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The hyperbolic metric and geometric function theory

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Abstract. The goal is to present an introduction to the hyperbolic metric and various forms of the Schwarz-Pick Lemma. As a consequence we obtain a number of results in geometric function theory.

Keywords. hyperbolic metric, Schwarz-Pick Lemma, curvature, Ahlfors Lemma.

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Contents

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1. Introduction

The authors are writing a book, *The hyperbolic metric in complex analysis*, that will include all of the material in this article and much more. The material presented here is a selection of topics from the book that relate to the Schwarz-Pick Lemma. Our goal is to develop the main parts of geometric function theory by using the hyperbolic metric and other conformal metrics. This paper is intended to be both an introduction to the hyperbolic metric and a concise treatment of a few recent applications of the hyperbolic metric to geometric function theory. There is no attempt to present a comprehensive presentation of the material here; rather we present a selection of several topics and then offer suggestions for further reading.

The first part of the paper (Sections 2-5) studies holomorphic self-maps of the unit disk $\mathbb{D}$ by using the hyperbolic metric. The unit disk with the hyperbolic metric and hyperbolic distance is presented as a model of the hyperbolic plane. Then Pick's fundamental invariant formulation of the Schwarz Lemma is presented. This is followed by various extensions of the Schwarz-Pick Lemma for holomorphic self-maps of $\mathbb{D}$, including a Schwarz-Pick Lemma for hyperbolic derivatives. The second part of the paper (Sections 6-9) is concerned with the investigation of holomorphic maps between simply connected proper subregions of the complex plane $\mathbb{C}$ using the hyperbolic metric, as well as a study of negatively curved metrics on simply connected regions. Here 'negatively curved' means metrics with curvature at most -1. The Riemann Mapping Theorem is used to transfer the hyperbolic metric to any simply connected region that is conformally equivalent to the unit disk. A version of the Schwarz-Pick Lemma is valid for holomorphic maps between simply connected proper subregions of the complex plane $\mathbb{C}$. The hyperbolic metric is explicitly determined for a number of special simply connected regions and estimates are provided for general simply connected regions. Then the important Ahlfors Lemma, which asserts the maximality of the hyperbolic metric among the family of metrics with curvature at most -1, is established; it provides a vast generalization of the Schwarz-Pick Lemma. The representation of metrics with constant curvature -1 by bounded holomorphic functions is briefly mentioned. The third part (Sections 10-13) deals with holomorphic maps between hyperbolic regions; that is, regions whose complement in the extended complex plane $\mathbb{C}_\infty$ contains at least three points, and negatively curved metrics on such regions. The Planar Uniformization Theorem is utilized to transfer the hyperbolic metric from the unit disk to hyperbolic regions. The Schwarz-Pick and Ahlfors Lemmas extend to this context. The hyperbolic metric for punctured disks and annuli are explicitly calculated. A new phenomenon, rigidity theorems, occurs for multiply connected regions; several examples of rigidity theorems are presented. The final section offers some suggestions for further reading on topics not included in this article.

2. The unit disk as the hyperbolic plane

We assume that the reader knows that the most general conformal automorphism of the unit disk $\mathbb{D}$ onto itself is a Möbius map of the form

$$z \mapsto \frac{az+\bar{c}}{cz+\bar{a}}, \quad a, c \in \mathbb{C}, \quad |a|^2 - |c|^2 = 1, \tag{2.1}$$

or of the equivalent form

$$z \mapsto e^{i\theta}\frac{z-a}{1-\bar{a}z}, \quad \theta \in \mathbb{R}, \quad a \in \mathbb{D}. \tag{2.2}$$

It is well known that these maps form a group $\mathcal{A}(\mathbb{D})$ under composition, and that $\mathcal{A}(\mathbb{D})$ acts transitively on $\mathbb{D}$ (that is, for all z and w in $\mathbb{D}$ there is some g in $\mathcal{A}(\mathbb{D})$ such that $g(z) = w$). Also, $\mathcal{A}(\mathbb{D}, 0)$, the subgroup of conformal automorphisms that fix the origin, is the set of rotations of the complex plane about the origin.

The *hyperbolic plane* is the unit disk $\mathbb{D}$ with the *hyperbolic metric*

$$\lambda_{\mathbb{D}}(z)|dz| = \frac{2\,|dz|}{1-|z|^2}.$$

This metric induces a *hyperbolic distance* $d_{\mathbb{D}}(z, w)$ between two points z and w in $\mathbb{D}$ in the following way. We join z to w by a smooth curve γ in $\mathbb{D}$, and define the *hyperbolic length* $\ell_{\mathbb{D}}(\gamma)$ of γ by

$$\ell_{\mathbb{D}}(\gamma) = \int_\gamma \lambda_{\mathbb{D}}(z)\,|dz|.$$

Finally, we set

$$d_{\mathbb{D}}(z, w) = \inf_\gamma \ \ell_{\mathbb{D}}(\gamma),$$

where the infimum is taken over all smooth curves γ joining z to w in $\mathbb{D}$.

It is immediate from the construction of $d_{\mathbb{D}}$ that it satisfies the requirements for a distance on $\mathbb{D}$, namely
(a) $d_{\mathbb{D}}(z, w) \geq 0$ with equality if and only if $z = w$;
(b) $d_{\mathbb{D}}(z, w) = d_{\mathbb{D}}(w, z)$;
(c) for all u, v, w in $\mathbb{D}$, $d_{\mathbb{D}}(u, w) \leq d_{\mathbb{D}}(u, v) + d_{\mathbb{D}}(v, w)$.

The hyperbolic area of a Borel measurable subset of $\mathbb{D}$ is

$$a_{\mathbb{D}}(E) = \int\int_E \lambda_{\mathbb{D}}^2(z)dxdy.$$

We need to identify the isometries of both the hyperbolic metric and the hyperbolic distance. A holomorphic function $f : \mathbb{D} \to \mathbb{D}$ is an *isometry of the metric* $\lambda_{\mathbb{D}}(z)\,|dz|$ if for all z in $\mathbb{D}$,

$$\lambda_{\mathbb{D}}\big(f(z)\big)|f'(z)| = \lambda_{\mathbb{D}}(z), \tag{2.3}$$

and it is an *isometry of the distance* $d_{\mathbb{D}}$ if, for all z and w in $\mathbb{D}$,

$$d_{\mathbb{D}}\big(f(z), f(w)\big) = d_{\mathbb{D}}(z, w). \tag{2.4}$$

In fact, the two classes of isometries coincide, and each isometry is a Möbius transformation of $\mathbb{D}$ onto itself.

