Region of variability for exponentially convex univalent functions

A. Vasudevarao
(A joint work with S. Ponnusamy and M. Vuorinen)

Department of Mathematics, IIT Madras, India.
Email- alluvasu@iitm.ac.in

International Conference on Complex Analysis and Related Topics
(The 12^{th} Romanian–Finnish Seminar), 17–21 August, 2009.

File: vasu_Region_Turku_new.tex, printed: 2009-9-19, 14.46

Organization

Notations and preliminaries

- $\mathbb{D} := \{z : |z| < 1\}$ denotes the unit disk in the complex plane $\mathbb{C}$.
- $\mathcal{H}$ denotes the space of all analytic functions in $\mathbb{D}$.
- $\mathcal{B}_0 = \{f : \mathbb{D} \to \overline{\mathbb{D}} : f \text{ is analytic and } f(0) = 0\}$.
- $\mathcal{A} := \{f \in \mathcal{H} : f(0) = f'(0) - 1 = 0\}$
- $\mathcal{S}$ denotes the class of univalent functions in $\mathcal{A}$.
- $\mathcal{S}^*$ denotes the class of univalent starlike functions

Notations and preliminaries

- $\mathcal{C}$ denotes the class of univalent convex functions

Definition [AMRu] (1997)

- A univalent function f in the unit disk $\mathbb{D}$ is called exponentially convex if e^f maps $\mathbb{D}$ onto a convex domain.

Definition [AMRu] (1997)

- Let $\alpha \in \mathbb{C} \setminus \{0\}$. We say that $f \in \mathcal{S}$ belongs to the class $\mathcal{E}(\alpha)$, of α- exponentially convex functions, if $F(\mathbb{D})$ is a convex set, where

$$F(z) = e^{\alpha f(z)}.$$

- $\mathcal{F}_\alpha = \mathcal{A} \cap \mathcal{E}(\alpha)$

Theorem A[Du, Go]

For $f \in \mathcal{S}$, we have

1. $\left| \frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2} \right| \leq \frac{4r}{1-r^2}, \quad |z| = r < 1$
2. $\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \quad |z| = r < 1$
3. $\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \quad |z| = r < 1.$

For each $z \in \mathbb{D}$, $z \neq 0$, equality occurs if and only if f is a suitable rotation of the Koebe function.

Theorem B[Du, Go]

If $f \in \mathcal{C}$ then

1. $\frac{1}{(1+r)^2} \leq |f'(z)| \leq \frac{1}{(1-r)^2}$
2. $\frac{r}{1+r} \leq |f(z)| \leq \frac{r}{1-r}, \quad |z| = r < 1$

Work of Yanagihara [Ya1, Ya2] (2005–2006)

1. $\mathcal{B}(\lambda) =$
$\{f \in \mathcal{H} : |f'(z)| \leq 1, f(0) = 0, f'(0) = \lambda, \quad \text{with } |\lambda| \leq 1\}$
2. $\mathcal{P}(\lambda) =$
$\{f \in \mathcal{H} : \mathrm{Re}\, f'(z) > 0, f(0) = 0, f'(0) = \lambda, \quad \text{with } \mathrm{Re}\, \lambda > 0\}$
3. $\mathcal{C}(\lambda) =$
$\left\{f \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \;\text{ and } f''(0) = 2\lambda \;\text{ with } |\lambda| \leq 1\right\}$

For each fixed $z_0 \in \mathbb{D}$ and λ

1. $V_{\mathcal{B}}(z_0, \lambda) = \{f(z_0) : f \in \mathcal{B}(\lambda)\}$ Theorem-01
2. $V_{\mathcal{P}}(z_0, \lambda) = \{f(z_0) : f \in \mathcal{P}(\lambda)\}$ Theorem-02
3. $V_{\mathcal{C}}(z_0, \lambda) = \{\log f'(z_0) : f \in \mathcal{C}(\lambda)\}$ Theorem-03

Exponentially convex univalent functions

Theorem C ([AMRu, Theorem 1])

Let $\alpha \in \mathbb{C} \setminus \{0\}$. A function f is in $\mathcal{F}_\alpha$ if and only if $\mathrm{Re}\, P_f(z) > 0$ in $\mathbb{D}$, where

$$P_f(z) = 1 + \frac{zf''(z)}{f'(z)} + \alpha z f'(z).$$

Theorem D ([AMRu, Theorem 1])

Let $f \in \mathcal{E}(\alpha)$. Then $f(\mathbb{D})$ is convex in the $\overline{\alpha}-$ direction (and therefore close-to-convex). It is not necessarily starlike univalent.

