# Function Theories in Higher dimensions

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# Outline

1. Complex Plane

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2. Key Ideas in Higher Dimensions

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3. Quaternions

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5. Hypermonogenic functions
6. Integral formulas
7. Riemannian manifolds

# Complex Numbers

- Extend real numbers to two dimensional numbers

$$z = x + yi$$

where $x_0, x_1 \in \mathbb{R}$and $i$ is a new number with the length 1 obtained by rotating 1 counter clockwise by the angle $\frac{\pi}{2}$ .

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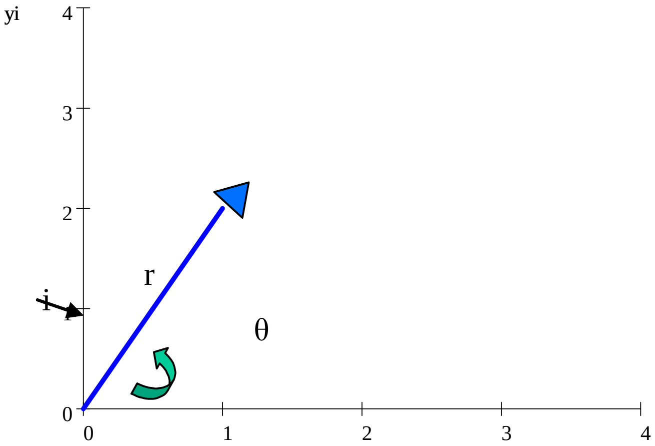
- $z = r\left(\cos\theta + i\sin\theta\right)$


yi
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4

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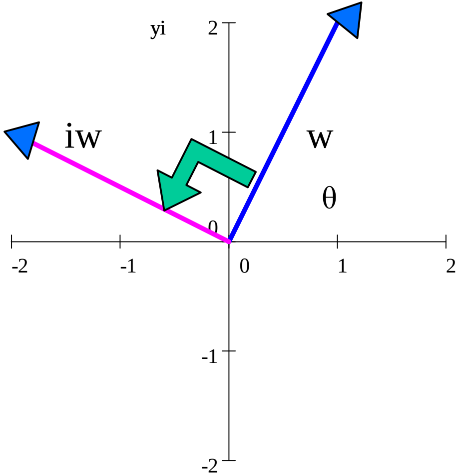


$$i^2 = -1$$
$$i\left(x_0 + y_0 i\right) = -y_0 + x_0 i$$

- generally

$$(x_0 + y_0 i)(x_1 + y_1 i) = x_0 (x_1 + y_1 i) + y_0 i (x_1 + y_1 i)$$

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- if $z = \cos\varphi + i\sin\varphi$ and $w = \cos\theta + i\sin\theta$ are unit complex numbers then $zw$ is obtained by rotating $w$ counter clockwise by the angle $\varphi$.

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- If $f : \Omega \to \mathbb{C}$ is holomorphic, then $f$ can be written locally in the form
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- He introduced quaternions in 1843

# Quaternions

- The set $\mathbb{H}$ of quaternions is be the real associative algebra generated by $e_1$, $e_2$ satisfying

$$\begin{aligned} e_1^2 &= e_2^2 = -1, \\ e_1 e_2 &= -e_2 e_1 \end{aligned}$$

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- Setting

$$e_{12} = e_1 e_2$$

any element in $\mathbb{H}$ may be written as

$$x = x_0 + x_1 e_1 + x_2 e_2 + x_{12} e_{12}$$

for $x_0, x_1, x_2, x_{12} \in \mathbb{R}$.

## Theorem (Hamilton)

*If $T$ is a rotation the space*

$$\mathrm{im}\ \mathbb{H} = \{xi + yj + zk \mid x, y, z \in \mathbb{R}\}$$

*with the angle $\theta$ and the axes $h \in \mathrm{im}\ \mathbb{H}$, $|h| = 1$, then $T(q) = aq\bar{a}$ for any $q \in \mathrm{im}\,\mathbb{H}$ where $a = \cos\frac{\theta}{2} + h \sin\frac{\theta}{2}$.*

# Pairs of complex numbers

- Any quaternion $x$ may be presented as

$$x = z_1 + z_2 e_2$$

where $z_1$ and $z_2$ are complex numbers and $\mathbb{C}$ is identified with the set