Theorem 2.1. *For any holomorphic map $f : \mathbb{D} \to \mathbb{D}$ the following are equivalent:*
(a) *f is a conformal automorphism of $\mathbb{D}$;*
(b) *f is an isometry of the metric $\lambda_{\mathbb{D}}$;*
(c) *f is an isometry of the distance $d_{\mathbb{D}}$.*

Proof. First, (a) implies (b). Indeed, if (a) holds, then f is of the form (2.1), and a calculation shows that

$$\frac{|f'(z)|}{1-|f(z)|^2} = \frac{1}{1-|z|^2},$$

so (b) holds. Next, (b) implies (a). Suppose that (b) holds; that is, f is an isometry of the hyperbolic metric. Then for any conformal automorphism g of $\mathbb{D}$, $h = g \circ f$ is again an isometry of the hyperbolic metric. If we choose g so that $h(0) = g(f(0)) = 0$, then

$$2|h'(0)| = \lambda_{\mathbb{D}}(h(0)|h'(0)| = \lambda_{\mathbb{D}}(0) = 2.$$

Thus, h is a holomorphic self-map of $\mathbb{D}$ that fixes the origin and $|h'(0)| = 1$, so Schwarz's Lemma implies $h \in \mathcal{A}(\mathbb{D}, 0)$. Then $f = g^{-1} \circ h$ is in $\mathcal{A}(\mathbb{D})$. We have now shown that (a) and (b) are equivalent.

Second, we prove (a) and (c) are equivalent. If $f \in \mathcal{A}(\mathbb{D})$, then f is an isometry of the metric $\lambda_{\mathbb{D}}$. Hence, for any smooth curve γ in $\mathbb{D}$,

$$\ell_{\mathbb{D}}\big(f \circ \gamma\big) = \int_{f\circ\gamma} \lambda_{\mathbb{D}}(w)\,|dw| = \int_{\gamma} \lambda_{\mathbb{D}}\big(f(z)\big)|f'(z)|\,|dz| = \ell_{\mathbb{D}}(\gamma).$$

This implies that for all $z, w \in \mathbb{D}$, $d_{\mathbb{D}}(f(z), f(w)) \le d_{\mathbb{D}}(z, w)$. Because $f \in \mathcal{A}(\mathbb{D})$, the same argument applies to f^{-1}, and hence we may conclude that f is a $d_{\mathbb{D}}$–isometry. Finally, we show that (c) implies (a). Take any $f : \mathbb{D} \to \mathbb{D}$ that is holomorphic and a $d_{\mathbb{D}}$–isometry. Choose any g of the form (2.1) that maps $f(0)$ to 0 and put $h = g \circ f$. Then h is holomorphic, a $d_{\mathbb{D}}$–isometry, and $h(0) = 0$. Thus $d_{\mathbb{D}}\big(0, h(z)\big) = d_{\mathbb{D}}\big(h(0), h(z)\big) = d_{\mathbb{D}}(0, z)$. This implies that $|h(z)| = |z|$ and hence, that $h(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$. Thus $h \in \mathcal{A}(\mathbb{D}, 0)$ and, as $f = g^{-1} \circ h$, f is also in $\mathcal{A}(\mathbb{D})$. ∎

In summary, relative to the hyperbolic metric and the hyperbolic distance, the group $\mathcal{A}(\mathbb{D})$ of conformal automorphism of the unit disk becomes a group of isometries.

Theorem 2.2. *The hyperbolic distance $d_{\mathbb{D}}(z, w)$ in $\mathbb{D}$ is given by*

$$d_{\mathbb{D}}(z, w) = \log \frac{1 + p_{\mathbb{D}}(z, w)}{1 - p_{\mathbb{D}}(z, w)} = 2\tanh^{-1} p_{\mathbb{D}}(z, w), \tag{2.5}$$

where the pseudo-hyperbolic distance $p_{\mathbb{D}}(z, w)$ is given by

$$p_{\mathbb{D}}(z, w) = \left| \frac{z - w}{1 - z\bar{w}} \right|. \tag{2.6}$$

Proof. First, we prove that if $-1 < x < y < 1$ then

$$d_{\mathbb{D}}(x,y) = \log\left(\frac{1+\frac{y-x}{1-xy}}{1-\frac{y-x}{1-xy}}\right). \tag{2.7}$$

Consider a smooth curve γ joining x to y in $\mathbb{D}$, and write $\gamma(t) = u(t) + iv(t)$, where $0 \le t \le 1$. Then

$$\ell_{\mathbb{D}}(\gamma) = \int_0^1 \frac{2\,|\gamma'(t)|\,dt}{1-|\gamma(t)|^2} \ge \int_0^1 \frac{2\,u'(t)\,dt}{1-u(t)^2}$$

because $|\gamma(t)|^2 \ge |u(t)|^2 = u(t)^2$ and $|\gamma'(t)| \ge |u'(t)| \ge u'(t)$. The second integral can be evaluated directly and gives

$$\ell_{\mathbb{D}}(\gamma) \ge \log\left(\frac{1+y}{1-y}\,\frac{1-x}{1+x}\right) = \log\left(\frac{1+\frac{y-x}{1-xy}}{1-\frac{y-x}{1-xy}}\right).$$

Because equality holds here when $\gamma(t) = x + t(y-x)$, $0 \le t \le 1$, we see that (2.7) holds, so (2.5) is valid for $-1 < x < y < 1$.

Now we have to extend (2.5) to any pair of points z and w in $\mathbb{D}$. Theorem 2.1 shows that each Euclidean rotation about the origin is a hyperbolic isometry and this implies that, for all z, $d_{\mathbb{D}}(0,z) = d_{\mathbb{D}}(0,|z|)$. Now take any z and w in $\mathbb{D}$, and let $f(z) = (z-w)/(1-z\bar{w})$. Then f is a conformal automorphism of $\mathbb{D}$, and so is a hyperbolic isometry. Thus

$$\begin{aligned} d_{\mathbb{D}}(z,w) &= d_{\mathbb{D}}(w,z) \\ &= d_{\mathbb{D}}\big(f(w), f(z)\big) \\ &= d_{\mathbb{D}}\big(0, f(z)\big) \\ &= d_{\mathbb{D}}\big(0, |f(z)|\big) \\ &= d_{\mathbb{D}}\big(0, p_{\mathbb{D}}(z,w)\big), \end{aligned}$$

which, from (2.7) with $x = 0$ and $y = p_{\mathbb{D}}(z,w)$, gives (2.5). ■

Note that (2.5) produces

$$d_{\mathbb{D}}(0,z) = \log\frac{1+|z|}{1-|z|}, \quad d_{\mathbb{D}}(0,z) = 2\tanh^{-1}|z|.$$

Also,

$$\lim_{z\to w}\frac{d_{\mathbb{D}}(z,w)}{|z-w|} = \lambda_{\mathbb{D}}(w) = 2\lim_{z\to w}\frac{p_{\mathbb{D}}(z,w)}{|z-w|}.$$

A careful examination of the proof of (2.5) shows that if γ is a smooth curve that joins x to y, where $-1 < x < y < 1$, then $\ell_{\mathbb{D}}(\gamma) = d_{\mathbb{D}}(0,x)$ if and only if γ is the simple arc from x to y along the real axis. As hyperbolic isometries map circles into circles, map the unit circle onto itself, and preserve orthogonality, we can now make the following definition.