Theorem E ([AMRu, Theorem 1])

For $\alpha \in \mathbb{C} \setminus \{0\}$ we have

$$\mathcal{E}(\alpha) = \left\{ \frac{1}{\alpha} \log(1 + \alpha g) \colon g \in \mathcal{C}(\alpha) \right\}.$$

Theorem F ([AMRu, Theorem 1])

$\mathcal{E}(\alpha)$ is empty for $|\alpha| > 2$, and for $|\alpha| = 2$ it contains only of the function

$$f(z) = \frac{1}{\alpha} \log \frac{2 + \alpha z}{2 - \alpha z}.$$

Herglotz representation

If $f \in \mathcal{F}_\alpha$, then there exists a unique positive unit measure μ on $(-\pi, \pi]$ such that

$$1 + \frac{zf''(z)}{f'(z)} + \alpha z f'(z) = \int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, d\mu(t), \quad z \in \mathbb{D}.$$

A computation gives that

$$\log f'(z) + \alpha f(z) = \int_{-\pi}^{\pi} \log \left(\frac{1}{1 - ze^{-it}} \right)^2 d\mu(t),$$

or equivalently we can write

$$\log f'(z) + \alpha f(z) = 2 \int_0^1 \frac{\omega(tz)}{1 - \omega(tz)} \frac{dt}{t} \quad \text{for some } \omega \in \mathcal{B}_0.$$

Fixing second coefficient

• For each $f \in \mathcal{F}_\alpha$ there exists an $\omega_f \in \mathcal{B}_0$ of the form

$$\omega_f(z) = \frac{P_f(z) - 1}{P_f(z) + 1}, \quad z \in \mathbb{D},$$

and conversely.

• It is a simple exercise to see that

$$P_f'(0) = 2\omega_f'(0) = f''(0) + \alpha.$$

• An application of the Schwarz lemma (see for example [Dinn, Du, Po, PoSi]) shows that

$$|P_f'(0)| = |f''(0) + \alpha| \leq 2.$$

That is

$$f''(0) = 2\lambda - \alpha \qquad \text{for some } \lambda \in \overline{\mathbb{D}}.$$

- For $\omega_f \in \mathcal{B}_0$ and $\lambda \in \overline{\mathbb{D}}$, we define the function $g : \mathbb{D} \to \overline{\mathbb{D}}$ by

$$g(z) = \frac{\frac{\omega_f(z)}{z} - \lambda}{1 - \overline{\lambda}\frac{\omega_f(z)}{z}}.$$

- For $f \in \mathcal{F}_\alpha$ we have

$$|g'(0)| \leq 1 \quad \text{if and only if} \quad f'''(0) = 2[(1-|\lambda|^2)a + (\lambda-\alpha)^2 - \lambda\alpha]$$

for some $|a| \leq 1$.

PROBLEM:

For fixed $z_0 \in \mathbb{D}$ and $\lambda \in \mathbb{D}$ we introduce

$$\begin{aligned} \mathcal{F}_\alpha(\lambda) &= \left\{ f \in \mathcal{F}_\alpha \colon f''(0) = 2\lambda - \alpha \right\} \\ V(z_0, \lambda) &= \{ \log f'(z_0) + \alpha f(z_0) \colon f \in \mathcal{F}_\alpha(\lambda) \}. \end{aligned}$$

AIM:

To determine explicitly the region of variability $V(z_0, \lambda)$ of $\log f'(z_0) + \alpha f(z_0)$ when f ranges over the class $\mathcal{F}_\alpha(\lambda)$.

Key Lemma

- For a positive integer p, let

$$(\mathcal{S}^*)^p = \{f = f_0^p : \, f_0 \in \mathcal{S}^*\}.$$

Lemma [Ya2]

Let f be an analytic function in $\mathbb{D}$ with $f(z) = z^p + \cdots$. If

$$\mathrm{Re}\left(z\frac{f''(z)}{f'(z)}\right) > -1, \quad z \in \mathbb{D},$$

then $f \in (\mathcal{S}^*)^p$.

BASIC PROPERTIES OF $V(z_0, \lambda)$

Proposition 1

We have

1. $V(z_0, \lambda)$ is Compact
2. $V(z_0, \lambda)$ is Convex
3. For $|\lambda| = 1$ or $z_0 = 0$,
$$V(z_0, \lambda) = \{-2 \log(1 - \lambda z_0)\}.$$
4. For $|\lambda| < 1$ and $z_0 \neq 0$, the set $V(z_0, \lambda)$ has $-2 \log(1 - \lambda z_0)$ as an interior point.