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- if $z \in \mathbb{C}$, then

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- if $z_1, \ldots, z_4 \in \mathbb{C}$, then

$$(z_1 + z_2 e_2)(z_3 + z_4 e_2) = z_1 z_3 - z_2 \bar{z}_4 + (z_1 z_4 + z_2 \bar{z}_3) e_2.$$

# Higher Dimensions

- Let $V$ be an $n-$ dimensional vector space with a non-generated symmetric bilinear form $B : V \times V \to \mathbb{R}$. The universal Clifford algebra $Cl(V, B)$ is the free associative algebra with a unit generated by $V$ satisfying the relation

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- It is always possible to find an orthonormal bases basis $e_1, ..., e_{p+q}$ satisfying

$$B\left(e_i, e_i\right) = \begin{cases} -1, & i = 1, ..., p \\ 1, & i = 1, ..., q \end{cases}$$

for some unique $p$ and $q$ with $p + q = n$.(Sylvester)

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- Clifford algebras were introduced by William Clifford in 1878.

The Clifford algebra $Cl_{p,q}$ is formed by the elements presented as

$$x = \sum_A x_A e_A,$$

where $x_A$ are real numbers,

$$A = \{i_1, i_2, \cdots, i_k \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\},$$

and

$$e_A = e_{i_1} e_{i_2} \cdots e_{i_k}, \qquad e_\emptyset = e_0 = 1.$$

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- $C\ell_{3,0} \simeq \mathrm{Mat}(2, \mathbb{C})$, $C\ell_{0,3} = \mathbb{H} \oplus \mathbb{H}$,

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| involutions | $i = 1, ..., n$ | products |
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| main or grade $'$ | $e_i' = -e_i$ | $(ab)' = a'b'$ |
| reversion $^*$ | $e_i^* = e_i$ | $(ab)^* = b^*a^*$ |
| conjugation $-$ | $\overline{e}_i = -e_i$ | $\overline{ab} = \overline{b}\overline{a}$ |
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- In the complex plane the main and the conjugation are the usual complex conjugation.

# Generalizations of Real and Imaginary Parts

- Any element $a \in C\ell_{0,n}$ may be uniquely decomposed as

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for $b, c \in C\ell_{0,n-1}$(the Clifford algebra generated by $e_1, ..., e_{n-1}$).

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  for $b, c \in C\ell_{0,n-1}$(the Clifford algebra generated by $e_1, ..., e_{n-1}$).
- The mappings $P : C\ell_{0,n} \to C\ell_{0,n-1}$ and $Q : C\ell_{0,n} \to C\ell_{0,n-1}$ are defined by

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- When $n = 1$, $P$ is the real part and $Q$ the imaginary part of the complex number.

# Calculation Rules and Notations

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- An element $a$ is a paravector in $Cl_{0,n}$ if and only if
$$\sum_{i=0}^{n} e_i a e_i + (n-1)\,a' = 0.$$

# Which Metric?

## Theorem

*The group of orientation preserving Möbius transformations mapping the upper half space onto itself is the group of isometries of the upper half space model, that is mappings f satisfying*

$$d(f(v), f(w)) = d(v, w).$$

*(that is translations, dilatations, special orthogonal transformations and the inversion with respect to the sphere mapping the upper half onto itself and their compositions)*

# Hyperbolic Harmonic Functions

- We consider harmonic functions on a Riemannian space

$$\mathbb{R}_{+}^{n+1} = \{(x_0, ..., x_n) \mid x_n > 0\}$$

with respect to the hyperbolic metric

$$ds^2 = \frac{dx_0^2 + ... + dx_n^2}{x_n^2}.$$

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- This equation and the metric have been studied for example by A. Weinstein in 1948–1953, A. Huber in 1953, Ahlfors in 1981, Loo-keng Hua in 1969.

