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Hyperbolic-type metrics

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Abstract. The article is a status report on the contemporary research of hyperbolic-type metrics, and considers progress in the study of the classes of isometry- and bilipschitz mappings with respect to some of the presented metrics. Also, the Gromov hyperbolicity question is discussed.

Keywords. Hyperbolic-type metric, intrinsic metric, isometry problem, bilipschitz-mapping, Gromov hyperbolic space.

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1. Introduction

In geometric function theory there are many different distance functions around, which — to a greater or lesser degree — resemble the classical hyperbolic metric. Some of these are defined by geometric means, some by implicit formulas, and many by integrating over certain weight functions.

What all these metrics have in common, is that they are defined in some proper subdomain $D \subsetneq \mathbb{R}^n$, and are strongly affected by the geometry of the domain boundary. Thus we should actually speak of families of metrics $\{d_D\}_{D\subsetneq\mathbb{R}^n}$, since the metric looks different in each domain, even though the defining formula might be the same. In the literature, however, one usually abuses notation and speaks only of "the metric d", which we will do here also. The metrics typically have negative curvature, ie. the geodesics, if they exist, avoid the boundary. Most of the metrics described here also have an invariance property in the sense that

$$d_D(x, y) = d_{f(D)}(f(x), f(y)), \qquad (1.1)$$

for mappings f belonging to some fixed class, say similarities, Möbius transformations, or conformal mappings.

Many of the metrics, especially those with simple explicit formulas, have been developed as tools for estimating other, more hard-to-handle metrics, such as the quasihyperbolic metric, which is probably the one most commonly used metric presented in this text. It has found applications in many branches of analysis, and is a very natural generalization of the classical hyperbolic metric to any domain D and dimension $n \geq 2$. It has some flaws though, in most cases one cannot compute it, and actually very little is known about the metric itself. The difficulty of explicit computation is typical also for some other metrics, and for this reason we have a lot of "similar" metrics around, which in many cases are equivalent to each other; a handy feature, if one metric is suited for your study, but the other is not. Here we will try to give a survey on some of these metrics.

2. The metrics

The classical starting point is the hyperbolic geometry developed by Poincaré and Lobachevsky in the early 19:th century. Poincaré used the unit ball as domain for his model, and Lobachevsky used the half space. These models turned out to be equivalent in the sense that Möbius transformations between them are isometries.

2.1. Definition. Let $D \in \{\mathbf{H}^n, \mathbf{B}^n\}$, and define a *weight* (or *density*) function $w \colon D \to \mathbb{R}$ by

$$w(z) = \frac{1}{\operatorname{dist}(z, \partial D)}, \quad \text{for } D = \mathbf{H}^n \quad \text{and} \quad w(z) = \frac{2}{1 - |z|^2}, \quad \text{for } D = \mathbf{B}^n.$$

Then the *hyperbolic length* $\ell_\rho(\gamma)$ of a curve γ is defined by

$$\ell_\rho(\gamma) = \ell_{\rho,D}(\gamma) = \int_\gamma w(z)\, |dz|, \tag{2.2}$$

where $|dz|$ denotes the length element. After this, the *hyperbolic distance* ρ_D is defined for all $x, y \in D$ by

$$\rho_D(x, y) = \inf_{\gamma \in \Gamma_{xy}} \ell_{\rho,D}(\gamma) = \inf_{\gamma \in \Gamma_{xy}} \int_\gamma w(z)\, |dz|, \tag{2.3}$$

where Γ_{xy} is the family of all rectifiable curves joining x and y within D.

The above method to define metrics is frequently used. In fact, to get a completely new metric, the only thing that needs to be changed is the weight function. After that, the length and the new distance function are defined as in (2.2) and (2.3), respectively. The benefit of defining a metric d like this is that it will automatically be *intrinsic*, in other words, it will be its own *inner metric* $\hat{d}$. This means that

$$d_D(x, y) = \hat{d}_D(x, y) := \inf_{\gamma \in \Gamma_{xy}} \ell_{d,D}(\gamma). \tag{2.4}$$

