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Circle Packings, Quasiconformal Mappings, and Applications

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Abstract. We provide an overview of the connections between circle packings and quasiconformal mappings, with particular attention to applications to string theory and image recognition.

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Contents

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1. Introduction

The deep connections between the combinatorial and geometric properties of circle packings and the analytic properties of the maps they induce have been the subject of intense study in recent years. In 1985, William Thurston conjectured, and Burt Rodin and Dennis Sullivan proved, that maps between circle packings were nearly analytic [Thu85, RS87]. Since then the study of circle packings has exploded to impact a great many other fields including conformal mapping [HS93, HS96, Ste05], complex analysis [BS91, DS95a, Ste97, Ste02, Ste03] Teichmüller theory [BS90, Bro96, Wil01b, BW02, Wil03, BS04b], brain mapping [Bea99, Kra99], random walks [Ste96, Dub97, HS95, McC98, DW05], tilings [BS97, Rep98], minimal surfaces and integrable systems [BS04a], numerical analysis [Moh93, CS99], metric measure spaces [BK02] and much more.

The fundamental folk theorem of circle packing is that "packings desperately want to be conformal." They react to combinatorial or geometric changes in precisely the same way as conformal maps. Maps between packings seem determined to approximate conformal maps. There is, however, much to be said about the relationships between circle packings and *quasiconformal* maps. It is principally with these connections and the applications arising from them that we will concern ourselves in this paper.

After some initial background on quasiconformal mappings in Section 2, we describe the crucial concept of conformal welding in Section 3. We review the fundamental concepts of circle packing in Section 4, and then describe three applications of circle packings and quasiconformal maps in Section 5. Namely, we discuss the use of packings in image recognition, in implementing Radnell-Schippers quantum field theory, and in constructing quasiconformal maps.

2. Quasiconformal Maps

2.1. Analytic Definition of Quasiconformality. Quasiconformal mappings form the heart of Teichmüller theory as developed in the 1950's and 1960's. They are the natural generalization of analytic functions. For more detailed explanations, a number of excellent resources are available, including [Ahl66, LV73, Leh87, Nag88, IT92, GL00].

Definition 2.1. A homeomorphism $f \in L^2$ is **quasiconformal** if

$$\partial_{\bar{z}} f = \mu \partial_z f \tag{2.1}$$

for some $\mu \in L^\infty$, $||\mu||_\infty < 1$. Recall the complex partial derivatives are defined by

$$\partial_{\bar{z}} f = \frac{1}{2} \left(\partial_x f + \partial_y f \right)$$
$$\partial_z f = \frac{1}{2} \left(\partial_x f - \partial_y f \right).$$

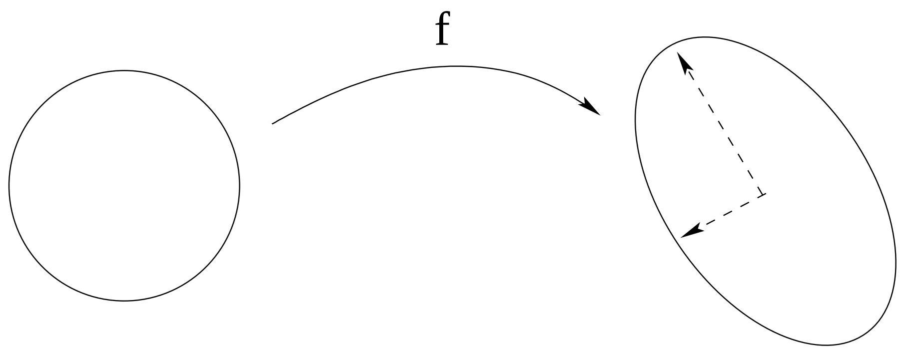

Figure 1. A geometric measure of quasiconformality. The quotient of the length of the dashed lines measures how close the curve on the right is to being a circle.

Equation 2.1 is called the **Beltrami equation** and μ, a **Beltrami differential**. Notice that when $\mu \equiv 0$, the Beltrami equation becomes $\partial_{\bar{z}} f \equiv 0$, which when separated into real and imaginary parts is precisely the familiar Cauchy-Riemann equations. Thus μ determines how "quasi" a quasiconformal map really is. This measure of the "quasi-ness," or **distortion** of a map is most often expressed in terms of the **dilatation**

$$K = \frac{1 + ||\mu||_\infty}{1 - ||\mu||_\infty} \geq 1$$

of the map. A quasiconformal map f with dilatation K is called a K-quasiconformal map; a 1-quasiconformal map is thus conformal.

The Beltrami differential μ corresponding to a quasiconformal map is often called its **complex dilatation**. Notice, however, that the complex dilatation is a complex function and actually measures the distortion of f at every point in its domain. The dilatation, on the other hand, is a single real number and provides a global bound on the distortion of f over the entire domain.

2.2. Geometric Definition of Quasiconformality. An equivalent measure of the distortion of a quasiconformal map is provided by the **dilatation quotient**

$$D_f(z) = \limsup_{r \to 0^+} \frac{\sup_\theta |f(z + re^{i\theta}) - f(z)|}{\inf_\theta |f(z + re^{i\theta}) - f(z)|}.$$

The dilatation quotient has a simple geometric interpretation. If we consider a small circle of radius r about z in the domain, it will be mapped to some curve about $f(z)$ in the range. The dilatation quotient is then the ratio of the maximal to the minimal distance from $f(z)$ to this curve. See Figure 1.

Recall that conformal maps preserve angles; moreover, if $f'(z) = re^{i\theta} \neq 0$, then

$$df = f'(z)\, dz = re^{i\theta}\, dz.$$

Thus infinitesimally, f acts geometrically like

$$z \mapsto re^{i\theta} z + C$$

for some number C; that is, f acts like the composition of a scaling, rotation, and translation. This not only explains the reason analytic maps with non-vanishing derivative preserve angles, but also implies that they must map infinitesimal circles to infinitesimal circles. Consequently, the dilatation quotient of a conformal map is identically 1.