# Invariant Laplace

## Theorem

*Let* $\Omega$ *be an open subset of the upper half space* $\mathbb{R}^{n+1}_{+}$. *If* $f : \Omega \to \mathbb{R}$ *is a function satisfying*

$$x_n \triangle f - (n-1)\frac{\partial f}{\partial x_n} = 0$$

*then* $f \circ T$ *satisfies the same equation on* $T^{-1}(\Omega)$ *for any orientation preserving Möbius transformation* $T$ *mapping* $\mathbb{R}^{n+1}_{+}$ *onto itself.*

# Hyperbolic metric

- The hyperbolic surface measure and the volume measure are

$$d\sigma_h = \frac{d\sigma}{x_n^n}, dx_h = \frac{dx}{x_n^{n+1}}.$$

and the hyperbolic normal derivative is

$$\frac{\partial}{\partial n_h} = x_n \frac{\partial}{\partial n}.$$

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- geodesics are circular arcs perpendicular to the hyperplane $x_n = 0$ (half-circles whose origin is on $x_n = 0$ and straight vertical lines ending on the hyperplane $x_n = 0$.

- The hyperbolic ball $B_h(a, R_h)$ with the hyperbolic center $a$ and the radius $R_h$ is an euclidean ball with the euclidean center

$$a_e = a_0 + a_1 e_1 + \ldots + a_{n-1} e_{n-1} + e_n a_n \cosh\, R_h$$

and the euclidean radius

$$R_e = a_n \sinh\, R_h.$$

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and the euclidean radius

$$R_e = a_n \sinh R_h.$$

- the hyperbolic distance $d_h(x, a)$ between the points $x = x_0 + e_1 x_2 + ... + x_n e_n$ and $a = a_0 + e_1 a_1 + ... + a_n e_n$ is

$$\begin{aligned} R_h &= d_h(x, a) = \operatorname{arcosh} \delta(x, a), \\ \delta(x, a) &= \frac{|x - a|^2 + 2a_n x_n}{2x_n a_n}. \end{aligned}$$

We map $P$ and $Q$ to the points $e_n$ and $\lambda e_n$. Then

$$\begin{aligned} d_h(P,Q) &= d_h\left(e_n, \lambda e_n\right) = \int_1^\lambda \frac{dt}{t} = \ln \lambda, \\ \delta(P,Q) &= \delta\left(e_n, \lambda e_n\right) = \frac{1}{2}(\lambda + \frac{1}{\lambda}) \\ \cosh d_h(P,Q) &= \cosh \ln \lambda = \frac{1}{2}(e^{\ln \lambda} + e^{-\ln \lambda}) = \delta(P,Q). \end{aligned}$$

- Hyperbolic function theory was initiated by Heinz Leutwiler around 1990.

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- For any $m \in \mathbb{N}$, the power function

$$f(x) = x^m$$

is paravector-valued for any paravector $x = x_0 + x_1 e_1 + ... + x_n e_n$.

# In Higher Dimensions

- For any $m \in \mathbb{N}$, the power function $x^m$ satisfies the property

$$x^m = \frac{\partial h}{\partial x_0} - \sum_{i=1}^{n} e_i \frac{\partial h}{\partial x_i} \tag{1}$$

where

$$h(x) = \frac{1}{m+1} \operatorname{Re} x^{m+1}.$$

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where

$$h(x) = \frac{1}{m+1} \operatorname{Re} x^{m+1}.$$

- $h$ satisfies the preceding hyperbolic equation

$$x_n \triangle f - (n-1) \frac{\partial f}{\partial x_n} = 0.$$

# H-solutions

- Heinz Leutwiler initiated in 1992 the research of the functions $f$, called $H$-solutions, admitting locally the preceding property for some function $h$ satisfying the equality

- Another characterization of $H$-solutions is that they are paravector-valued solutions $f = u_0 + u_1 e_1 + ... + u_n e_n$ of the following generalized Cauchy-Riemann equation

$$\begin{aligned} & x_n \left( \frac{\partial u_0}{\partial x_0} - \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} \right) + (n-1)\, u_n = 0, \\ & \frac{\partial u_i}{\partial x_k} = \frac{\partial u_k}{\partial x_i}, \qquad i, k = 1, ..., n, \\ & \frac{\partial u_0}{\partial x_k} = -\frac{\partial u_k}{\partial x_0}, \qquad k = 1, ..., n. \end{aligned} \tag{H}$$

# General assumptions

- Let $\Omega$ be an open subset of $\mathbb{R}^{n+1}$.