2.5. Geodesics. When a metric is defined in the way described above, one might ask how to find the curve $\gamma \in \Gamma_{xy}$ giving the desired infimum (which — if it is found — is in fact a minimum). In general, this can be far from trivial, even if such a curve exists. Curves minimizing the distance in this way are called *geodesics* or *geodesic segments*. Another way of characterizing a geodesic, is that it satisfies the triangle inequality with equality, ie. the curve $\gamma \in \Gamma_{xy}$ is a geodesic, if for all $u, v, w \in |\gamma|$ properly ordered, we have

$$d_D(u,w) = d_D(u,v) + d_D(v,w).$$

We denote by $J_{d_D}[x,y]$ the geodesic segment between x and y in (D,d). This segment may, however, not be unique, and no particular choice is made here. A metric space in which geodesic segments exist between any two given points, is called a *geodesic metric space*. If, in addition, the geodesic is unique, the space is *totally geodesic*. Naturally a geodesic metric is always intrinsic.

2.6. Hyperbolic metric in G. It is also possible to define the hyperbolic metric in a general simply connected subdomain G of the plane, since by the Riemann mapping theorem there exists a conformal mapping $f\colon G \to fG = \mathbf{B}^2$. Then the metric density is defined by

$$\rho_G(z) = \rho_{\mathbf{B}^2}(f(z))|f'(z)|.$$

From the Schwarz lemma it follows that ρ_G is independent of the choice of f. We then define the hyperbolic metric h_G by (2.3) using the density ρ_G. This definition automatically gives the hyperbolic metric the invariance property of (1.1) for the class of conformal mappings. Note, that while in the classical cases we use the traditional notation $\rho_{\mathbf{B}^n}$ and $\rho_{\mathbf{H}^n}$ for the hyperbolic metric, in general domains we use h_G. Also, note that when the dimension $n \geq 3$, every conformal mapping is a Möbius mapping, so it is not possible to extend the definition to general simply connected domains like above. In fact, for $n \geq 3$ the hyperbolic metric is defined only in $\mathbf{B}^n$ and $\mathbf{H}^n$.

The hyperbolic metric is well understood, and the geodesic flow is known. In fact, in the classical models $\mathbf{B}^n$ and $\mathbf{H}^n$ the geodesics are known to be circular arcs orthogonal to the boundary, and in other domains the geodesics simply are induced by the conformal mapping. Moreover, for the classical cases there are explicit formulas to calculate the value of the hyperbolic metrics in terms of euclidean distances. For a comprehensive study on the classical cases, see the book by Beardon [Be1]. The hyperbolic metric in an arbitrary domain has been studied by F. Gehring, K. Hag and A. Beardon, see eg. the articles [Be3] and [GeHa1].

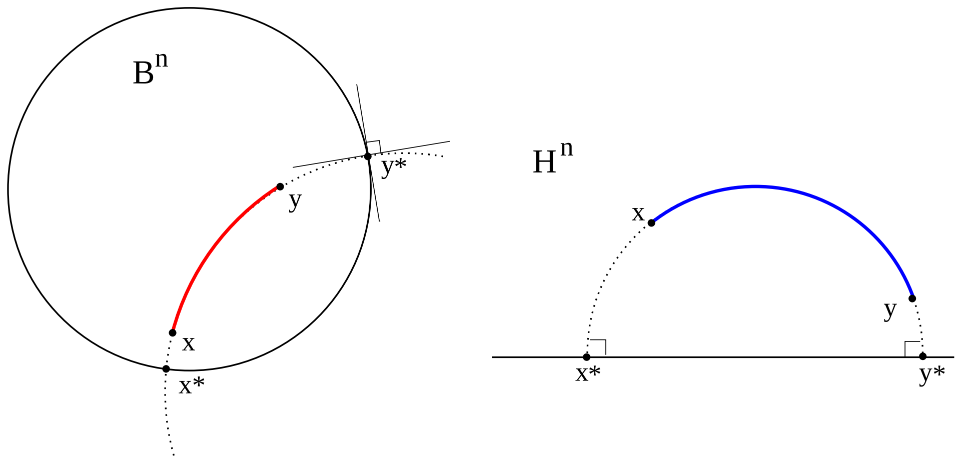

Figure 1: Hyperbolic geodesics in $\mathbf{B}^n$ *and* $\mathbf{H}^n$*.*

One way to calculate the hyperbolic distance, is to use the *absolute cross-ratio* defined by

$$|a, b, c, d| = \frac{|a-c||b-d|}{|a-b||c-d|}, \quad a, b, c, d \in \mathbb{R}^n.$$

One can prove that, if C is the circle containing $J_{\rho_{\mathbf{B}^n}}[x, y]$ or $J_{\rho_{\mathbf{H}^n}}[x, y]$ and $\{x^*, y^*\} = C \cap \partial\mathbf{B}^n$ or $\{x^*, y^*\} = C \cap \partial\mathbf{H}^n$ in the same order as in Figure 1, then

$$\rho_{\mathbf{B}^n}(x, y) = \log |x^*, x, y, y^*| = \rho_{\mathbf{H}^n}(x, y). \tag{2.7}$$

Other explicit formulas have also been derived, see the book [Be1].