Definition 2.3. Suppose that z and w are in $\mathbb{D}$. Then the *(hyperbolic) geodesic* through z and w is $C \cap \mathbb{D}$, where C is the unique Euclidean circle (or straight line) that passes through z and w and is orthogonal to the unit circle $\partial\mathbb{D}$. If γ is any smooth curve joining z to w in $\mathbb{D}$, then the hyperbolic length of γ is $d_{\mathbb{D}}(z, w)$ if and only if γ is the simple arc of C in $\mathbb{D}$ that joins z and w.

The unit disk $\mathbb{D}$ together with the hyperbolic metric is called the *Poincaré model of the hyperbolic plane*. The "lines" in the hyperbolic plane are the hyperbolic geodesics and the angle between two intersecting lines is the Euclidean angle between the Euclidean tangent lines at the point of intersection. The hyperbolic plane satisfies all of the axioms for Euclidean geometry with the exception of the Parallel Postulate. It is easy to see that if γ is a hyperbolic geodesic in $\mathbb{D}$ and $a \in \mathbb{D}$ is a point not on γ, then there are infinitely many geodesics through a that do not intersect γ and so are parallel to γ.

We shall now show that the hyperbolic distance $d_{\mathbb{D}}$ *is additive along geodesics.* By contrast, the pseudo-hyperbolic distance $p_{\mathbb{D}}$ is never additive along geodesics.

Theorem 2.4. *If u, v and w are three distinct points in $\mathbb{D}$ that lie, in this order, along a geodesic, then $d_{\mathbb{D}}(u, w) = d_{\mathbb{D}}(u, v) + d_{\mathbb{D}}(v, w)$. For any three distinct points u, v and w in $\mathbb{D}$, $p_{\mathbb{D}}(u, w) < p_{\mathbb{D}}(u, v) + p_{\mathbb{D}}(v, w)$.*

Proof. Suppose that u, v and w lie in this order, along a geodesic. Then there is an isometry f that maps this geodesic to the real diameter $(-1, 1)$ of $\mathbb{D}$, with $f(v) = 0$. Let $x = f(u)$ and $y = f(w)$, so that $-1 < x < 0 < y < 1$. It is sufficient to show that $d_{\mathbb{D}}(x, 0) + d_{\mathbb{D}}(0, y) = d_{\mathbb{D}}(x, y)$; this is a direct consequence of (2.7).

It is easy to verify that $p_{\mathbb{D}}$ a distance function on $\mathbb{D}$, except possibly for the verification of the triangle inequality. This holds because, for any distinct u, v and w,

$$\begin{aligned}
p_{\mathbb{D}}(u, w) &= \tanh \frac{1}{2} d_{\mathbb{D}}(u, w) \\
&\le \tanh \frac{1}{2}[d_{\mathbb{D}}(u, v) + d_{\mathbb{D}}(v, w)] \\
&= \frac{\tanh \frac{1}{2} d_{\mathbb{D}}(u, v) + \tanh \frac{1}{2} d_{\mathbb{D}}(v, w)}{1 + \tanh \frac{1}{2} d_{\mathbb{D}}(u, v) \tanh \frac{1}{2} d_{\mathbb{D}}(v, w)} \\
&< \tanh \frac{1}{2} d_{\mathbb{D}}(u, v) + \tanh \frac{1}{2} d_{\mathbb{D}}(v, w) \\
&= p_{\mathbb{D}}(u, v) + p_{\mathbb{D}}(v, w).
\end{aligned}$$

This also shows that there is always a strict inequality in the triangle inequality for $p_{\mathbb{D}}$ for any three distinct points. ■

The following example illustrates how the hyperbolic distance compares with the Euclidean distance in $\mathbb{D}$.

Example 2.5. The Poincaré model of the hyperbolic plane does not accurately reflect all of the properties of the hyperbolic plane. For example, the hyperbolic plane is homogeneous; this means that for any pair of points a and b in $\mathbb{D}$ there is an isometry f with $f(a) = b$. Intuitively this means that the hyperbolic plane looks the same at each point just as the Euclidean plane does. However, with our Euclidean eyes, the origin seems to occupy a special place in the hyperbolic plane. In fact, in the hyperbolic plane the origin is no more special than any point $a \neq 0$.

Here is another way in which the Poincaré model deceives our Euclidean eyes. Let $x_0, x_1, x_2, \ldots$ be the sequence $0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$, so that $x_n = (2^n - 1)/2^n$, and x_{n+1} is halfway between x_n and 1 in the Euclidean sense. A computation using (2.5) shows that $d_{\mathbb{D}}(0, x_n) = \log(2^{n+1} - 1)$. We conclude that $d_{\mathbb{D}}(x_n, x_{n+1}) \to \log 2$ as $n \to \infty$; thus the points x_n are, for large n, essentially equally spaced in the hyperbolic sense along the real diameter of $\mathbb{D}$. Moreover, in any figure representing the Poincaré model the points x_n, for $n \geq 30$, are indistinguishable from the point 1 which does not lie in the hyperbolic plane. In brief, although the hyperbolic plane contains arbitrarily large hyperbolic disks about the origin, our Euclidean eyes can only see hyperbolic disks about the origin with a moderate sized hyperbolic radius.

Let us comment now on the various formulae that are available for $d_{\mathbb{D}}(z, w)$. It is often tempting to use the pseudohyperbolic distance $p_{\mathbb{D}}$ rather than the hyperbolic distance $d_{\mathbb{D}}$ (and many authors do) because the expression for $p_{\mathbb{D}}$ is algebraic whereas the expression for $d_{\mathbb{D}}$ is not. However, this temptation should be resisted. The distance $p_{\mathbb{D}}$ is not additive along geodesics, and it does not arise from a Riemannian metric. Usually, the solution is to use the following functions of $d_{\mathbb{D}}$, for it is these that tend to arise naturally and more frequently in hyperbolic trigonometry:

$$\sinh^2 \frac{1}{2} d_{\mathbb{D}}(z, w) = \frac{|z - w|^2}{(1 - |z|)^2(1 - |w|^2)} = \frac{1}{4}|z - w|^2 \lambda_{\mathbb{D}}(z) \lambda_{\mathbb{D}}(w), \tag{2.8}$$

and

$$\cosh^2 \frac{1}{2} d_{\mathbb{D}}(z, w) = \frac{|1 - z\bar{w}|^2}{(1 - |z|^2)(1 - |w|^2)} = \frac{1}{4}|1 - z\bar{w}|^2 \lambda_{\mathbb{D}}(z) \lambda_{\mathbb{D}}(w).$$

These can be proved directly from (2.5), and together they give the familiar formula

$$\tanh \frac{1}{2} d_{\mathbb{D}}(z, w) = \left| \frac{z - w}{1 - z\bar{w}} \right| = p_{\mathbb{D}}(z, w).$$

We investigate the topology defined on the unit disk by the hyperbolic distance. For this we study hyperbolic disks since they determine the topology. The hyperbolic circle C_r given by $\{z \in \mathbb{D} : d_{\mathbb{D}}(0, z) = r\}$ is a Euclidean circle with Euclidean center 0 and Euclidean radius $\tanh \frac{1}{2} r$. Now let C be any hyperbolic circle, say of hyperbolic radius r and hyperbolic center w. Then there is a hyperbolic isometry f with $f(w) = 0$, so that $f(C) = C_r$. As C_r is a Euclidean circle, so is $f^{-1}(C_r)$, which is C. Conversely, suppose that C is a Euclidean circle in $\mathbb{D}$.