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- Let $\Omega$ be an open subset of $\mathbb{R}^{n+1}$.
- We consider function mapping

$$f : \Omega \longrightarrow C\ell_{0,n},$$

whose components are continuously differentiable.

# Dirac operators

- The left Dirac operators in $C\ell_{0,n}$ is defined by

$$D_l f = \sum_{i=0}^{n} e_i \frac{\partial f}{\partial x_i}, \qquad \overline{D_l} f = \sum_{i=0}^{n} \overline{e_i} \frac{\partial f}{\partial x_i},$$

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- The right Dirac operators by

$$D_r f = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} e_i, \qquad \overline{D_r} f = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} \overline{e_i}.$$

## Theorem (Cauchy formula)

*Let $\Omega \in \mathbb{R}^{n+1}$ be an open, $K \subset \Omega$ a compact set with smooth boundary $\delta K$ and $\nu$ its outer unit normal field. Then for any monogenic function $f : \Omega \to Cl_{0,n}$, that is $D_l f = 0$, we have*

$$f(q) = \frac{1}{\omega_{n+1}} \int_{\delta K} \frac{(p-q)^{-1}}{|p-q|^{n-1}} \nu(p) f(p)\, dS,$$

*for all $q \in \overset{\circ}{K}$, where $\omega_{n+1}$is the surface measure of the unit ball in $\mathbb{R}^{n+1}$.*

# Modified Dirac Operators

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- Let $\Omega$ be an open subset of $\mathbb{R}^{n+1} \setminus \{x_n = 0\}$.
- The modified Dirac operators $M_k^l$, $M_k^r$, $\overline{M}_k^l$ and $\overline{M}_k^r$ are introduced by

$$M_k^l f(x) = D_l f(x) + k\frac{Q'f}{x_n}, \quad M_k^r f(x) = D_r f(x) + k\frac{Qf}{x_n}$$
$$\overline{M}_k^l f(x) = \overline{D}_l f(x) - k\frac{Q'f}{x_n}, \quad \overline{M_k}^r f(x) = \overline{D}_r f(x) - k\frac{Qf}{x_n}.$$

where $f \in \mathcal{C}^1(\Omega, C\ell_{0,n})$.

# Hypermonogenic functions

## Definition

Let $\Omega \subset \mathbb{R}^{n+1}$ be open. A function $f : \Omega \to C\ell_{0,n}$ is *left hypermonogenic function*, if $f \in \mathcal{C}^1(\Omega)$ and

$$M_k^l f(x) = 0$$

for any $x \in \Omega \setminus \{x_n = 0\}$. The *right k-hypermonogenic functions* are defined similarly. If $k = n - 1$ left hypermonogenic functions are called *hypermonogenic functions.*

- Hypermonogenic functions were introduced by H. Leutwiler and S.-L. E in 2000.

# Basic properties

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- If $f : \Omega \rightarrow C\ell_{0,n}$ is $k-$hypermonogenic, then $\widehat{f}\left(\widehat{x}\right)$ is $k$-hypermonogenic in $\widehat{\Omega} = \left\{x \in \mathbb{R}^{n+1} \mid \widehat{x} \in \Omega\right\}$.

# Notably studied by

The theory is notably studied by

| | | |
|---|---|---|
| H. Leutwiler | S.-L. Eriksson | S. Krausshar |
| J. Cnops | Kettunen, J. | E. Lehman |
| P. Cerejeiras | Hirvonen, J. | I. Ramadanoff |
| Th. Hempfling | H. Orelma | Bernstein, Sw. |
| P. Zeilinger | Ryan, J. | Laville, G. |
| Pernas, L. | Qiao, Yuying | Xiaoli Bian |
| Junxia Li | | |

# Generalized Cauchy-Riemann equations

## Theorem

*Let $\Omega$ be an open subset of $\mathbb{R}^{n+1}$ and $f : \Omega \longrightarrow C\ell_{0,n}$ be a mapping with continuous partial derivatives. The equation $Df + k\frac{Q'f}{x_n} = 0$ is equivalent with the following system of equations*

$$D_{n-1}(Pf) - \frac{\partial (Q'f)}{\partial x_n} + k\frac{Q'f}{x_n} = 0,$$
$$D_{n-1}(Qf) + \frac{\partial P'(f)}{\partial x_n} = 0.$$