2.8. The Apollonian metric. The formula in (2.7) makes one wonder whether a similar approach could be generalized to any domain $D \subsetneq \mathbb{R}^n$. It turns out that this is very much possible; the *Apollonian distance* in a domain D is defined by

$$\alpha_D(x, y) = \sup_{z,w \in \partial D} \log \frac{|z-x|}{|z-y|} \frac{|w-y|}{|w-x|}, \tag{2.9}$$

for all $x, y \in D$. This is a metric, unless the boundary is the subset of a circle or a line, in which case it is only a *pseudo-metric*, ie. the metric axiom $d(x, y) = 0 \Rightarrow x = y$ need not hold.

Geometrically the Apollonian metric can be thought of in the following way: an *Apollonian circle* (or sphere, when $n \geq 3$) with respect to the pair (x, y), is a set

$$B_{x,y,q} = \left\{ z \in \mathbb{R}^n \;\middle|\; \frac{|z-x|}{|z-y|} = q \right\}.$$

Then the Apollonian metric is

$$\alpha_D(x, y) = \log q_x q_y,$$

where q_x and q_y are the ratios of the largest possible balls B_{x,y,q_x} and B_{y,x,q_y} still contained in D.

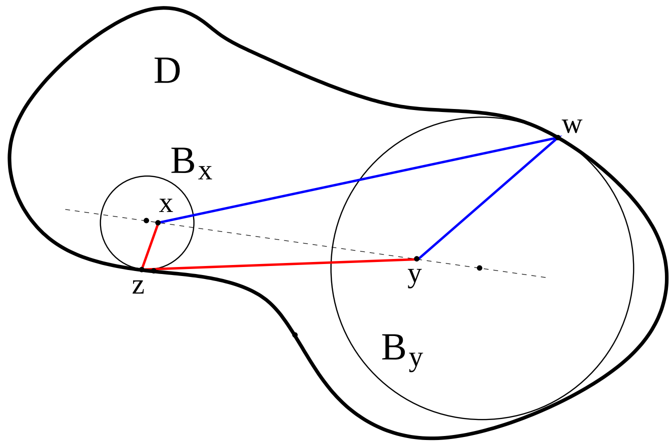

Figure 2: The Apollonian balls approach.

The Apollonian metric is invariant in Möbius mappings in the sense of (1.1). It is an easy exercise in geometry to show that in the case $D = \mathbf{H}^n$ the points z and w are actually the points x^* and y^* in (2.7), and thus $\rho_{\mathbf{H}^n} = \alpha_{\mathbf{H}^n}$.

The Apollonian metric has been studied in [GeHa2] and [Se], but especially by P. Hästö and Z. Ibragimov in a series of articles, see e.g. [Hä1],[Hä2],[HäIb] and [Ib].

The Apollonian metric is in a way a convenient construction with a clear geometric interpretation, but as a shortcoming it has its lack of geodesics. In the article [HäLi] some work is done to overcome this problem, by introducing the *half-Apollonian metric*, defined by

$$\eta_D(x,y) = \sup_{z \in \partial D} \left| \log \frac{|x-z|}{|y-z|} \right|, \tag{2.10}$$

for all $x, y \in D$. The geometric intuition here is the same as for the Apollonian metric, Indeed, instead of $\log q_x q_y$ we have

$$\eta_D(x,y) = \log \max\{q_x, q_y\}.$$

This metric is a only similarity invariant, but instead it has more geodesics than the Apollonian metric. It is also bilipschitz equivalent to the Apollonian metric, in fact

$$\frac{1}{2}\, \alpha_D(x,y) \le \eta_D(x,y) \le \alpha_D(x,y).$$

2.11. The quasihyperbolic metric. The *quasihyperbolic metric* is perhaps the most well-known and frequently used of the metrics considered here. It was developed by F. Gehring and his collaborators in the 70's. It is defined by the method of 2.1 using

$$w(z) = \frac{1}{\mathrm{dist}(z, \partial D)}, \quad z \in D$$

as weight function. It is immediate that for $D = \mathbf{H}^n$ the quasihyperbolic metric coincides with the hyperbolic metric $\rho_{\mathbf{H}^n}$. The quasihyperbolic metric is invariant under the class of similarity mappings.