Then there is a hyperbolic isometry f such that $f(C)$ is a Euclidean circle with center 0, so that $f(C) = C_r$ for some r. Thus, as f is a hyperbolic isometry, $f^{-1}(C_r) = C$, is also a hyperbolic circle. This shows that *the set of hyperbolic circles coincides with the set of Euclidean circles in* $\mathbb{D}$. As the same is obviously true for open disks (providing that the closed disks lie in $\mathbb{D}$), we see that *the topology induced by the hyperbolic distance on* $\mathbb{D}$ *coincides with the Euclidean topology on the unit disk.*

Theorem 2.6. *The topology induced by* $d_{\mathbb{D}}$ *on* $\mathbb{D}$ *coincides with the Euclidean topology. The space* $\mathbb{D}$ *with the distance* $d_{\mathbb{D}}$ *is a complete metric space.*

Proof. We have already proved the first statement. Suppose, then, that (z_n) is a Cauchy sequence with respect to the distance $d_{\mathbb{D}}$. Then (z_n) is a bounded sequence with respect to $d_{\mathbb{D}}$ and, as we have seen above, this means that the (z_n) lie in a compact disk K that is contained in $\mathbb{D}$. As $\lambda_{\mathbb{D}} \geq 2$ on $\mathbb{D}$, we see immediately from (2.8) that (z_n) is a Cauchy sequence with respect to the Euclidean metric, so that $z_n \to z^*$, say, where $z^* \in K \subset \mathbb{D}$. It is now clear that $d_{\mathbb{D}}(z_n, z^*) \to 0$ so that $\mathbb{D}$ with the distance $d_{\mathbb{D}}$ is complete. ■

The Euclidean metric on $\mathbb{D}$ arises from the fact that $\mathbb{D}$ is embedded in the larger space $\mathbb{C}$ and is not complete on $\mathbb{D}$. By contrast, an important property of the distance $d_{\mathbb{D}}$ is that $d_{\mathbb{D}}(0, |z|) \to +\infty$ as $|z| \to 1$; informally, the boundary $\partial\mathbb{D}$ of $\mathbb{D}$ is 'infinitely far away' from each point in $\mathbb{D}$. This is a consequence of the fact that $\mathbb{D}$ equipped with the hyperbolic distance $d_{\mathbb{D}}$ is a complete metric space and is another reason why $d_{\mathbb{D}}$ should be preferred to the Euclidean metric on $\mathbb{D}$.

Exercises.

1. Verify that (2.1) and (2.2) determine the same subgroup of Möbius transformations.
2. Suppose equality holds in the triangle inequality for the hyperbolic distance; that is, suppose u, v, w in $\mathbb{D}$ and $d_{\mathbb{D}}(u, w) = d_{\mathbb{D}}(u, v) + d_{\mathbb{D}}(v, w)$. Prove that u, v and w lie on a hyperbolic geodesic in this order.
3. Verify that the hyperbolic disk $D_{\mathbb{D}}(a, r)$ is the Euclidean disk with center c and radius R, where
$$c = \frac{a\left(1 - \tanh^2(r/2)\right)}{1 - |a|^2 \tanh^2(r/2)} \quad \text{and} \quad R = \frac{(1 - |a|^2)\tanh(r/2)}{1 - |a|^2 \tanh^2(r/2)}\,.$$
4. (a) Prove that the hyperbolic area of a hyperbolic disk of radius r is $4\pi \sinh^2(r/2)$.
(b) Show that the hyperbolic length of a hyperbolic circle with radius r is $2\pi \sinh r$.

3. The Schwarz-Pick Lemma

We begin with a statement of the classical Schwarz Lemma.

Theorem 3.1 (Schwarz's Lemma). *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and that $f(0) = 0$. Then either*
(a) $|f(z)| < |z|$ *for every non-zero z in $\mathbb{D}$, and $|f'(0)| < 1$, or*
(b) *for some real constant θ, $f(z) = e^{i\theta} z$ and $|f'(0)| = 1$.*

The Schwarz Lemma is proved by applying the Maximum Modulus Theorem to the holomorphic function $f(z)/z$ on the unit disk $\mathbb{D}$. It says that if a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ fixes 0 then either (a) $f(z)$ is closer to 0 than z is, or (b) f is a rotation of the plane about 0. Although both of these assertions are true in the context of Euclidean geometry, they are only invariant under conformal maps when they are interpreted in terms of *hyperbolic geometry*. Moreover, as Pick observed in 1915, in this case the requirement that f has a fixed point in $\mathbb{D}$ is redundant. We can now state Pick's invariant formulation of Schwarz's Lemma [33].

Theorem 3.2 (The Schwarz-Pick Lemma). *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic. Then either*
(a) *f is a hyperbolic contraction; that is, for all z and w in $\mathbb{D}$,*

$$d_{\mathbb{D}}\big(f(z), f(w)\big) < d_{\mathbb{D}}(z, w), \quad \lambda_{\mathbb{D}}\big(f(z)\big)|f'(z)| < \lambda_{\mathbb{D}}(z), \tag{3.1}$$

or
(b) *f is a hyperbolic isometry; that is, $f \in \mathcal{A}(\mathbb{D})$ and for all z and w in $\mathbb{D}$,*

$$d_{\mathbb{D}}\big(f(z), f(w)\big) = d_{\mathbb{D}}(z, w), \quad \lambda_{\mathbb{D}}\big(f(z)\big)|f'(z)| = \lambda_{\mathbb{D}}(z) \tag{3.2}$$

