quad \forall a \hat{\in} D.$$

- ▶ **Proposition:** Let $\{q_z\}$ be a principal quasi-Newtonian potential on \overline{D} and $\psi \in C^{1,1}(\overline{D})$. (1) If $\psi \in \mathfrak{H}(D)$, then ψ is of GM type with respect to $\{q_z\}$ if and only if ψ is a constant. (2) If $\psi \in \mathcal{O}(D^*)$, then, for each $a \in D$, ψ is either a -pseudospherical or a -pseudoradial with respect to $\{q_z\}$ if and only if ψ is a constant.
- ▶ $\mathcal{A} \setminus p^{-1}(a')$: a finite set. \Rightarrow every a -pseudospherical function $\psi \in C^\lambda(\overline{D} \setminus \mathcal{A}) \cap C^0(\overline{D})$ has the generalized mean-value property (1).
- ▶ Pseudospherical, resp., pseudoradial, functions: characterized in terms of (centralized) radial, resp., spherical, derivatives.
- ▶ (Theorem) If a function $\psi \in C^\lambda(D)$ is a -radially symmetric for all points a off some thin analytic subset of D , then ψ is semiharmonic in D .
- ▶ Thus the a -pseudospherical semiharmonic functions may be thought of a generalization of the spherical harmonics in \mathbb{C}^m .

Assume that X is a normal Riemann domain:

- ▶ **Proposition:** $\psi \in C^{1,1}(\overline{D}) \cap \mathfrak{H}(D)$. Then: ψ is of GM type on \overline{D} wrt some $\{q_z\}$ (as above) $\Leftrightarrow \psi = \text{const}$.
- ▶ **Theorem:** If $\psi : \overline{D} \rightarrow \mathbb{C}$ is locally Lipschitz, then: $\psi = \text{const}$. \Leftrightarrow locally at each $a \in D^*$, ψ is a.e. a -radially and a -spherically symmetric.
- ▶ **Open question:** Are the constant functions the sole functions of GM type?

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