It turns out that the dilatation of a quasiconformal map is nothing more than the supremum of the dilatation quotient over the domain. Thus we have the following equivalent definition of quasiconformality.

Definition 2.2. A homeomorphism f is K-quasiconformal if it is absolutely continuous on lines and

$$D_f(z) \leq K$$

for all z in its domain.

2.3. An Important Example. If we think of the complex plane as $\mathbb{R}^2$ and $x + iy$ as $\begin{pmatrix} x \\ y \end{pmatrix}$, then it is natural to consider the effect of linear and affine transformations. Suppose

$$f(x+iy) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}, \tag{2.2}$$

where $ad - bc \neq 0$.

A moment's linear algebra shows f can be re-written as

$$f(x+iy) = \begin{pmatrix} ax + by + e \\ cx + dy + f \end{pmatrix}. \tag{2.3}$$

Then

$$\partial_z f = \frac{1}{2}\left(\begin{pmatrix} a \\ c \end{pmatrix} - \begin{pmatrix} -d \\ b \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} a+d \\ c-b \end{pmatrix} \tag{2.4}$$

$$\partial_{\bar{z}} f = \frac{1}{2}\left(\begin{pmatrix} a \\ c \end{pmatrix} + \begin{pmatrix} -d \\ b \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} a-d \\ c+b \end{pmatrix}.$$

Notice that $\partial_{\bar{z}} f = 0$ if and only if $a = d$ and $c = -b$, in which case, multiplication by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is equivalent to multiplication by the complex number $a + ib$.

In general, however, we will have

$$\mu = \frac{\partial_{\bar{z}} f}{\partial_z f} = \frac{(a-d) + i(c+b)}{(a+d) + i(c-b)},$$

and f will be quasiconformal.

Geometrically, f will map the basis vectors 1 and i to $a + ic$ and $b + id$, respectively, and then translate by $e + if$. It is easy to check that $\mu = 0$ if and only if the new basis vectors $a+ic$ and $b+id$ are perpendicular, and $|\mu|$ increases toward 1 as the angle decreases toward 0.

Notice that affine maps have constant complex dilatation; conversely, if μ is constant, it is a simple exercise to solve for the affine map whose dilatation is μ.

The importance of this example becomes apparent when we consider the infinitesimal behavior of any quasiconformal map. Just as we observed that conformal maps act infinitesimally by rotation, scaling, and translation, quasiconformal maps act infinitesimally as affine maps.

3. Conformal Welding

3.1. Quasisymmetries and Quasicircles. We continue our exploration of quasiconformal maps with an investigation of their boundary values [BA56, LV73, DE86, LP88, GL00]. Note that when maps extend continuously or smoothly to the boundary, we will use same notation for the extended maps.

Definition 3.1. A homeomorphism $\varphi : \partial\mathbb{D} \to \partial\mathbb{D}$ is **quasisymmetric** or a **quasisymmetry** if it is the boundary function of some quasiconformal map of $\mathbb{D}$ onto itself.

As might be expected, quasisymmetries have a beautiful geometric characterization as well [BA56, LV73, Leh87, Krz87].

Definition 3.2. An orientation preserving homeomorphism $\varphi : \partial\mathbb{D} \to \partial\mathbb{D}$ is a k-quasisymmetry if

$$\frac{1}{k} \leq \frac{|\varphi(I)|_{\mathbb{D}}}{|\varphi(J)|_{\mathbb{D}}} \leq k$$

for any two adjacent intervals (subarcs) I and J of $\partial\mathbb{D}$ having equal length $|I|_{\mathbb{D}} = |J|_{\mathbb{D}}$.

Essentially, this definition says quasisymmetries can't map adjacent symmetric intervals to extremely non-symmetric intervals.

Next, we temporarily leave quasisymmetries to consider the effect of quasiconformal maps on circles. However, as we will see, these quasicircles are intimately connected to quasisymmetries.

Definition 3.3. A Jordan curve Γ is a K**-quasicircle** if it is the image of the unit circle under a K-quasiconformal map of $\mathbb{C}$ onto itself.

As might be expected by now, quasicircles have both analytic and geometric definitions [Ahl63, Ahl66].

Definition 3.4. A Jordan curve Γ is a quasicircle if there exists $R > 1$ so that for all points $x, y \in \Gamma$

$$\text{diam}(\Gamma_{x,y}) \leq R|x - y|,$$

where $\Gamma_{x,y}$ is the sub-arc of Γ connecting x and y which has the smaller diameter.

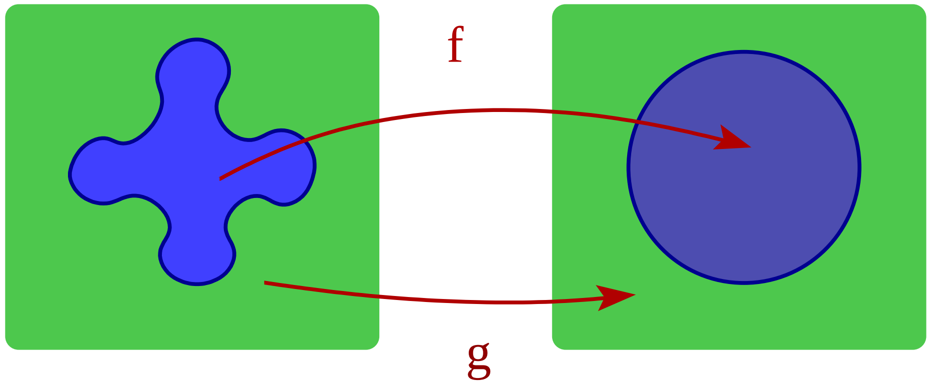

FIGURE 2. If Γ is a Jordan curve, then the Riemann Mapping Theorem promises the existence of a conformal map f from the inside of Γ to the inside of the unit disc $\mathbb{D}$. Similarly, there exists a conformal map g from the outside of Γ to the outside of the unit disc.