Proof. By Theorem 2.1, f is an isometry if and only if one, and hence both, of the conditions in (3.2) hold. Suppose now that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic but not an isometry. Select any two points z_1 and z_2 in $\mathbb{D}$. Here is the intuitive idea behind the proof. Because the hyperbolic plane is homogeneous, we may assume without loss of generality that both z_1 and $f(z_1)$ are at the origin. In this special situation (3.1) follows directly from part (b) of Theorem 3.1. Now we write out a formal argument. Let g and h be conformal automorphisms (and hence isometries) of $\mathbb{D}$ such that $g(z_1) = 0$ and $h\big(f(z_1)\big) = 0$. Let $F = hfg^{-1}$; then F is a holomorphic self-map of $\mathbb{D}$ that fixes 0. As g and h are isometries, F is not an isometry or else f would be too. Therefore, by Schwarz's Lemma, for all z, $d_{\mathbb{D}}\big(0, F(z)\big) < d_{\mathbb{D}}(0, z)$ and $|F'(0)| < 1$. Thus, as $Fg = hf$ and g, h are hyperbolic isometries,

$$\begin{aligned} d_{\mathbb{D}}\big(f(z_1), f(z_2)\big) &= d_{\mathbb{D}}\big(hf(z_1), hf(z_2)\big) \\ &= d_{\mathbb{D}}\big(Fg(z_1), Fg(z_2)\big) \\ &= d_{\mathbb{D}}\big(0, Fg(z_2)\big) \\ &< d_{\mathbb{D}}\big(0, g(z_2)\big) \\ &= d_{\mathbb{D}}\big(g(z_1), g(z_2)\big) \\ &= d_{\mathbb{D}}(z_1, z_2). \end{aligned}$$

This is the first inequality in (3.1). To obtain the second inequality, we apply the Chain Rule to each side of $Fg = hf$ and obtain

$$|F'(0)| = \frac{|f'(z_1)|(1-|z_1|^2)}{1-|f(z_1)|^2} < 1.$$

This gives the second inequality in (3.1) at an arbitrary point z_1. ∎

Often the Schwarz-Pick Lemma is stated in the following form: Every holomorphic self-map of the unit disk is a contraction relative to the hyperbolic metric. That is, if f is a holomorphic self-map of $\mathbb{D}$, then

$$d_{\mathbb{D}}\big(f(z), f(w)\big) \le d_{\mathbb{D}}(z, w), \quad \lambda_{\mathbb{D}}\big(f(z)\big)|f'(z)| \le \lambda_{\mathbb{D}}(z). \tag{3.3}$$

If equality holds in either inequality, then f is a conformal automorphism of $\mathbb{D}$. One should note that the two inequalities in (3.3) are equivalent. If the first inequality holds, then

$$\lambda_{\mathbb{D}}\big(f(z)\big)|f'(z)| = \lim_{w\to z} \frac{d_{\mathbb{D}}(f(z), f(w))}{|f(z)-f(w)|}\frac{|f(z)-f(w)|}{|z-w|} \le \lim_{w\to z}\frac{d_{\mathbb{D}}(z,w)}{|z-w|} = \lambda_{\mathbb{D}}(z).$$

On the other hand, if the second inequality holds, then integration over any path γ in $\mathbb{D}$ gives $\ell_{\mathbb{D}}(f\circ\gamma) \le \ell_{\mathbb{D}}(\gamma)$. This implies the first inequality in (3.3).

Hyperbolic geometry had been used in complex analysis by Poincaré in his proof of the Uniformization Theorem for Riemann surfaces. The work of Pick is a milestone in geometric function theory, it shows that the hyperbolic metric, not the Euclidean metric, is the natural metric for much of the subject. The definition of the hyperbolic metric might seem arbitrary. In fact, up to multiplication by a positive scalar it is the only metric on the unit disk that makes every holomorphic self-map a contraction, or every conformal automorphism an isometry.

Theorem 3.3. *For a metric $\rho(z)|dz|$ on the unit disk the following are equivalent:*
(a) *For any holomorphic self-map of $\mathbb{D}$ and all $z \in \mathbb{D}$, $\rho(f(z))|f'(z)| \le \rho(z)$;*
(b) *For any $f \in \mathcal{A}(\mathbb{D})$ and all $z\in\mathbb{D}$, $\rho(f(z))|f'(z)| = \rho(z)$;*
(c) *$\rho(z) = c\lambda_{\mathbb{D}}$ for some $c > 0$.*

Proof. (a)⇒(b) Suppose $f \in \mathcal{A}(\mathbb{D})$. Then the inequality in (a) holds for f. The inequality in (a) also holds for f^{-1}; this gives $\rho(z) \le \rho(f(z))|f'(z)|$. Hence, every conformal automorphism of $\mathbb{D}$ is an isometry relative to $\rho(z)|dz|$.

(b)⇒(c) Define $c > 0$ by $\rho(0) = c\lambda_{\mathbb{D}}(0)$. Now, consider any $a \in \mathbb{D}$. Let f be a conformal automorphism of $\mathbb{D}$ with $f(0) = a$. Then because f is an isometry relative to both $\rho(z)|dz|$ and the hyperbolic metric,

$$\begin{aligned}\rho(a)|f'(0)| &= \rho(0)\\ &= c\lambda_{\mathbb{D}}(0)\\ &= c\lambda_{\mathbb{D}}(a)|f'(0)|.\end{aligned}$$

Hence, $\rho(a) = c\lambda_{\mathbb{D}}(a)$ for all $a \in \mathbb{D}$.

(c)⇒(a) This is an immediate consequence of the Schwarz-Pick Lemma. ∎

Exercises.

1. Suppose f is a holomorphic self-map of the unit disk. Prove $|f'(0)| \leq 1$. Determine a necessary and sufficient condition for equality.
2. If a holomorphic self-map of the unit disk fixes two points, prove it is the identity.
3. Let a and b be distinct points in $\mathbb{D}$.
(a) Show that there exists a conformal automorphism f of $\mathbb{D}$ that interchanges a and b; that is, $f(a) = b$ and $f(b) = a$.
(b) Suppose a holomorphic self-map f of $\mathbb{D}$ interchanges a and b; that is, $f(a) = b$ and $f(b) = a$. Prove f is a conformal automorphism with order 2, or $f \circ f$ is the identity.

4. An extension of the Schwarz-Pick Lemma

Recently, the authors [8] established a multi-point version of the Schwarz-Pick Lemma that unified a number of known variations of the Schwarz and Schwarz-Pick Lemmas and also has many new consequences. A selection of results from [8] are presented in this and the next section; for more results of this type, consult the original paper.