The quasihyperbolic metric is well-behaved in many senses: the weight function is quite simple and it is a natural generalization of the hyperbolic metric. Also, it is known to be geodesic for any domain $D \subsetneq \mathbb{R}^n$ [GeOs]. One of the shortcomings of the metric is that in general the geodesics are not easy to determine. Besides the half-space $\mathbf{H}^n$, the geodesics are known in the punctured space $\mathbb{R}^n \setminus \{z\}$ and in the ball $\mathbf{B}^n$, see [MaOs]. Recently the geodesics were determined also for the punctured ball $\mathbf{B}^n \setminus \{0\}$, and planar angular domains

$$S_\varphi = \{(r,\theta) \mid 0 < \theta < \varphi\}, \;\; 0 < \varphi < 2\pi,$$

see [Li1].

2.12. Distance-ratio metrics. As the quasihyperbolic metric cannot be explicitly evaluated in the case of general domains, a typical way to overcome this problem is to approximate it by another metric, often one of the *distance-ratio metrics* or *j-metrics.* (Actually, by their construction also the Apollonian and half-Apollonian metrics could be described as "distance-ratio metrics"). There are two versions of these. The first, introduced by F. Gehring, is defined by

$$\widetilde{j}_D(x,y) = \log\left(1 + \frac{|x-y|}{\operatorname{dist}(x,\partial D)}\right)\left(1 + \frac{|x-y|}{\operatorname{dist}(y,\partial D)}\right), \;\; x,y,\in D. \tag{2.13}$$

The other one is defined by

$$j_D(x,y) = \log\left(1 + \frac{|x-y|}{\operatorname{dist}(x,\partial D) \wedge \operatorname{dist}(y,\partial D)}\right). \;\; x,y,\in D. \tag{2.14}$$

is a modification due to M. Vuorinen.

The two metrics have much in common, but also important differences, which will be discussed further in Sections 2 and 3. Both are similarity invariant, and can be used to estimate the quasihyperbolic metric. The metrics satisfy the relation

$$j_D(x,y) \le \widetilde{j}_D(x,y) \le 2\, j_D(x,y), \;\; x,y \in D.$$

The lower bound for the quasihyperbolic metric is given by the inequality

$$j_D(x,y) \le k_D(x,y)$$

proved in [GePa], which holds for points x, y in any proper subdomain D. The upper bound holds for so called *uniform domains*, which is a wide class of domains introduced in [MaSa].

2.15. Definition. A domain $D \subsetneq \mathbb{R}^n$ is called *uniform* or *A-uniform*, if there exists a number $A \ge 1$ such that the inequality

$$k_D(x,y) \le A\, j_D(x,y)$$

holds for all $x, y \in D$.

There are many definitions for uniform domains around, see eg. [Ge], so often many "nice" domains can be shown to be uniform by other means, and so one has access to the inequality in 2.15. However, typically very little can be said about the constant A. These matters have been studied in [Li1].

The j-metric defined in (2.14) has another important connection to the quasihyperbolic metric. The quasihyperbolic metric is namely the inner metric of the j-metric, in the sense of (2.4). In other words

$$k_D(x,y) = \inf_{\gamma\in\Gamma_{xy}} \ell_{j,D}(\gamma).$$

Since the j-metric fails to be intrinsic, it cannot be geodesic either. In fact, the j-metric has geodesics only in some special cases, see [HäIbLi, 3.7]. Very little is known about the geodesic segments of the $\widetilde{j}$-metric, although it can be conjectured that there is not much of them either.

3. Isometries and bilipschitz-mappings

As pointed out earlier, most of the hyperbolic-type metrics defined in this article satisfy some kind of invariance property, that is, they satisfy the equality (1.1) for some class of mappings f. Typically this invariance property follows almost directly from the definition of the metric, for instance, it is easy to see from the formulas (2.9) and (2.10) that the Apollonian metric is Möbius-invariant and the half-Apollonian metric is similarity invariant. The interesting question mostly regards the other implication. Is the class of "natural candidates" the only mappings which give isometries in the metric in question? And what are the "near-isometries", that is, the bilipschitz mappings? There are still many open ends regarding these questions, though some progress has been made recently.