• From $(1) - (2)$ and 4 : the boundary $\partial V(z_0, \lambda)$ is a Jordan curve and $V(z_0, \lambda)$ is the union of $\partial V(z_0, \lambda)$ and its inner domain.

EXTREMAL FUNCTIONS:

For $\lambda \in \mathbb{D}$ and $a \in \overline{\mathbb{D}}$, we introduce

- $$\delta(z,\lambda) = \frac{z+\lambda}{1+\overline{\lambda}z}, \quad z \in \mathbb{D}.$$
- $$\log H'_{a,\lambda}(z) + \alpha H_{a,\lambda}(z) = \int_0^z \frac{2\delta(a\zeta,\lambda)}{1-\delta(a\zeta,\lambda)\zeta}\, d\zeta, \quad z \in \mathbb{D}.$$

Main Results

Theorem 1

For $\lambda \in \mathbb{D}$, $\alpha \in \mathbb{C} \setminus \{0\}$ and $z_0 \in \mathbb{D} \setminus \{0\}$, the boundary $\partial V(z_0, \lambda)$ is the Jordan curve given by

$$\begin{aligned}(-\pi, \pi] \ni \theta \quad &\mapsto \quad \log H'_{e^{i\theta},\lambda}(z_0) + \alpha H_{e^{i\theta},\lambda}(z_0) \\ &= \quad \int_0^{z_0} \frac{2\delta(e^{i\theta}\zeta, \lambda)}{1 - \delta(e^{i\theta}\zeta, \lambda)\zeta}\, d\zeta.\end{aligned}$$