We begin with a brief discussion of Blaschke products. A function $F : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is a *(finite) Blaschke product* if it is holomorphic in $\mathbb{D}$, continuous in $\overline{\mathbb{D}}$ (the closed unit disk), and $|F(z)| = 1$ when $|z| = 1$. If F is a Blaschke product then so are the compositions $g(F(z))$ and $F(g(z))$ for any conformal automorphism g of $\mathbb{D}$. In addition, it is clear that any finite product of conformal automorphisms of $\mathbb{D}$ is a Blaschke product. We shall now show that the converse is true. Suppose that F is a Blaschke product. If F has no zeros in $\mathbb{D}$ then, by the Minimum Modulus Theorem, F is a constant, which must be of modulus one. Now suppose that F does have a zero in $\mathbb{D}$. Then it can only have a finite number of zeros in $\mathbb{D}$, say $a_1, \dots, a_k$ (which need not be distinct), and

$$F(z) \Big/ \prod_{m=1}^{k} \left(\frac{z - a_m}{1 - \bar{a}_m z} \right)$$

is a Blaschke product with no zeros in $\mathbb{D}$. This shows that F *is a Blaschke product if and only if it is a finite product of automorphisms of* $\mathbb{D}$. We say that F is of *degree* k if this product has exactly k non-trivial factors.

We now discuss the complex pseudo-hyperbolic distance in $\mathbb{D}$, and the hyperbolic equivalent of the usual Euclidean difference quotient of a function.

Definition 4.1. The *complex pseudo-hyperbolic distance* $[z, w]$ between z and w in $\mathbb{D}$ is given by

$$[z, w] = \frac{z - w}{1 - \overline{w} z}.$$

We recall that the pseudo-hyperbolic distance is $|[z, w]|$; see (2.6).

The complex pseudo-hyperbolic distance is an analog for the hyperbolic plane $\mathbb{D}$ of the real directed distance $x - y$ from y to x, for points on the real line $\mathbb{R}$.

Definition 4.2. Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, and that $z, w \in \mathbb{D}$ with $z \neq w$. The *hyperbolic difference quotient* $f^*(z, w)$ is given by

$$f^*(z, w) = \frac{[f(z), f(w)]}{[z, w]}. \qquad \square$$

If we combine (2.6) with the Schwarz-Pick Lemma we see that

$$p_{\mathbb{D}}\big(f(z_1), f(z_2)\big) \leq p_{\mathbb{D}}(z_1, z_2),$$

and that equality holds for one pair z_1 and z_2 of distinct points if and only if f is a conformal automorphism of $\mathbb{D}$ (in which case, equality holds for all z_1 and z_2). It follows that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then either f is a hyperbolic isometry and $|f^*(z, w)| = 1$ for all z and w, or f is not an isometry and $|f^*(z, w)| < 1$ for all z and w.

We shall now discuss the hyperbolic difference quotient $f^*(z, w)$. This is a function of two variables but, unless we state explicitly to the contrary, *we shall regard it as a holomorphic function of the single variable* z. Note that $f^*(z, w)$ is *not* holomorphic as a function of the second variable w. The basic properties of $f^*(z, w)$ are given in our next result.

Theorem 4.3. *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, and that $w \in \mathbb{D}$.*
(a) *The function $z \mapsto f^*(z, w)$ is holomorphic in $\mathbb{D}$.*
(b) *If f is not a conformal automorphism of $\mathbb{D}$, then $z \mapsto f^*(z, w)$ is a holomorphic self-map $\mathbb{D}$.*
(c) *The map $z \mapsto f^*(z, w)$ is a conformal automorphism of $\mathbb{D}$ if and only if f is a Blaschke product of degree two.*

Proof. Part (a) is obvious as w is a removable singularity of the function

$$f^*(z, w) = \left(\frac{f(z) - f(w)}{1 - \overline{f(w)}f(z)}\right)\left(\frac{z - w}{1 - \overline{w}z}\right)^{-1}.$$

Now suppose that f is not a conformal automorphism of $\mathbb{D}$. Then, as we have seen above, $|f^*(z, w)| < 1$ and this proves (b).

To prove (c) we note first that there are conformal automorphisms g and h (that depend on w) of $\mathbb{D}$ such that $f^*(z, w) = g(f(z))/h(z)$ or, equivalently, $f(z) = g^{-1}\big(f^*(z, w)h(z)\big)$. Clearly, if $f^*(z, w)$ is an automorphism then f is a Blaschke product of degree two. Conversely, suppose that f is a Blaschke product of degree two. Then $g\big(f(z)\big)$ is also a Blaschke product, say B, of degree two and $f^*(z, w) = B(z)/h(z)$. As $f^*(z, w)$ is holomorphic in z, we see that $B(z) = h(z)h_1(z)$ for some automorphism h_1. Thus $f^*(z, w) = h_1(z)$ as required. $\blacksquare$

We shall now derive a *three-point* version of the Schwarz-Pick Lemma. Because it involves three points rather than two points as in the Schwarz-Pick Lemma, the following theorem has extra flexibility and it includes all variations and extensions of the Schwarz-Pick Lemma that are known to the authors. We stress, though, that this theorem contains much more than simply the union of all such known results. Although several Euclidean variations of the Schwarz-Pick Lemma are known, in our view much greater clarity is obtained by a strict adherence to hyperbolic geometry. This and other stronger versions of the Schwarz-Pick Lemma appear in [8].

Theorem 4.4 (Three-point Schwarz-Pick Lemma). *Suppose that f is holomorphic self-map of $\mathbb{D}$, but not an automorphism of $\mathbb{D}$. Then, for any z, w and v in $\mathbb{D}$,*

$$d_{\mathbb{D}}\big(f^*(z,v), f^*(w,v)\big) \le d_{\mathbb{D}}(z,w). \tag{4.1}$$

Further, equality holds in (4.1) *for some choice of z, w and v if and only if f is a Blaschke product of degree two.*

Proof. As f is holomorphic in $\mathbb{D}$, but not an automorphism, Theorem 4.3(b) shows that the left-hand side of (4.1) is defined. The inequality (4.1) now follows by applying the Schwarz-Pick Lemma to the holomorphic self-map $z \mapsto f^*(z,v)$ of $\mathbb{D}$. The Schwarz-Pick Lemma also implies that equality holds in (4.1) if and only if $f^*(z,w)$ is a conformal automorphism of $\mathbb{D}$ and, by Theorem 4.3(c), this is so if and only if f is a Blaschke product of degree two. ∎

Theorem 4.4 is a genuine improvement of the Schwarz-Pick Lemma. Suppose, for example that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, but not an automorphism, and that $f(0) = 0$. Then the Schwarz-Pick Lemma tells us only that $f(z)/z$ lies in the hyperbolic plane $\mathbb{D}$, and that $|f'(0)| < 1$. However, it we put $w = 0$ in (4.1), and then let $v \to 0$, we obtain the stronger conclusion that $f(z)/z$ lies in the hyperbolic disk with center $f'(0)$ and hyperbolic radius $d_{\mathbb{D}}(0,z)$.

Exercises.

1. If $f(z)$ is a Blaschke product of degree k, prove that $f^*(z,w)$ is a Blaschke product of degree $k-1$.
2. Verify the following Chain Rule for the $*$-operator: For all z and w in $\mathbb{D}$, and all holomorphic maps f and g of $\mathbb{D}$ into itself,
$$(f \circ g)^*(z,w) = f^*\big(g(z), g(w)\big)g^*(z,w).$$

5. Hyperbolic derivatives

Since the hyperbolic metric is the natural metric to study holomorphic self-maps of the unit disk, one should also use derivatives that are compatible with this metric. We begin with the definition of a hyperbolic derivative; just as the Euclidean difference quotient leads to the usual Euclidean derivative, the hyperbolic difference quotient results in the hyperbolic derivative.