3.1. Definition. Let D and $D' = f(D)$ be domains such that equipped with distances d_D and $d_{D'}$ they are metric spaces. Then a continuous mapping $f\colon D \to D'$ is said to be *L-bilipschitz in (or with respect to) the metric d* if for all $x,y\in D$ we have

$$\frac{1}{L}\, d_D(x,y) \le d_{D'}(f(x),f(y)) \le L\, d_D(x,y)$$

for some $L\ge 1$. If the above inequality holds with $L=1$, f is a *d-isometry*.

3.2. "One-point" and "two-point" metrics. In general, the hyperbolic-type metrics can be divided into length-metrics, defined by means of integrating a weight function, and point-distance metrics. The point-distance metrics may again be classified by the number of boundary points used in their definition. So for instance the j-metric and the half-Apollonian metric would be 'one-point metrics", whereas the $\widetilde{j}$, and the Apollonian metrics are "two-point metrics".

Actually also the length metrics can be characterized in the same way, by looking at their weight function. Then the quasihyperbolic metric is a one-point metric. An example of a two-point length metric is the so called *Ferrand metric* σ_D, see [Fe1]. It is defined for a domain $D \subsetneq \overline{\mathbb{R}}^n$ with $\operatorname{card}\partial D\ge 2$, using the weight function

$$w_D(x) = \sup_{a,b\in\partial D} \frac{|a-b|}{|x-a||x-b|},\quad x\in D\setminus\{\infty\}.$$

This metric is Möbius invariant and coincides with the hyperbolic metric on $\mathbf{H}^n$ and $\mathbf{B}^n$. Moreover, it is bilipschitz equivalent to the quasihyperbolic metric by the inequality

$$k_D(x,y) \le \sigma_D(x,y) \le 2\ k_D(x,y), \quad x,y \in D. \tag{3.3}$$

Naturally one would expect the one-point point-distance metrics to be the easiest ones to study. In fact, much can be said about these metrics when it comes to the isometry question. The half-Apollonian metric has recently been studied in [HäLi]. A point $x \in D$ is called *circularly accessible* if there exists a ball $B \subset G$ such that $x \in \partial B$. If x is circularly accessible by two distinct balls whose surfaces intersect at more than one point, it is called a *corner point*, otherwise a *regular point*.

3.4. Theorem. *Let $D \subsetneq \mathbb{R}^n$ be a domain which has at least n regular boundary points which span a hyperplane. Then $f\colon D \to \mathbb{R}^n$ is a homeomorphic η-isometry if and only if it is a similarity mapping.* □

Furthermore, it was shown that Möbius mappings are in fact 2-bilipschitz with respect to η_D.

For the j-metric, some results can be found in [HäIbLi], and in fact in a slightly more general setting. The implications for the j-metric can be expressed as follows;

3.5. Corollary. *Let $D \subsetneq \mathbb{R}^n$. Then $f\colon D \to \mathbb{R}^n$ is a j-isometry if and only if*
(1) *f is a similarity, or*
(2) *$D = \mathbb{R}^n \setminus \{a\}$ and, up to similarity, f is the inversion in a sphere centered at a.* □

Since $\hat{j}_D = k_D$, it immediately follows that every isometry of the j-metric is an isometry of the quasihyperbolic metric, of course in this case that does not provide us with very much new information. However, a similar relation is true for the *Seittenranta metric* δ_D defined in [Se] by

$$\delta_D(x,y) = \log\left(1 + \sup_{a,b\in\partial D} \frac{|x-y||a-b|}{|a-x||b-y|}\right), \quad x,y \in D,$$

which is also studied in [HäIbLi]. Namely, here we have that $\hat{\delta}_D = \sigma_D$, so we directly see that this is a Möbius invariant metric. In [Se] it is proved that at least Euclidean bilipschitz mappings are bilipschitz with respect to δ. The converse is not true, as can be shown by the counterexample

$$f\colon \mathbf{B}^2 \setminus \{0\} \to \mathbf{B}^2 \setminus \{0\}, \quad f(x) = |x| \cdot x.$$

However, in [Se] it was shown that every bilipschitz δ-mapping is a quasiconformal mapping, and that every δ-isometry is conformal with respect to the Euclidean metric (and thus Möbius for $n \ge 3$). In [HäIbLi] it was shown that also for $n = 2$ in fact the δ-isometries are exactly the Möbius mappings.