# Hyperbolic harmonic functions

## Theorem

*Let $f : \Omega \to C\ell_{0,n}$ be twice continuously differentiable. Then*

$$P\left(M_k\overline{M}_k f\right) = \triangle Pf - \frac{k}{x_n}\frac{\partial Pf}{\partial x_n}$$

$$Q\left(M_k\overline{M}_k f\right) = \triangle Qf - \frac{k}{x_n}\frac{\partial Qf}{\partial x_n} + k\frac{Qf}{x_n^2}$$

*If $f$ is k-hypermonogenic, then*

$$\begin{aligned} \triangle Pf - \frac{k}{x_n}\frac{\partial Pf}{\partial x_n} &= 0 \\ \triangle Qf - \frac{k}{x_n}\frac{\partial Qf}{\partial x_n} + k\frac{Qf}{x_n^2} &= 0. \end{aligned}$$

# Laplace Beltrami equations

These are Laplace-Beltrami equations with respect to the Riemannian metric

$$ds^2 = \frac{\sum_{i=0}^{n} dx_i^2}{x_n^{\frac{2k}{n-1}}}.$$

# k-hyperbolic harmonic functions

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- Let $f : \Omega \to Cl_{0,n}$ be twice continuously differentiable. Then $f$ is $k$-hypermonogenic if and only if $f$ and $xf$ are $k$-hyperbolic harmonic functions.

# k-hyperbolic harmonic functions

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- Let $f : \Omega \to Cl_{0,n}$ be twice continuously differentiable. Then $f$ is $k$-hypermonogenic if and only if $f$ and $xf$ are $k$-hyperbolic harmonic functions.
- Let $\Omega$ be an open subset of $\mathbb{R}^{n+1}$and $f : \Omega \to C\ell_{0,n}$ be twice continuously differentiable. Then $f$ is $k$-hypermonogenic if and only if there exists locally a $k$-hyperbolic harmonic mapping $H$ with values in $C\ell_{0,n-1}$ satisfying $\overline{D}H = f$.

## Theorem

*Let $U \subset \mathbb{R}_{+}^{n+1}$ be open. The following properties are equivalent:*

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1. *$h$ is hyperbolic harmonic on $U$.*
2. *$h \in \mathcal{C}^2(U)$ and*

$$h(a) = \frac{1}{\sigma_n \sinh^n R_h.} \int_{\partial B_h(a,R_h)} h d\sigma_h$$

*for all hyperbolic balls satisfying $\overline{B_h(a, R_h)} \subset U$ .*

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3. $h \in \mathcal{C}^2(U)$ *and*

$$h(a) = \frac{1}{V_h(B_h(a, R_h))} \int_{B_h(a,R_h)} f dx_h,$$

*for all hyperbolic balls satisfying* $\overline{B_h(a, R_h)} \subset U$, *where* $V_h(B_h(a, R_h)) = \sigma \int_0^{R_h} \sinh^n t dt$.

# Examples

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- $\cos x = \sum \frac{1}{(2k)!} (-1)^k x^{2k}$
- If $f(z) = \sum a_k z^k$ is holomorphic and $a_k \in \mathbb{R}$, then $f(x) = \sum a_k x^k$ is hypermonogenic.

# Fueter Construction

If $f = u + iv$ is holomorphic in an open set $\Omega \subset \mathbf{C}$, then

$$\widetilde{f}(x) = u\left(x_0, \sqrt{x_1^2 + ... + x_n^2}\right) + \frac{x_1 e_1 + .. + x_n e_n}{\sqrt{x_1^2 + .. + x_n^2}} v\left(x_0, \sqrt{x_1^2 + ... + x_n^2}\right)$$

is hypermonogenic.

- The $k$-hypermonogenic functions in an open subset $\Omega$ of $\mathbb{R}^{n+1}$ form a right $C\ell_{0,n-1}$-module.

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- If $f$ is a $k$-hypermonogenic function, then the function $fe_n$ is hypermonogenic if and only if $f = 0$.
- A function $f : \Omega \to C\ell_{0,n}$ is $k$-hypermonogenic if and only if the function $\frac{fe_n}{x_n^k}$ is $-k$-hypermonogenic.