Definition 5.1. Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, but not an isometry of $\mathbb{D}$. The *hyperbolic derivative* $f^h(w)$ of f at w in $\mathbb{D}$ is

$$f^h(w) = \lim_{z \to w} \frac{[f(z), f(w)]}{[z, w]} = \frac{(1-|w|^2)f'(w)}{1-|f(w)|^2}.$$

The *hyperbolic distortion* of f at w is

$$|f^h(w)| = \lim_{z \to w} \frac{d_{\mathbb{D}}(f(z), f(w))}{d_{\mathbb{D}}(z, w)}.$$

By Theorem 4.3, $|f^h(z)| \le 1$, and equality holds for some z if and only if equality holds for all z, and then f is a conformal automorphism of $\mathbb{D}$. Theorem 4.4 leads to the following upper bound on the magnitude of the hyperbolic difference quotient in terms of $d_{\mathbb{D}}(z, w)$ and the derivative at any point v between z and w.

Theorem 5.2. *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic. Then, for all z and w in $\mathbb{D}$, and for all v on the closed geodesic arc joining z and w,*

$$d_{\mathbb{D}}\big(0, f^*(z, w)\big) \le d_{\mathbb{D}}\big(0, f^h(v)\big) + d_{\mathbb{D}}(z, w). \tag{5.1}$$

Proof. First, it is clear that for any z and w, $|f^*(z, w)| = |f^*(w, z)|$. Thus

$$d_{\mathbb{D}}\big(0, f^*(z, w)\big) = d_{\mathbb{D}}\big(0, f^*(w, z)\big).$$

Next, Theorem 4.4 (applied twice) gives

$$\begin{aligned} d_{\mathbb{D}}\big(0, f^*(z, w)\big) &\le d_{\mathbb{D}}\big(0, f^*(v, w)\big) + d_{\mathbb{D}}\big(f^*(v, w), f^*(z, w)\big) \\ &\le d_{\mathbb{D}}\big(0, f^*(v, w)\big) + d_{\mathbb{D}}(z, v) \\ &= d_{\mathbb{D}}\big(0, f^*(w, v)\big) + d_{\mathbb{D}}(z, v) \\ &\le d_{\mathbb{D}}\big(0, f^*(u, v)\big) + d_{\mathbb{D}}(w, u) + d_{\mathbb{D}}(z, v). \end{aligned}$$

We now let $u \to v$, where v lies on the geodesic between z and w, and as $d_{\mathbb{D}}(z, v) + d_{\mathbb{D}}(v, w) = d_{\mathbb{D}}(z, w)$, we obtain (5.1). ■

Our next task is to transform (5.1) into a more transparent inequality about f. This is the next result which we may interpret as a *Hyperbolic Mean Value Inequality*, a result from [7].

Theorem 5.3 (Hyperbolic Mean Value Inequality)**.** *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic. Then, for all z and w in $\mathbb{D}$, and for all v on the closed geodesic arc joining z and w,*

$$d_{\mathbb{D}}\big(f(z), f(w)\big) \le \log\Big(\cosh d_{\mathbb{D}}(z, w) + |f^h(v)| \sinh d_{\mathbb{D}}(z, w)\Big). \tag{5.2}$$

This inequality is sharper than the Schwarz-Pick inequality for if we use $|f^h(v)| \le 1$ and the identity $\cosh t + \sinh t = e^t$, we recapture the Schwarz-Pick inequality. It is known that equality holds in (5.2) if and only if f is a Blaschke product of degree two and has a unique critical point c, such that either c, $z = v$, w, or c, $w = v$, z, lie in this order along a geodesic. We refer the reader to [8] for a proof of this, and for the fact that a Blaschke product of degree two has exactly one critical point in $\mathbb{D}$.

Proof. First, we note that, for all u and v,

$$\tanh \frac{1}{2} d_{\mathbb{D}}(0,u) = |u|,$$
$$\tanh \frac{1}{2} d_{\mathbb{D}}(u,v) = p_{\mathbb{D}}(u,v) = \tanh \frac{1}{2} d_{\mathbb{D}}(0,[u,v]).$$

Next, using the definition of $f^*(z,w)$, the inequality in Theorem 5.2, and the addition formula for $\tanh(s+t)$, we have

$$\begin{aligned}\tanh \frac{1}{2} d_{\mathbb{D}}(f(z),f(w)) &= p_{\mathbb{D}}(f(z),f(w)) \\ &= p_{\mathbb{D}}(z,w)|f^*(z,w)| \\ &= p_{\mathbb{D}}(z,w)\ \tanh \frac{1}{2} d_{\mathbb{D}}\big(0,f^*(z,w)\big) \\ &\le p_{\mathbb{D}}(z,w)\ \tanh\Big[\frac{1}{2} d_{\mathbb{D}}\big(0,f^h(v)\big) + \frac{1}{2} d_{\mathbb{D}}(z,w)\Big] \\ &= p_{\mathbb{D}}(z,w)\ \left(\frac{p_{\mathbb{D}}(z,w)+|f^h(v)|}{1+p_{\mathbb{D}}(z,w)|f^h(v)|}\right).\end{aligned}$$

Now the increasing function $x \mapsto \tanh(\frac{1}{2}x)$ has inverse $x \mapsto \log(1+x)/(1-x)$, so we conclude that, with $p = p_{\mathbb{D}}(z,w)$ and $d = |f^h(v)|$,

$$d_{\mathbb{D}}(f(z),f(w)) \le \log\left(\frac{1+pd+p(d+p)}{1+pd-p(d+p)}\right) = \log\left(\frac{1+p^2}{1-p^2} + d\frac{2p}{1-p^2}\right),$$

which is (5.2). ∎

Next, we provide a Schwarz-Pick type of inequality for hyperbolic derivatives; recall that the hyperbolic derivative is not holomorphic. This result is based on the observation that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, but not a conformal automorphism of $\mathbb{D}$, then $f^h(z)$ and $f^h(w)$ lie in $\mathbb{D}$ so that we can measure the hyperbolic distance between these two hyperbolic derivatives.