For the j-metric there are still many open problems regarding the bilipschitz question. It is well known (see [Vu]), that an Euclidean L-bilipschitz mapping is L^2-bilipschitz with respect to the j (and k) metric.

For the Apollonian metric the isometry and bilipschitz questions have been studied by several authors. The work was started by Beardon in [Be2], and continued by Gehring and Hag in [GeHa2] where they studied Apollonian bilipschitz mappings. They proved the following theorem.

3.6. Theorem. *Let $D \subsetneq \mathbb{R}^2$ be a quasidisk and $f\colon D \to D'$ be an Apollonian bilipschitz mapping.*
(1) *If D' is a quasidisk, then f is quasiconformal in D and $f = g|_D$, where $g\colon \overline{\mathbb{R}}^2 \to \overline{\mathbb{R}}^2$ is quasiconformal.*
(2) *If f is quasiconformal in D, then D' is a quasidisk and $f = g|_D$, where $g\colon \overline{\mathbb{R}}^2 \to \overline{\mathbb{R}}^2$ is quasiconformal.*

In [Hä2] the above property (1) was generalized to hold also for $n \geq 3$. In the same article also a condition was introduced which determines when a Euclidean bilipschitz mapping is also Apollonian bilipschitz. In the article [HäIb] it is shown that for $n = 2$ the Apollonian isometries are exactly restrictions of Möbius mappings.

For the quasihyperbolic metric the question regarding the isometries has long been open. In [MaOs] it was shown that every k_D-isometry is a conformal mapping. A similar proof gives the same result for Ferrand's metric σ_D. However, in [Hä3] it is shown that if the boundary of the domain is regular enough (C^3, or C^2 unless the domain is either strictly convex or has strictly convex complement), then the quasihyperbolic isometries are exactly the similarity mappings.

3.7. Conformal modulus. We conclude by introducing two new metrics which are particularly interesting regarding the question of bilipschitz mappings. Let Γ be a family of curves in $\overline{\mathbb{R}}^n$. By $\mathcal{F}(\Gamma)$ we denote the family of *admissible functions*, that is, non-negative Borel-measurable functions $\rho\colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}$ such that

$$\int_\gamma \rho \, ds \geq 1$$

for each locally rectifiable curve $\gamma \in \Gamma$. The *n-modulus* or the *conformal modulus* of Γ is defined by

$$\mathsf{M}(\Gamma) = \mathsf{M}_n(\Gamma) = \inf_{\rho \in \mathcal{F}(\Gamma)} \int_{\mathbb{R}^n} \rho^n \, dm,$$

where m is the n-dimensional Lebesgue measure. It is a conformal invariant, i.e. if $f\colon G \to G'$ is a conformal mapping and Γ is a curve family in G, then $\mathsf{M}(\Gamma) = \mathsf{M}(f\Gamma)$.

For $E, F, G \subset \overline{\mathbb{R}}^n$ we denote by $\Delta(E, F; D)$ the family of all closed non-constant curves joining E and F in D, that is, $\gamma\colon [a, b] \to \overline{\mathbb{R}}^n$ belongs to $\Delta(E, F; D)$ if one of $\gamma(a), \gamma(b)$ belongs to E and the other to F, and furthermore $\gamma(t) \in D$ for all $a < t < b$.

Now we will define two new conformal invariants in the following way. For $x, y \in D \subsetneq \overline{\mathbb{R}}^n$ λ_D is defined by

$$\lambda_D(x,y) = \inf_{C_x, C_y} \mathsf{M}\big(\Delta(C_x, C_y; D)\big),$$

where $C_z = \gamma_z[0,1)$ and $\gamma_z \colon [0,1] \to D$ is a curve such that $z \in |\gamma_z|$ and $\gamma_z(t) \to \partial D$ when $t \to 1$ and $z = x, y$. Correspondingly,

$$\mu_D(x,y) = \inf_{C_{xy}} \mathsf{M}\big(\Delta(C_{xy}, \partial D; D)\big),$$

where C_{xy} is such that $C_{xy} = \gamma[0,1]$ and γ is a curve with $\gamma(0) = x$ and $\gamma(1) = y$.