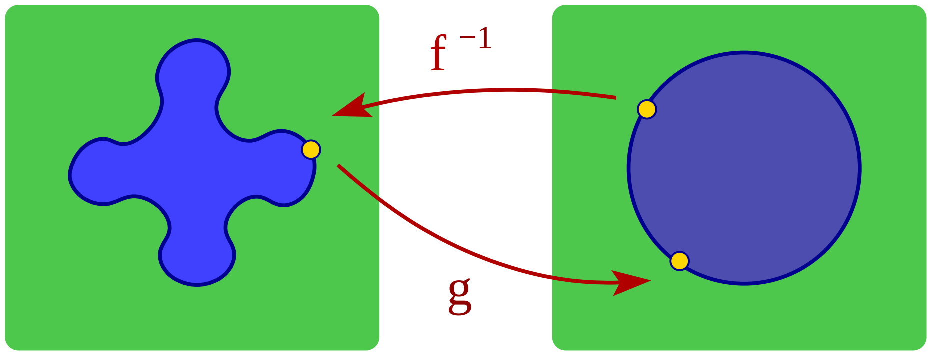

FIGURE 3. Since f and g extend to the boundary, they induce a homeomorphism $\varphi = g \circ f^{-1} : \partial\mathbb{D} \to \partial\mathbb{D}$.

Loosely speaking, this condition limits "pinching" – a quasicircle cannot visit a point x, wander far away, and then return to a point very near x. Fred Gehring's monograph [Geh82] contains an extensive list of these and other characterizations of quasicircles.

3.2. Conformal Welding Theorem. The intimate connection between quasisymmetries and quasicircles is illustrated by the following two theorems [Pfl51, LV73, Leh87, GL00].

Theorem 3.5. *Suppose Γ is Jordan curve dividing the plane into complementary components Ω and Ω^*. Let $f : \Omega \to \mathbb{D}$ and $g : \Omega^* \to \mathbb{D}^*$ be conformal homeomorphisms, the existence of which are promised by the Riemann Mapping Theorem. Then f and g extend to homeomorphisms of the boundary and*

$$g \circ f^{-1} : \partial\mathbb{D} \to \partial\mathbb{D}$$

is a quasisymmetry if Γ is a quasicircle. See Figures 2 and 3.

The converse is also true. Given a quasisymmetry $\varphi : \partial\mathbb{D} \to \partial\mathbb{D}$, we can glue $\mathbb{D}$ and $\mathbb{D}^*$ together by attaching points $e^{i\theta} \in \partial\mathbb{D}$ to their image points $\varphi\left(e^{i\theta}\right) \in \partial\mathbb{D}^*$. The result is a topological sphere. As $\mathbb{D}$ and $\mathbb{D}^*$ struggle to fit together after the welding, the "seam" between them will be pushed and pulled into a quasicircle.

Conformal Welding Theorem. *Let $\varphi : \partial\mathbb{D} \to \partial\mathbb{D}$ be a quasisymmetry. Then φ induces a* ***conformal welding*** *of $\mathbb{D}$ and $\mathbb{D}^*$. That is, there exist conformal maps $f : \Omega \to \mathbb{D}$ and $g : \Omega^* \to \mathbb{D}^*$ of complementary Jordan domains in $\mathbb{C}$ with boundary values satisfying*

$$g \circ f^{-1}(e^{i\theta}) = \varphi(e^{i\theta}).$$

Moreover, the Jordan curve $\Gamma = f^{-1}(\partial\mathbb{D}) = g^{-1}(\partial\mathbb{D}^*)$ is unique up to Möbius transformations.

For quasisymmetries defined on $\partial\mathbb{D}$, it is customary to normalize our welding maps so that $f^{-1}(1) = g^{-1}(1) = \varphi(1) = 1$, $f(0) = 0$, and $g(\infty) = \infty$. With these normalizations, the maps f and g and the curve Γ are unique.

4. Circle Packing

4.1. Definitions and Examples. Since William Thurston's work in the mid-1980's, the connections between circle packings and analytic functions have been widely studied. More detailed information is contained in the rapidly expanding literature, including several recent survey articles [DS95b,Ste97,Ste02,Ste03] and Ken Stephenson's excellent new book [Ste05].

Definition 4.1. A **CP-complex** $\mathcal{K}$ is an abstract simplicial 2-complex such that

1. $\mathcal{K}$ is simplicially equivalent to a triangulation of an (orientable) surface.
2. Every boundary vertex of $\mathcal{K}$ has an interior neighbor.
3. The collection of interior vertices is nonempty and edge-connected.
4. There is an upper bound on the degree of vertices in $\mathcal{K}$.

The restrictions imposed by conditions 2 through 4 are extremely mild and are met by most any reasonable triangulation.

Notice that a CP-complex is a purely combinatorial object. It possesses no geometric structure until it is embedded in a surface by a circle packing. To emphasize this fact, we will often refer to a CP-complex simply as an **abstract triangulation**.