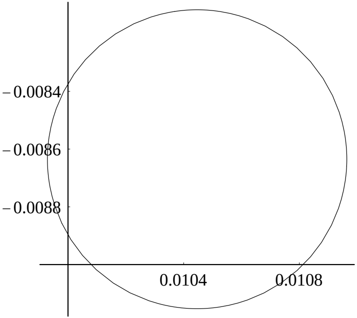
If

$$f(z_0) = H_{e^{i\theta},\lambda}(z_0)$$

for some $f \in \mathcal{F}_\alpha(\lambda)$ and $\theta \in (-\pi, \pi]$, then

$$f = H_{e^{i\theta},\lambda}.$$

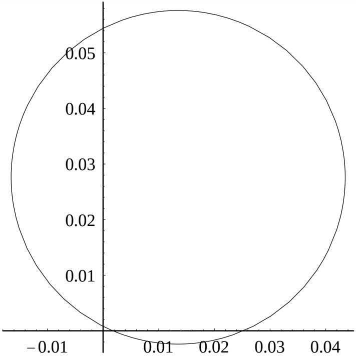
Geometric view of Theorem 1



$$\partial V(z_0, \lambda)$$

$$z_0 = 0.0230875 + 0.00517512i$$
$$\lambda = 0.175557 - 0.225417i$$

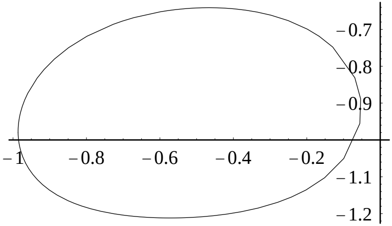
Geometric view of Theorem 1



$\partial V(z_0, \lambda)$

$z_0 = 0.147076 + 0.0913164i$
$\lambda = 0.0748874 + 0.0476965i$

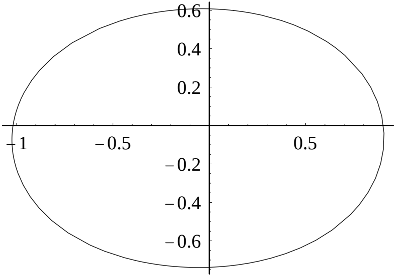
Geometric view of Theorem 1



$$\partial V(z_0, \lambda)$$

$z_0 = -0.819143 - 0.551002i$
$\lambda = 0.722765 + 0.433556i$

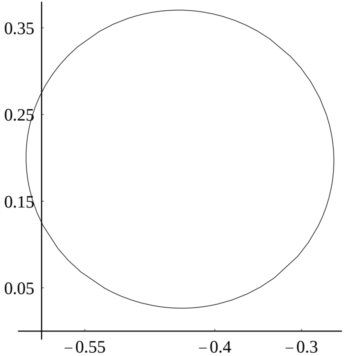
Geometric view of Theorem 1



$$\partial V(z_0, \lambda)$$

$$z_0 = 0.757794 - 0.598957i$$
$$\lambda = -0.308071 - 0.32103i$$

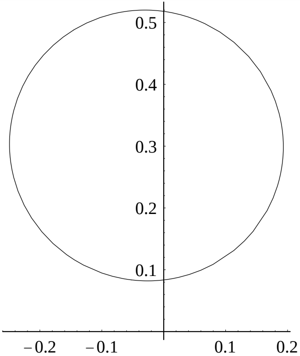
Geometric view of Theorem 1



$$\partial V(z_0, \lambda)$$

$$z_0 = -0.414782 - 0.377338i$$
$$\lambda = 0.196381 - 0.500501i$$

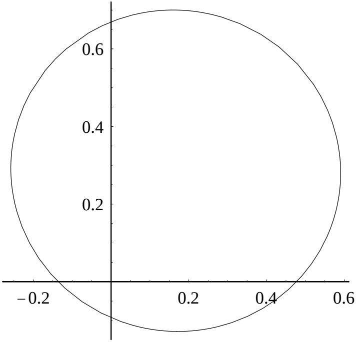
Geometric view of Theorem 1



$$\partial V(z_0, \lambda)$$

$$z_0 = 0.386456 - 0.316514i$$
$$\lambda = -0.236285 + 0.235873i$$

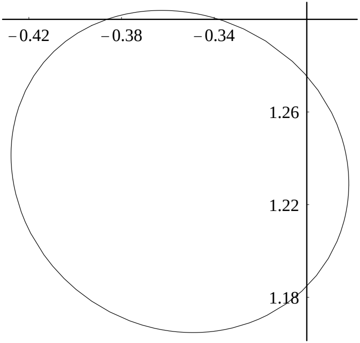
Geometric view of Theorem 1



$$\partial V(z_0, \lambda)$$

$$z_0 = 0.419565 + 0.478471i$$
$$\lambda = 0.242605 + 0.097106i$$

Geometric view of Theorem 1



$$\partial V(z_0, \lambda)$$

$$z_0 = 0.754872 + 0.0830025i$$
$$\lambda = 0.130907 + 0.931628i$$

Preparation for the proof of theorem 1

Proposition 2

For $f \in \mathcal{F}_\alpha(\lambda)$ and $\lambda \in \mathbb{D}$ we have

$$\left|\frac{f''(z)}{f'(z)} + \alpha f'(z) - c(z,\lambda)\right| \leq r(z,\lambda), \quad z \in \mathbb{D},$$

where

$$\begin{aligned} c(z,\lambda) &= \frac{2[\lambda(1-|z|^2) + \overline{z}(|z|^2 - |\lambda|^2)]}{(1-|z|^2)(1+|z|^2 - 2\mathrm{Re}\,(\lambda z))}, \quad \text{and} \\ r(z,\lambda) &= \frac{2(1-|\lambda|^2)|z|}{(1-|z|^2)(1+|z|^2 - 2\mathrm{Re}\,(\lambda z))}. \end{aligned}$$

For each $z \in \mathbb{D} \setminus \{0\}$, equality holds if and only if $f = H_{e^{i\theta},\lambda}$ for some $\theta \in \mathbb{R}$.

Preparation for the proof of theorem 1

Corollary 3

For $f \in \mathcal{F}_\alpha(0)$ we have

$$\left| \frac{f''(z)}{f'(z)} + \alpha f'(z) - \frac{2|z|^2\overline{z}}{1-|z|^4} \right| \leq \frac{2|z|}{1-|z|^4}, \quad z \in \mathbb{D}. \tag{1}$$

For each $z \in \mathbb{D} \setminus \{0\}$, equality holds if and only if $f = H_{e^{i\theta},0}$ for some $\theta \in \mathbb{R}$.