It is not difficult to show that both quantities μ_D and λ_D are conformal invariants, and that μ_D is a metric (often called the *modulus metric*) when $\operatorname{cap} \partial D > 0$, see [Gá]. λ_D is not a metric, but $\lambda_D^* = \lambda_D^{1/(1-n)}$ introduced in [Fe2] is, as long as the boundary of the domain has more then two points.

One of the interesting feature regarding these metrics is that both are easily seen — by their definitions — to be conformal invariants. Moreover, the following can be shown (see [Vu, 10.19]);

3.8. Theorem. *If $f \colon D \to D' = fD$ is a quasiconformal mapping, then*
(1) $\mu_D(x,y)/L \le \mu_{fD}(f(x), f(y) \le L\, \mu_D(x,y)$,
(2) $\lambda_D^*(x,y)/L^{1/(n-1)} \le \lambda_{fD}^*(f(x), f(y)) \le L^{1/(n-1)}\, \lambda_D^*(x,y)$
hold for all $x, y \in D$, where $L = \max\{K_I(f), K_O(f)\}$ is the maximal dilatation of f. □

It is not known if the class of bilipschitz mappings with respect to μ or λ^* includes any other than quasiconformal mappings.

4. Gromov hyperbolicity

One way of telling "how hyperbolic" a metric in fact is, is to study whether it satisfies hyperbolicity in the sense of M. Gromov. Classically such spaces have been studied in the geodesic case, and then a space is said to be *Gromov δ-hyperbolic* if for all triples of geodesics $J_d[x,y]$, $J_d[y,z]$ and $J_d[x,z]$ we have that

$$\operatorname{dist}(w, J_d[y,z] \cup J_d[z,x]) \le \delta$$

for all $w \in J_d[x,y]$, i.e. if all geodesic triangles are *δ-thin.*

4.1. The Gromov product. In non-geodesic spaces, however, we are constrained to use the definition involving the *Gromov product*. This can be defined for two points $x, y \in D$ with respect to a base point w by setting

$$(x|y)_w = \frac{1}{2}\big(d(x,w) + d(y,w) - d(x,y)\big).$$

A space is then said to be *Gromov δ-hyperbolic* if it satisfies the inequality

$$(x|z)_w \geq (x|y)_w \wedge (y|z)_w - \delta$$

for all $x, y, z \in D$ and a base point $w \in D$. A space is said to be *Gromov hyperbolic* if it is Gromov δ-hyperbolic for some δ. Sometimes one wants to use the equivalent definition for Gromov hyperbolicity

$$d(x,z) + d(y,w) \leq \big(d(x,w) + d(y,z) \vee d(x,y) + d(z,w)\big) + 2\delta. \tag{4.2}$$

Recently the study of Gromov hyperbolicity has become quite popular, and even hyperbolicity results on particular metrics in geometric function theory have been developed by a number of authors. A systematic study of the different metrics is made easier by the fact that Gromov hyperbolicity is preserved by certain classes of mappings, so called rough isometries. We say that two metrics d and d' are *roughly isometric* if there exists a positive constant C such that

$$d(x,y) - C \leq d'(x,y) \leq d(x,y) + C.$$

It is immediately clear from the definition (4.2) that roughly isometric metrics are Gromov hyperbolic in the same domains. Moreover, we say that two metrics are (A,C)*-quasi-isometric* if there is $A \geq 1,\ C \geq 0$ such that

$$A^{-1} d(x,y) - C \leq d'(x,y) \leq A\, d(x,y) + C.$$

Also quasi-isometries (and thus bilipschitz mappings) are known to preserve Gromov hyperbolicity, provided that the spaces are geodesic.

Naturally we would want the hyperbolic metric itself to be Gromov hyperbolic also, and in fact it is, with constant $\delta = \log 3$, as is shown in [CoDePa]. One of the more interesting and general results is one from the comprehensive study of M. Bonk, J. Heinonen and P. Koskela [BoHeKo], where it is shown that for a uniform domain D the space (D, k_D) is always Gromov hyperbolic.