Definition 4.2. A **circle packing** is a configuration of circles with a specified pattern of tangencies. In particular, if $\mathcal{K}$ is a CP-complex, then a circle packing P for $\mathcal{K}$ is a configuration of circles such that

1. P contains a circle $\mathcal{C}_v$ for each vertex v in $\mathcal{K}$,
2. $\mathcal{C}_v$ is externally tangent to $\mathcal{C}_u$ if $[v, u]$ is an edge of $\mathcal{K}$,
3. $\langle \mathcal{C}_v, \mathcal{C}_u, \mathcal{C}_w \rangle$ forms a positively oriented mutually tangent triple of circles if $\langle v, u, w \rangle$ is a positively oriented face of $\mathcal{K}$.

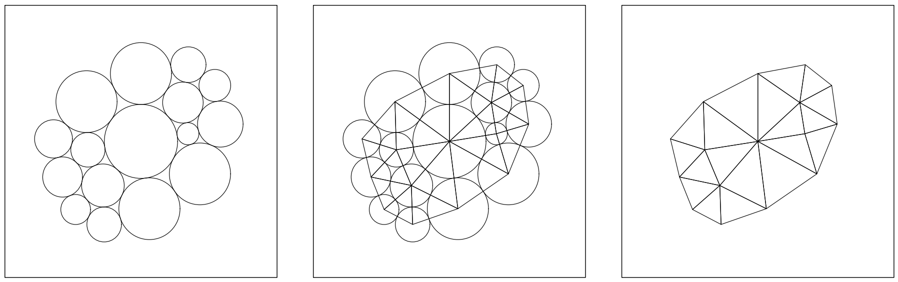

Figure 4. A finite circle packing (left). The underlying triangulation can be recovered by connecting centers of tangent circles with line segments (middle). The resulting collection of triangles forms the carrier of the packing (right).

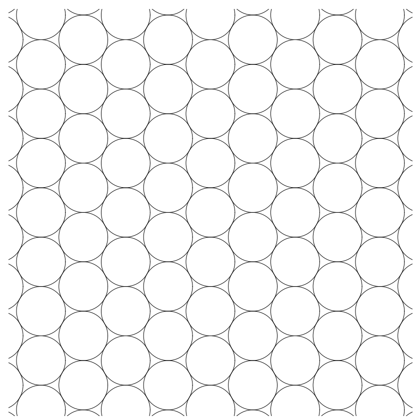

Figure 5. A portion of the "regular hex" packing. Notice that every circle has the same radius.

A packing is called **univalent** if none of its circles overlap, that is, if no pair of circles intersect in more than one point.

A univalent circle packing produces a geometric realization of its underlying complex. Vertices can be embedded as centers of their corresponding circles, and edges can be realized as geodesic segments joining centers of circles. The collection of triangles embedded in this way is called the **carrier** of the packing, written $\operatorname{carr} P$. See Figure 4.

Example 4.3. William Thurston's original interest in packings began with the infinite "regular hex packing" in which every circle touches exactly 6 others. He showed that the only univalent packing with this combinatorial pattern is the one in which every circle has the same radius. (It remains an open question to characterize the non-univalent ones.) See Figure 5.

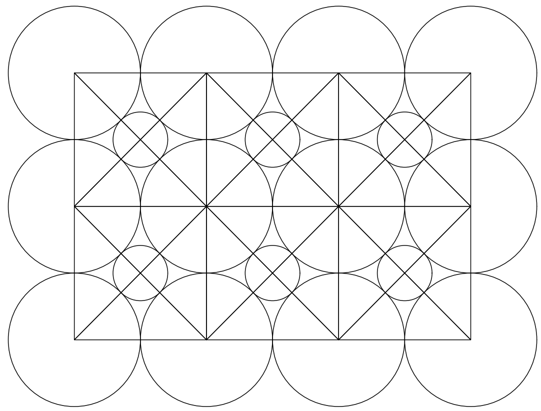

FIGURE 6. A portion of the "ball bearing" packing. The carrier has been drawn in to emphasize the lattice structure.

Example 4.4. Another useful infinite packing is the "ball bearing packing" named by Tomasz Dubejko and Ken Stephenson [DS95b]. The underlying triangulation is created from a lattice, and the original lattice structure is still apparent in the resulting packing. Consequently, the carrier of the packing can be decomposed into small squares. Moreover, there is a natural refinement of the triangulation and carrier created by replacing each square with four copies of the original. See Figure 6.

4.2. Packings and Maps. The connection between circle packings and function theory arises from the investigation of maps between the carriers of two different packings for the same abstract complex. That is, suppose P and $\widetilde{\mathcal{P}}$ are both Euclidean circle packings for the same underlying complex $\mathcal{K}$. Then every face in $\mathcal{K}$ is realized as both a Euclidean triangle T in $\operatorname{carr} P$ and a triangle $\widetilde{T}$ in $\operatorname{carr} \widetilde{\mathcal{P}}$. It is easy now to construct an affine map between triangles T and $\widetilde{T}$. If we translate one vertex of each to the origin, then the two edges meeting at the origin form a basis for $\mathbb{R}^2$ and can be mapped one onto the other by a linear map.

Thus the entire carrier of P can be mapped onto the carrier of $\widetilde{\mathcal{P}}$ by a piecewise affine map defined triangle by triangle. Notice that the individual triangle maps agree on adjacent edges, so the complete map is continuous. Circle packing maps constructed in this way are called **discrete conformal maps**. See Figure 7.

4.3. The Rodin-Sullivan Theorem. Recall from Section 2.3, that affine maps are quasiconformal. The dilation on each triangle will be constant and depend only by the difference between corresponding angles. If there are only finitely many circles in the packings, the dilatation of a discrete conformal map will be finite and depend only on the maximal difference in corresponding angles between triangles in the two carriers.