- The $k$-hypermonogenic functions in an open subset $\Omega$ of $\mathbb{R}^{n+1}$ form a right $C\ell_{0,n-1}$-module.
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- A function $f : \Omega \to C\ell_{0,n}$ is $k$-hypermonogenic if and only if the function $\frac{fe_n}{x_n^k}$ is $-k$-hypermonogenic.
- A function $f : \Omega \to C\ell_{0,n}$ is left $k$-hypermonogenic if and only if the function $fe_n$ satisfies the equation

$$D_l g - \frac{ke_n Pg}{x_n} = 0.$$

# Stokes theorem

## Theorem

*For $\Omega$ an open subset of $\mathbb{R}^{n+1}_{+}$ (or $\mathbb{R}^{n+1}_{-}$), $K \subset \Omega$ a smoothly bounded compact set with outer unit normal field $\nu$, and $f, g \in \mathcal{C}^1(\Omega, C\ell_{0,n})$,*

$$\begin{aligned}
\int_{\partial K} P(g\nu f)\frac{d\sigma}{x_n^k} &= \int_K P\left(\left(M_k^r g\right) f + gM_k^l f\right)\frac{dx}{x_n^k} \\
\int_{\partial K} Q(g\nu f)\, d\sigma &= \int_K Q\left(\left(M_{-k}^r g\right) f + gM_k^l f\right) dx.
\end{aligned}$$

## Theorem

*Let* $\Omega$ *be an open subset of* $\mathbb{R}^{n+1}_{+}$ *and* $\overline{K} \subset \Omega$ *a smoothly bounded compact set with outer unit normal field* $\nu$. *If* $f$ *is hypermonogenic in* $\Omega$ *and* $y \in K$, *then*

$$f(y) = \frac{2^{n-1}}{\omega_{n+1}} \int_{\partial K} \left( K_1(x,y)\,\nu(x)\,f(x) - K_2(x,y)\,\widehat{\nu(x)}\widehat{f(x)} \right) d\sigma.$$

*where*

$$K_1(x,y) = \frac{y_n^{n-1}(x-y)^{-1}}{|x-y|^{n-1}\,|x-\widehat{y}|^{n-1}}$$

$$K_2(x,y) = \frac{y_n^{n-1}(\widehat{x}-y)^{-1}}{|x-y|^{n-1}\,|x-\widehat{y}|^{n-1}}.$$

## Theorem

*Let $\Omega$ be an open subset of $\mathbb{R}_+^{n+1}$ and $\overline{K} \subset \Omega$ a smoothly bounded compact set with outer unit normal field $\nu$. If $f$ is hypermonogenic in $\Omega$ and $y \in K$*

$$f(x) = \frac{2^{n-1}}{\omega_{n+1}} \int_{\partial K} k(x,y) \left( Q\left( y\nu'(y) f'(y) \right) + xQ'\left( \nu(y) f(y) \right) \right)$$

*where*

$$\begin{aligned} k(x,y) &= \frac{1}{2^{2n-2} y_n^n} \overline{D}^x \left( \int_{\frac{|y-x|}{|x-\widehat{y}|}}^{1} \frac{\left(1-s^2\right)^{n-1}}{s^n} ds \right) \\ &= -\frac{x_n^{n-1}}{y_n} \left( \frac{(x-y)^{-1} - (x-\widehat{y})^{-1}}{|x-y|^{n-1} |x-\widehat{y}|^{n-1}} \right). \end{aligned}$$

*are hypermonogenic with respect to $x$ in $\mathbb{R}^{n+1} \setminus \{y, \widehat{y}\}$.*

## Theorem

*Let* $\Omega$ *be an open subset of* $\mathbb{R}^{n+1}_{+}$ *and* $\overline{K} \subset \Omega$ *a smoothly bounded compact set with outer unit normal field* $\nu$*. If* $f$ *is continuous on* $\Omega$ *then*

$$g(x) = \frac{2^{n-1}}{\omega_{n+1}} \int_{\partial K} k(x,y) \left( Q\left(y\nu'(y) f'(y)\right) + xQ'\left(\nu(y) f(y)\right)\right) d\sigma$$

*is hypermonogenic in* $\mathbb{R}^{n+1}_{+} \backslash \partial K$*.*

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