For many of the other metrics Gromov hyperbolicity is easily proved or disproved using the results from [Hä4]. Namely, it turns out that the $\widetilde{j}$-metric is Gromov hyperbolic in every proper subdomain of $\mathbb{R}^n$, whereas the j-metric is Gromov hyperbolic only in $\mathbb{R}^n \setminus \{a\}$. Then, using inequalities

$$j_D(x,y) - \log 3 \leq \eta_D(x,y) \leq j_D(x,y),$$

$$\tilde{j}_D(x,y) - \log 9 \leq \alpha_D(x,y) \leq \tilde{j}_D(x,y),$$

and

$$\alpha_D(x,y) \leq \delta_D(x,y) \leq \alpha_D(x,y) + \log 3$$

we immediately get some results by rough isometry, that is, the results in Table 1 regarding the Apollonian, half-Apollonian and Seittenranta metrics. For proving Gromov hyperbolicity of the Ferrand metric one can use geodesity, Gromov hyperbolicity of the quasihyperbolic metric, and the bilipschitz equivalence in (3.3).

Finally, for the μ and λ_* metrics positive results regarding Gromov hyperbolicity are shown in [Li2].

4.3. Theorem. *The metric space $(\mathbf{B}^n, \lambda^*_{\mathbf{B}^n})$ is Gromov δ-hyperbolic, with Gromov constant*

$$\delta \le \tfrac{1}{2}\left(\tfrac{\omega_{n-1}}{2}\right)^{\frac{1}{1-n}}\left(\log\tfrac{64}{3} + 4\log\lambda_n\right) \le \tfrac{1}{2}\left(\tfrac{\omega_{n-1}}{2}\right)^{\frac{1}{1-n}}\left(\log\tfrac{64}{3} + 4(\log 2 + n - 1)\right),$$

where ω_{n-1} denotes the $(n-1)$-dimensional surface area of S^{n-1} and λ_n is the Grötzsch constant. Also, any simply connected proper subdomain $D \subsetneq \mathbb{R}^2$ is Gromov δ-hyperbolic with respect to the metric λ^*_G, where

$$\delta \le \frac{\log 5462}{2\pi} \approx 1.3696.$$

$\square$

4.4. Theorem. *The metric space $(\mathbf{B}^n, \mu_{\mathbf{B}^n})$ is Gromov δ-hyperbolic, with Gromov constant*

$$\delta \le 2^{n-1} c_n \log 12,$$

where c_n is the spherical cap inequality constant, see [Vu]. *Especially, every simply connected domain $D \subsetneq \mathbb{R}^2$ is Gromov hyperbolic with*

$$\delta \le \frac{2\log 12}{\pi} \approx 1.5819.$$

$\square$

4.5. Theorem. *The metric space $(\mathbb{R}^n \setminus \{z\}, \lambda^*_{\mathbb{R}^n\setminus\{z\}})$ is Gromov hyperbolic, with*

$$\delta \le 2\omega_{n-1}^{\frac{1}{n-1}} \log 18\lambda_n^2 \le 2\omega_{n-1}^{\frac{1}{n-1}}\left(\log 72 + 2n - 2\right).$$

$\square$

As the below table indicates, the j-metric and the half-Apollonian metric are the only metrics of the ones discussed here which fail to be Gromov hyperbolic in most cases. These results indicate that these metrics are in a way “too easy”, or have too little structure for satisfying Gromov hyperbolicity. On the other hand, in other contexts that is one of their strongest features, as has been seen in earlier sections.

	Domain condition	Proved where
k_D	D uniform	[BoHeKo]
h_D	$n = 2$ all domains defined, $n \ge 3$, $D = \mathbf{B}^n, \mathbf{H}^n$	[CoDePa] and conf. invariance
α_D	All domains $D \subsetneq \mathbb{R}^n$	[Hä4] and rough isometry
η_D	Only $D = \mathbb{R}^n \setminus \{z\}, \delta = \log 9$	[Hä4],[HäLi]
j_D	Only $D = \mathbb{R}^n \setminus \{z\}, \delta = \log 9$	[Hä4]
$\tilde{j}_D$	All domains $D \subsetneq \mathbb{R}^n$	[Hä4]
δ_D	All domains $D \subsetneq \mathbb{R}^n$	[Hä4],[Se]
σ_D	D uniform, for $D = \mathbf{B}^n$ $\delta = \log 3$	[Fe1],[BoHeKo]
λ^*_D	$D = \mathbf{B}^n$, $\mathbb{R}^n_*$, $n = 2$ simply conn. domains	[Li2]
μ_D	$D = \mathbf{B}^n$, $n = 2$ simply conn. domains	[Li2]

Table 1: Gromov hyperbolicity of some metrics.

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