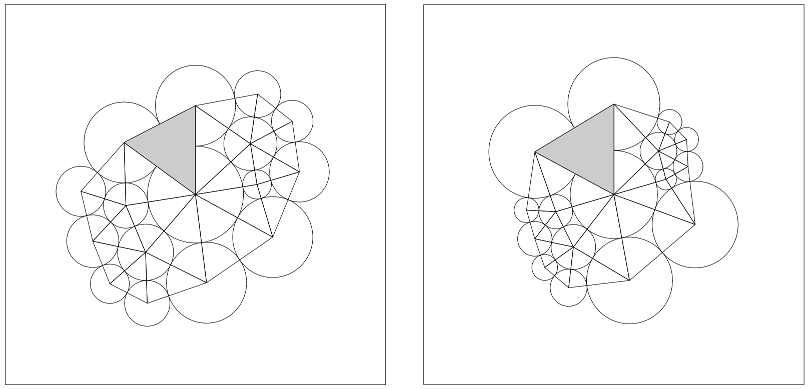

FIGURE 7. Two circle packings with the same underlying triangulation. The carrier for each is indicated and one pair of corresponding triangles are shaded. Each triangle in the carrier on the left can be mapped via an affine map to its corresponding triangle in the carrier on the right.

At this point in our story, we come to Burt Rodin and Dennis Sullivan's Ring Lemma, the first connection between the analytic properties of discrete conformal maps and the combinatorial properties of packings [RS87].

Ring Lemma. *In a univalent packing, there is a lower bound $\mathcal{C}_n$ on the ratio of the radius of any interior circle to the radius of any of its neighbors. This bound depends only on the degree n (the number of neighbors) of the circle.*

The sharp value of the bound $\mathcal{C}_n$ was determined by Dov Aharonov [Aha97].

Lemma 4.5. *If $\{a_n\}$ is the Fibonacci sequence, then*

$$\mathcal{C}_n = \frac{1}{a_{n-2}^2 + a_{n-1}^2 - 1}.$$

Moreover, $\dfrac{\mathcal{C}_n}{\mathcal{C}_{n+1}}$ converges to the square of the golden ratio.

The Ring Lemma thus connects a purely combinatorial property of the packing (the degree) with a geometric property of the packing (the ratio of the radii of adjacent circles). This geometric constraint on the circles implies angles in the carrier must be bounded away from 0 and π. Hence there is a uniform bound on the difference between corresponding angles in the carriers of two packings with the same underlying triangulation. Consequently, the associated discrete conformal map is quasiconformal with a bound on the dilatation determined only the degree. In this way, a combinatorial property of the triangulation leads directly to an analytic property of the associated discrete conformal maps.

In 1985, William Thurston conjectured the relationships between the combinatorics, geometry, and mapping properties of packings run much deeper [Thu85].

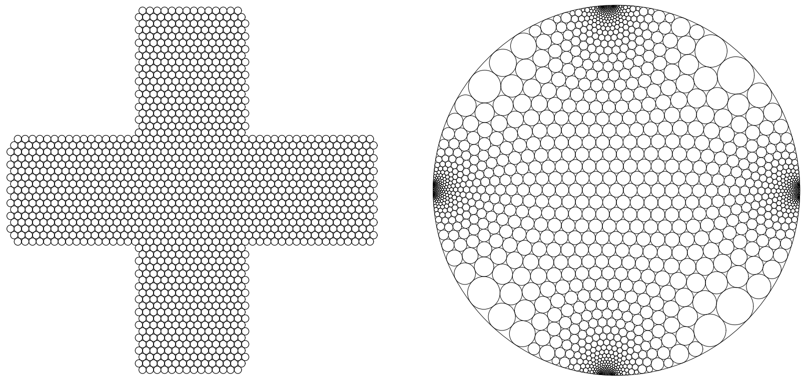

FIGURE 8. A cross-shaped packing (left) which has been re-packed in the unit disc (right). Since both packings share the same underlying triangulation, there is discrete conformal map between them which approximates the classical Riemann map.

Suppose $\Omega \subsetneq \mathbb{C}$ is a bounded simply connected region and $p, q \in \Omega$, $p \neq q$. The Riemann Mapping Theorem implies there is a unique conformal map $f : \Omega \to \mathbb{D}$ with $f(p) = 0$ and $f(q) > 0$.

Now suppose P_n is a sequence of packings in Ω with mesh (radius of the largest circle) decreasing to 0 and $\operatorname{carr} P_n \to \Omega$ as $n \to \infty$. Let $\mathcal{K}_n$ be the underlying triangulation of P_n. Paul Koebe [Koe36], E. M. Andreev [And70a, And70b], and William Thurston [Thu] independently proved that any finite, simply connected CP-complex (such as $\mathcal{K}_n$) can be realized by a packing in $\mathbb{D}$ which is "maximal" in the sense that boundary circles are tangent to $\partial\mathbb{D}$. This maximal, or Andreev, packing is unique up to disc automorphisms.

Thus for each $P_n \subset \Omega$, there is a maximal packing $\widetilde{\mathcal{P}}_n \subset \mathbb{D}$ with the same underlying triangulation $\mathcal{K}_n$ as P_n. Moreover, we can normalize $\widetilde{\mathcal{P}}_n$ so that if C_p and C_q are the nearest circles in P_n to p and q, respectively, then the corresponding circles $\widetilde{C}_p$ and $\widetilde{C}_q$ in $\widetilde{\mathcal{P}}_n$ are centered at 0 and on the positive real axis, respectively.

Since P_n and $\widetilde{\mathcal{P}}_n$ share the same underlying triangulation, there is a discrete conformal map

$$f_n : \operatorname{carr} P_n \to \operatorname{carr} \widetilde{\mathcal{P}}_n.$$

William Thurston conjectured that $f_n \to f$ locally uniformly on Ω as $n \to \infty$ [Thu85]. This was quickly proven by Burt Rodin and Dennis Sullivan [RS87]. See Figure 8.