If $f \in \mathcal{F}_\alpha(0)$, then by (1) we obtain

$$(1-|z|^2) \left| \frac{f''(z)}{f'(z)} + \alpha f'(z) \right| \leq 2|z|, \quad z \in \mathbb{D}.$$

Preparation for the proof of theorem 1

Corollary 4

Let $\gamma : z(t)$ $(0 \leq t \leq 1)$ be a C^1-curve in $\mathbb{D}$ with $z(0) = 0$ and $z(1) = z_0$. Then we have

$$V(z_0, \lambda) \subset \overline{\mathbb{D}}(C(\lambda, \gamma), R(\lambda, \gamma)),$$

where

$$\begin{aligned} C(\lambda, \gamma) &= \int_0^1 c(z(t), \lambda) z'(t)\, dt \quad \text{and} \\ R(\lambda, \gamma) &= \int_0^1 r(z(t), \lambda) |z'(t)|\, dt. \end{aligned}$$

Preparation for the proof of theorem 1

Lemma

For $\theta \in \mathbb{R}$ and $\lambda \in \mathbb{D}$ the function

$$G(z) = \int_0^z \frac{e^{i\theta}\zeta}{\{1 + \lambda(e^{i\theta} - 1)\zeta - e^{i\theta}\zeta^2\}^2}\, d\zeta, \quad z \in \mathbb{D},$$

has a double zero at the origin and no zeros elsewhere in $\mathbb{D}$. Furthermore there exists a starlike univalent function G_0 in $\mathbb{D}$ such that

$$G = e^{i\theta} G_0^2$$

and

$$G_0(0) = G_0'(0) - 1 = 0.$$

Preparation for the proof of theorem 1

Proposition 5

Let $z_0 \in \mathbb{D} \setminus \{0\}$. Then for $\theta \in (-\pi, \pi]$ we have

$$\log H'_{e^{i\theta},\lambda}(z_0) + \alpha H_{e^{i\theta},\lambda}(z_0) \in \partial V(z_0, \lambda).$$

Furthermore if

$$\log f'(z_0) + \alpha f(z_0) = \log H'_{e^{i\theta},\lambda}(z_0) + \alpha H_{e^{i\theta},\lambda}(z_0)$$

for some $f \in \mathcal{F}_\alpha(\lambda)$ and $\theta \in (-\pi, \pi]$, then

$$f = H_{e^{i\theta},\lambda}.$$

Outline of the Proof of Theorem 1

- We prove that the closed curve

$$(-\pi, \pi] \ni \theta \mapsto \log H'_{e^{i\theta},\lambda}(z_0) + \alpha H_{e^{i\theta},\lambda}(z_0). \qquad (2)$$

is simple.

- From Proposition 1, $\partial V(z_0, \lambda)$ is a simple closed curve.

- From Proposition 5 the curve $\partial V(z_0, \lambda)$ contains the curve (2)

- Since a simple closed curve cannot contain any simple closed curve other than itself. Thus, $\partial V(z_0, \lambda)$ is given by (2).

Our work on Region of Variability

- S. Ponnusamy and A. Vasudevarao, *Region of variability of two subclasses of univalent functions*, J. Math. Anal. Appl. **332**(2007), 1323–1334.
- S. Ponnusamy, A. Vasudevarao, and H. Yanagihara, *Region of variability of univalent functions $f(z)$ for which $zf'(z)$ is spirallike*, Houston J. Math. **34**(4)(2008), 1037–1048.
- S. Ponnusamy, A. Vasudevarao, and H. Yanagihara, Region of variability for close-to-convex functions, *Complex Var. Elliptic Equ.* **53**(8)(2008), 709 – 716.

- S. Ponnusamy, A. Vasudevarao, and M. Vuorinen, Region of variability for spirallike functions with respect to a boundary point, *Colloq. Math.* **116**(1) (2009), 31–46.
- S. Ponnusamy, A. Vasudevarao, and M. Vuorinen, Region of variability for certain classes of univalent functions satisfying differential inequalities, *Complex Var. Elliptic Equ.* **54**(10)(2009), 899–922.
- S. Ponnusamy, A. Vasudevarao, and H. Yanagihara, Region of variability for close-to-convex functions-II, *Appl. Math. Comput.* **215**(3)(2009), 901–915.

- S. Ponnusamy and A. Vasudevarao, Region of variability for functions with positive real part, (Communicated)
- S. Ponnusamy, A. Vasudevarao, and M. Vuorinen, Region of variability for exponentially univalent convex functions, (Under preparation)

References I

[AMRu] J.H. Arango, D. Mejia, and St. Ruscheweyh, Exponentially convex univalent functions, *Complex Var. Elliptic Equ.* **33**(1)(1997), 33–50.

[Dinn] S. Dineen *The Schwarz lemma.* Oxford Math. Monogr., Clarendon Press, Oxford, 1989.

[Du] P. L. Duren, Univalent Functions (Grundlehren der mathematischen Wissenschaften 259, New York, Berlin, Heidelberg, Tokyo), Springer-Verlag, 1983.