Rodin-Sullivan Theorem. *The discrete conformal maps described above converge locally uniformly to the conformal map $f : \Omega \to \mathbb{D}$ with $f(p) = 0$ and $f(q) > 0$.*

Recall that if the degree of $\mathcal{K}_n$ is uniformly bounded for all n (Thurston's original conjecture was for packings with degree 6), then the Ring Lemma implies each f_n will be K-quasiconformal, with K independent of n. It remains to show that the dilatation of f_n must actually decrease to 1 as $n \to \infty$. This follows from the uniqueness of infinite packings.

Theorem 4.6. *Every infinite, simply connected CP-complex has a packing in either $\mathbb{C}$ or $\mathbb{D}$. This packing is unique up to conformal automorphisms.*

Various versions of Theorem 4.6 have been proven. Thurston's original proof was only for the regular hex packing of Example 4.3 and relied on deep results from the theory of hyperbolic 3-manifolds [Thu]. Later improvements by Ken Stephenson [Ste96], Alan Beardon and Ken Stephenson [BS90], Yves Colin de Verdiére [dV89, dV91], Zheng-Xu He and Burt Rodin [HR93], and Zheng-Xu He and Oded Schramm [HS96, HS98] utilized probabilistic techniques, variational principles, the Perron method, or elementary topology.

The effect of Theorem 4.6 is to force the dilation of f_n to decrease to 1 as $n \to \infty$. Consider a circle C "deep inside" P_n, that is, separated from $\partial\Omega$ by a great many generations of other circles. If C is far enough from the boundary, it can hardly tell if it is part of a finite packing, or the unique infinite one. The same must be true for the corresponding circle $\widetilde{C}$ in $\widetilde{\mathcal{P}}_n \subset \mathbb{D}$. Thus triangles in $\operatorname{carr} P_n$ and $\operatorname{carr} \widetilde{\mathcal{P}}_n$ which are far from the boundary, must be nearly the same (up to scaling, translation, and rotation). In particular, the corresponding angles must be nearly the same, and the resulting affine map must be nearly conformal. This is usually stated as the Packing Lemma [Ste96, Ste05].

Packing Lemma. *Suppose K_n is a sequence of simply connected CP-complexes with uniformly bounded degree and having univalent packings P_n in a bounded simply connected domain Ω. If $\widetilde{\mathcal{P}}_n$ is any other sequence of univalent packings for K_n, then the maximum difference between corresponding angles in* $\operatorname{carr} P_n$ *and* $\operatorname{carr} \widetilde{\mathcal{P}}_n$ *goes to* 0 *locally uniformly as $n \to \infty$.*

Finally, recall that we assumed the mesh of P_n decreased to 0 as $n \to \infty$; thus on compact subsets of Ω, the number of generations of circles between the compact subset and the boundary must go uniformly to infinity as $n \to \infty$. Consequently, the dilatation of f_n will decrease to 1 uniformly on compact subsets of Ω.

5. Applications

5.1. Image Recognition. In work with Ken Stephenson, we have applied circle packing techniques to two-dimensional image recognition problems. David Mumford and Eitan Sharon have recently developed a technique for studying two-dimensional shapes (Jordan curves) by means of the Weil-Peterson metric on their associated welding homeomorphisms [MS04]. They restrict their attention to smooth curves which then produce diffeomorphisms of $\partial\mathbb{D}$. The Weil-Peterson metric on these diffeomorphisms is invariant under Möbius transformations; thus shapes which differ only by scaling or rotation are recognized as being the same.

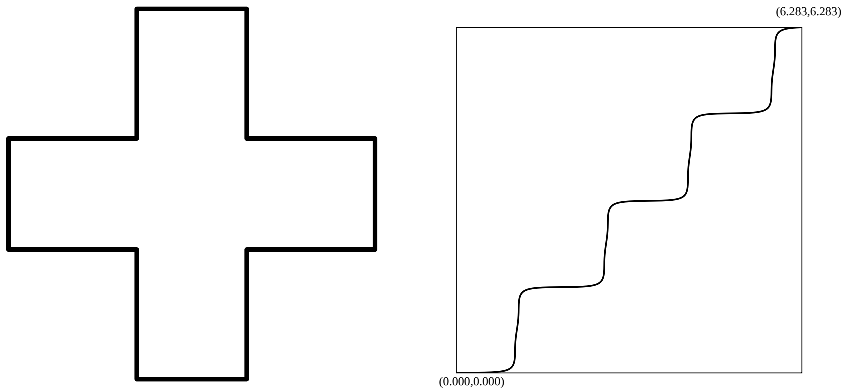

FIGURE 9. A cross-shaped quasicircle (left) and the graph of the resulting quasisymmetry, parametrized as a map from $[0, 2\pi]$ onto $[0, 2\pi]$.

It is relatively easy to extend their program to shapes bounded by quasicircles and to quasisymmetric maps on $\partial\mathbb{D}$. By packing both the inside Ω and outside Ω^* of a quasicircle Γ, then repacking in $\mathbb{D}$ and $\mathbb{D}^*$, respectively, we can create discrete analytic functions

$$f_n : \Omega_n \to \mathbb{D}$$
$$g_n : \Omega_n^* \to \mathbb{D}^*,$$

where $\Omega_n \to \Omega$ and $\Omega_n^* \to \Omega^*$.

It is much trickier to compare the boundary values of f_n and g_n since the packings in Ω and Ω^* don't necessarily match up on the boundary. However, it is possible with careful application of the geometry of quasicircles and a dash a topology to create a map

$$\varphi_n : \partial\mathbb{D} \to \partial\mathbb{D}$$

which is essentially given by $g_n \circ f_n^{-1}$. We then have the following theorem [Wil01a]:

Theorem 5.1. *The mappings φ_n converge uniformly to the quasisymmetry φ induced by the quasicircle Γ. Moreover, f_n and g_n converge locally uniformly to the Riemann maps $f : \Omega \to \mathbb{D}$ and $g : \Omega \to \mathbb{D}^*$, respectively.*

For example, consider the cross-shaped curve in Figure 9. Creating discrete conformal maps as described above (Recall Figure 8), we can approximate the corresponding quasisymmetry.

Repeating this procedure for a T-shaped curve and a hand-drawn cross in Figures 10 and 11, the similarities and differences with the straight-sided cross are easy to see.

A more difficult problem is to recover the shape given the map $\varphi : \partial\mathbb{D} \to \partial\mathbb{D}$. The Conformal Welding Theorem guarantees that this is possible, but is no help in actually computing the shape. Again, circle packing comes to the

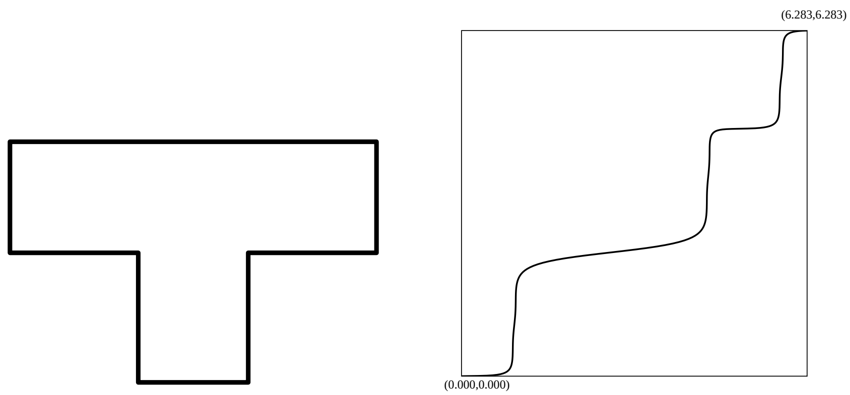

FIGURE 10. A T-shaped quasicircle (left) and the graph of the resulting quasisymmetry (right).

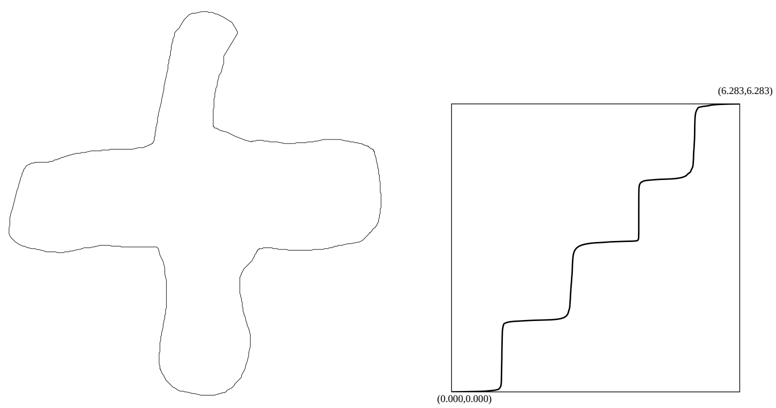

FIGURE 11. A hand-drawn cross (left) and the graph of the resulting quasisymmetry (right). Compare with Figures 9 and 10.

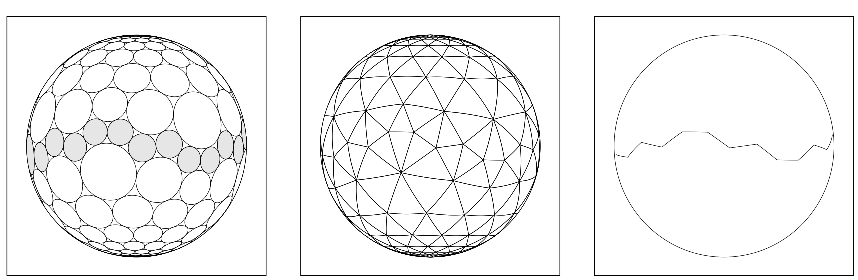

FIGURE 12. A discrete welding for the map $\varphi(e^{i\theta}) = e^{i\left(\theta + \frac{1}{3}\sin(3\theta)\right)}$. The circles corresponding to the "seam" in packing (left) are shaded. The packing provides a realization on S^2 of the welding triangulations (middle). The edges along the "seam" form the discrete welding curve (right).

rescue. Instead of welding $\mathbb{D}$ to $\mathbb{D}^*$, we will weld triangulations of discs. For example, if φ is a homeomorphism from the boundary of an triangulation $\mathcal{K}$ to the boundary of $\mathcal{K}^*$, we use φ to glue the triangulations together. After a few minor refinements and adjustments, we attach every boundary edge e of $\mathcal{K}$ to its image $\varphi(e)$. This **discrete welding** then yields a triangulation $\widetilde{\mathcal{K}}$ of a sphere. The welded triangulation $\widetilde{\mathcal{K}}$ can be realized by a unique circle packing on S^2. The uniqueness of this packing is exactly analogous to the uniqueness of the conformal structure on S^2. The circles must push and pull against each other to settle in locations compatible with the global pattern provided by $\widetilde{\mathcal{K}}$ in precisely the same way that two welded discs settle in locations compatible with the global conformal structure on S^2. This circle packing provides a geometric realization of the formerly purely combinatorial welding. In particular, the "seam" between the original triangulations is realized as a polygonal Jordan curve, a discretized version of the conformal welding curve.