[Du] A.W. Goodman, Univalent Functions, Vols. I and II. Mariner Publishing Co. Tampa, Florida, 1983.

References II

[HMac] D.J. Hallenback and T. H. Mac Gregor, Linear Porblems and Convexity Techniques in Geometric Function Theory, Pitman, London, 1984.

[Paa] V. Paatero, Über die konforme Abbildung von Gebieten deren Ränder von beschränkter Drehung sind, *Ann. Acad. Sci. Fenn. Ser. A***33**(1931), 1–78.

[Pin] B. Pinchuk, Extemal problems in the class of close-to-convex functions, *Trans. Amer. Math. Soc.* **129**(1967), 466–478.

[Po] S. Ponnusamy, Foundations of Complex Analysis, Alpha Science International Publishers, UK, 2005.

[PoSi] S. Ponnusamy and H. Silverman, Complex Variables with Applications, Birkhäuser, Boston, 2006.

References III

[Rusk] H. Ruskeepää, Mathematica Navigator: Mathematics, Statistics, and Graphics, 2nd Ed., Elsevier Academic Press, Burlington, MA, 2004.

[Scho] G. Schober, Univalent Functions-Selected Topics, Lecture Notes in Math., Vol. 478, Springer-Verlag, Berlin, 1975.

[Wall] H.S. Wall, Analytic Theory of Continued Fractions, Chelsea, Bronx, NY, 1973.

[Ya1] H. Yanagihara, Regions of variability for functions of bounded derivatives, *Kodai Math. J.* **28**(2005), 452–462.

[Ya2] H. Yanagihara, Regions of variability for convex function, *Math. Nachr.***279**(2006), 1723–1730.

Multumesc – Kiitos

Appendix Outline

Theorem 1

If $z_0 = 0$ or $|\lambda| = 1$, then $V_{\mathcal{B}}(z_0, \lambda) = \{\lambda z_0\}$. If $z_0 \neq 0$ and $|\lambda| < 1$, then $V_{\mathcal{B}}(z_0, \lambda)$ is the convex closed Jordan domain surrounded by the simply closed curve

$$\partial\mathbb{D} \ni c \mapsto f_c(z_0),$$

where

$$f_c(z) = \int_0^z \frac{c\zeta + \lambda}{1 + \overline{\lambda}c\zeta}\, d\zeta = \frac{z}{\overline{\lambda}} - \frac{1 - |\lambda|^2}{\overline{\lambda}^2 c} \log(1 + \overline{\lambda}cz), \quad z \in \mathbb{D}.$$

If $f(z_0) = f_c(z_0)$ for some $f \in \mathcal{B}(\lambda)$ and $c \in \partial\mathbb{D}$, then $f = f_c$.

Theorem 2

Let $z_0 \in \mathbb{D}$ and $\mathrm{Re}\,\alpha > 0$. If $z_0 = 0$, then $V_{\mathcal{P}}(z_0, \lambda) = \{0\}$. If $z_0 \neq 0$, then $V_{\mathcal{P}}(z_0, \lambda)$ is the convex closed Jordan domain surrounded by the simply closed curve

$$\partial\mathbb{D} \ni \theta \mapsto g_c(z_0),$$

where

$$g_c(z) = \int_0^z \frac{\lambda + \overline{\lambda} c\zeta}{1 - c\zeta}\, d\zeta = -\overline{\lambda} z - \frac{2\mathrm{Re}\,\alpha}{c} \log(1 - cz), \quad z \in \mathbb{D}.$$

If $f(z_0) = g_c(z_0)$ for some $f \in \mathcal{P}(\lambda)$ and $c \in \partial\mathbb{D}$, then $f = g_c$.

Theorem 3

For $0 \le \lambda \le 1$ and $z_0 \in \mathbb{D} \setminus \{0\}$, the boundary $\partial V_{\mathcal{C}}(z_0, \lambda)$ is the Jordan curve given by

$$(-\pi, \pi] \ni \theta \mapsto \log F'_{e^{i\theta},\lambda}(z_0) = \int_0^{z_0} \frac{2\delta(e^{i\theta}\zeta, \lambda)}{1 - \zeta\delta(e^{i\theta}\zeta, \lambda)}\, d\zeta$$

If

$$\log f'(z_0) = \log F'_{e^{i\theta},\lambda}(z_0)$$

for some $f \in \mathcal{C}(\lambda)$ and $\theta \in (-\pi, \pi]$, then

$$f(z) = F_{e^{i\theta},\lambda}(z).$$