Notice also that $\widetilde{\mathcal{K}}$ contains a copy of both $\mathcal{K}$ and $\mathcal{K}^*$. Thus we can define discrete analytic functions from $\mathcal{K}$ and $\mathcal{K}^*$ onto their copies in $\widetilde{\mathcal{K}}$. This is, of course, analogous to the existence of classical welding maps f and g onto complementary regions of S^2. Moreover, because of the way we used φ to weld $\widetilde{\mathcal{K}}$ together, a version of the welding condition $g \circ f^{-1} = \varphi$ also holds.

In fact, the discrete version is more than just analogous to the classical case – it converges to it as well. Welding finer and finer triangulations using the same quasisymmetric map produces discrete welding curves that converge uniformly to the classical conformal welding curve. Moreover, the discrete analytic functions converge locally uniformly to the classical conformal welding maps [Wil04].

Discrete Conformal Welding Theorem. *Given a quasisymmetric map $\varphi : \partial\mathbb{D} \to \partial\mathbb{D}$, our construction produces discrete analytic functions $\{f_n\}$ and $\{g_n\}$ converging locally uniformly to the conformal welding maps f and g induced by φ. Moreover, the discrete conformal welding curves Γ_n converge uniformly to the quasicircle Γ induced by φ.*

5.2. Radnell-Schippers Quantum Field Theory. Recently David Radnell and Eric Schippers [RS05] have developed a two-dimensional quantum field theory based on conformal welding and rigged Teichmüller spaces. Very briefly, one of fundamental ideas of string theory is that a one-dimensional closed string will sweep out a surface, called its **world sheet**, as it travels through time. As a string breaks apart and rejoins with itself, it alters the topology of the world sheet. See Figure 13. Dennis Sullivan and Moira Chas have in this manner described the topology of all world sheets in terms of the splitting and joining of strings [Sul01].

While the topology of the world sheet captures the splitting and re-joining of a string, it is the conformal structure of the world sheet that captures features such as the relative size of the string and length of time between splittings and joinings. Thus for many computations it is necessary to consider all possible conformal structures on all possible surfaces. The Universal Teichmüller space contains

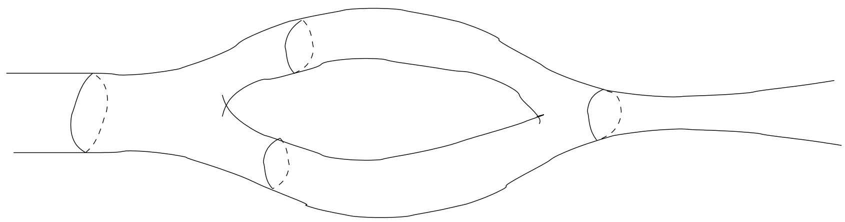

FIGURE 13. A depiction of a string traveling through time. As the string breaks into two pieces and then rejoins, a topological handle is created.

the Teichmüller spaces of all Riemann surfaces and as such has recently gained the attention of physicists as a possible setting for string theory computations [Pek94, Pek95].

Two common models for the Universal Teichmüller space are the space of normalized quasicircles and the space of normalized quasisymmetries. The process of conformal welding described in Section 3.2 provides the mechanism for switching between the two models [Leh87, Krz95]. Our method of discrete conformal welding described above provides the means for actually computing this correspondence as well [Wil01a, Wil04].

In the Radnell-Schippers model of quantum field theory, the ends of the world sheets are parametrized ("rigged") by quasisymmetric maps. The interaction between two strings then corresponds to the welding of the two worldsheets via the rigging [RS05]. These operations can be carried out using circle packings to approximate the world sheets. The packable surfaces are dense [Bro86, Bro92, Bro96, BS92, BS93, Wil03] in the moduli space of all surfaces, so nothing is lost in this approach, while much is gained by the ability to actually compute the new welded surface.

5.3. Circle Packing Measurable Riemann Mapping Theorem. Recall that the distortion of a quasiconformal map f is described by its complex dilatation μ, defined by the Beltrami equation

$$\overline{\partial} f = \mu\, \partial f. \tag{5.1}$$

The classical Measurable Riemann Mapping Theorem asserts that given a Beltrami differential μ on a simply connected domain $\Omega \subsetneq \mathbb{C}$, there is a corresponding quasiconformal map f^μ from Ω to the unit disc $\mathbb{D}$ having μ as its complex dilatation. If f^μ is normalized to send two points $p, q \in \Omega$, $p \neq q$, to 0 and the positive real axis, respectively, then f^μ is unique [LV73, Leh87, GL00].. The original Riemann Mapping Theorem follows from the special case $\mu = 0$.

Circle packings have been used previously by Zheng-Xu He [He90] to solve Beltrami differential equations, but they appear indirectly. By applying our discrete conformal welding technique, however, we can create quasiconformal maps directly from their complex dilatation.

Given a Beltrami differential μ on a bounded simply connected region $\Omega \subsetneq \mathbb{C}$, we pack Ω with a "ball bearing" packing. See Figure 6. The carrier divides Ω into small squares. We approximate μ by a constant function on each square. Recall from Section 2.3 that a map with constant dilatation is affine.

In work with Roger Barnard, we showed that the conformal structure on any compact torus can be transformed into any other by cutting it open appropriately and welding it back together [BW02]. However, the conformal structures of compact tori can also be distorted by affine maps. Thus our work on welding tori provides the mechanism for creating the effect of affine maps.

By refining our ball-bearing packing and performing a discrete conformal welding on each of the small squares in Ω, we can create a normalized discrete quasiconformal map f_n whose dilatation is approximately equal to μ on each square [Wil].

Circle Packing Measureable Riemann Mapping Theorem. *As the packings are refined, the discrete quasiconformal maps f_n converge to the similarly normalized quasiconformal map $f^\mu : \Omega \to \mathbb{D}$ with dilation μ.*

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