Theorem 5.4. *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic but not a conformal automorphism of $\mathbb{D}$. Then, for all z and w in $\mathbb{D}$,*

$$d_{\mathbb{D}}\big(f^h(z),f^h(w)\big) \le 2d_{\mathbb{D}}(z,w) + d_{\mathbb{D}}\big(f^*(z,w),f^*(w,z)\big). \tag{5.3}$$

Proof. Theorem 4.4 implies that for all z, w and v,

$$d_{\mathbb{D}}\big(f^*(z,w),f^*(v,w)\big) \le d_{\mathbb{D}}(z,v).$$

We let $v \to w$ and obtain

$$d_{\mathbb{D}}\big(f^*(z,w),f^h(w)\big) \le d_{\mathbb{D}}(z,w)$$

and (by interchanging z and w),

$$d_{\mathbb{D}}\big(f^*(w,z),f^h(z)\big) \le d_{\mathbb{D}}(z,w).$$

These last two inequalities and the triangle inequality yields (5.3). ∎

It is easy to see that if $f(0) = 0$ then $f^*(z, 0) = f^*(0, z) = f(z)/z$. Thus we have the following corollary originally established in [6].

Corollary 5.5. *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic but not a conformal automorphism of $\mathbb{D}$, and that $f(0) = 0$. Then, for all z,*

$$d_{\mathbb{D}}\big(f^h(0), f^h(z)\big) \le 2d_{\mathbb{D}}(0, z), \tag{5.4}$$

and the constant 2 is best possible.

Example 5.6. The preceding corollary is sharp for $f(z) = z^2$. Note that $f^h(z) = 2z/(1 + |z|^2)$ and $d_{\mathbb{D}}(f^h(z), f^h(w)) = 2d_{\mathbb{D}}(z, w)$ whenever z, w lie on the same hyperbolic geodesic through the origin. Thus, $z \mapsto f^h(z)$ doubles all hyperbolic distances along geodesics through the origin; this doubling is not valid in general because in hyperbolic geometry there are no similarities except isometries. Moreover, it is possible to verify that there is no finite K such that $d_{\mathbb{D}}(f^h(z), f^h(w)) \le K d_{\mathbb{D}}(z, w)$ for all z, $w \in \mathbb{D}$, so $z \mapsto f^h(z)$ does not even satisfy a hyperbolic Lipschitz condition, so (5.4) is no longer valid when the origin is replaced by an arbitrary point of the unit disk.

In spite of Example 5.6 a full-fledged result of Schwarz-Pick type is valid for the hyperbolic distortion.

Corollary 5.7 (Schwarz-Pick Lemma for Hyperbolic Distortion)**.** *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic but not a conformal automorphism of $\mathbb{D}$. Then for all $z, w \in \mathbb{D}$, $d_{\mathbb{D}}\big(|f^h(z)|, |f^h(w)|\big) \le 2d_{\mathbb{D}}(z, w)$.*

Proof. Note that, from the proof of Theorem 5.4,

$$d_{\mathbb{D}}\big(|f^*(z, w)|, |f^h(w)|\big) \le d_{\mathbb{D}}\big(f^*(z, w), f^h(w)\big) \le d_{\mathbb{D}}(z, w),$$

and, similarly, $d_{\mathbb{D}}\big(|f^*(w, z)|, |f^h(z)|\big) \le d_{\mathbb{D}}(z, w)$. As $|f^*(w, z)| = |f^*(z, w)|$, the desired inequality follows. ■

Exercises.

1. Verify the claims in Example 5.6.
2. Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic but not a conformal automorphism of $\mathbb{D}$. Prove that for all conformal automorphisms S and T of $\mathbb{D}$, and all z and w in $\mathbb{D}$,
$$|(S \circ f \circ T)^*(z, w)| = |f^*(T(z), T(w))|.$$
In particular, deduce that the hyperbolic derivative is invariant in the sense that
$$|(S \circ f \circ T)^h(z)| = |f^h(T(z))|.$$

6. The hyperbolic metric on simply connected regions

There are several equivalent definitions of what it means for a region in the complex plane to be simply connected. A region Ω in $\mathbb{C}$ is *simply connected* if and only if any one of the following (equivalent) conditions hold:

(a) the set $\mathbb{C}_\infty \backslash \Omega$ is connected;
(b) if f is holomorphic and never zero in Ω, then there is a single-valued holomorphic choice of $\log f$ in Ω;
(c) each closed curve in Ω can be continuously deformed within Ω to a point of Ω.

A region in $\mathbb{C}_\infty$ is simply connected if (a) or (c) holds. The regions $\mathbb{D}$, $\mathbb{C}$ and $\mathbb{C}_\infty$ are all simply connected; an annulus is not.

Two subregions regions of $\mathbb{C}$ are *conformally equivalent* if there is a holomorphic bijection of one onto the other. This is an equivalence relation on the class of subregions of $\mathbb{C}$, and the fundamental result about simply connected regions is the Riemann Mapping Theorem.

Theorem 6.1 (The Riemann Mapping Theorem)**.** *A subregion of $\mathbb{C}$ is conformally equivalent to $\mathbb{D}$ if and only if Ω is a simply connected proper subregion of $\mathbb{C}$. Moreover, given $a \in \Omega$ there is a unique conformal mapping $f : \Omega \to \mathbb{D}$ such that $f(a) = 0$ and $f'(a) > 0$.*

The Riemann Mapping Theorem enables us to transfer the hyperbolic metric from $\mathbb{D}$ to any simply connected proper subregion Ω of $\mathbb{C}$.

Definition 6.2. Suppose that f is a conformal map of a simply connected plane region Ω onto $\mathbb{D}$. Then the hyperbolic metric $\lambda_\Omega(z)|dz|$ of Ω is defined by

$$\lambda_\Omega(z) = \lambda_{\mathbb{D}}\big(f(z)\big)|f'(z)|. \tag{6.1}$$

The hyperbolic distance d_Ω is the distance function on Ω derived from the hyperbolic metric.

We need to show λ_Ω is independent of the choice of the conformal map f that is used in (6.1), for this will imply that λ_Ω is determined by Ω alone. Suppose, then, that f is a conformal map of Ω onto $\mathbb{D}$. Then the set of all conformal maps of Ω onto $\mathbb{D}$ is given by $h \circ f$, where h ranges over $\mathcal{A}(\mathbb{D})$. Any conformal automorphism h of $\mathbb{D}$ is a hyperbolic isometry, so that for all w in $\mathbb{D}$,

$$\lambda_{\mathbb{D}}(w) = \lambda_{\mathbb{D}}\big(h(w)\big)|h'(w)|.$$

If we now let $g = h \circ f$, $w = f(z)$ and use the Chain Rule we find that

$$\begin{aligned}\lambda_{\mathbb{D}}\big(g(z)\big)|g'(z)| &= \lambda_{\mathbb{D}}\big(h(f(z)\big)|h'(f(z))||f'(z)| \\ &= \lambda_{\mathbb{D}}(f(z))|f'(z)|\end{aligned}$$

so that λ_Ω as defined in (6.1) is independent of the choice of the conformal map f.

Thus, Definition 6.2 converts every conformal map of a simply connected proper subregion of $\mathbb{C}$ onto the unit disk into an isometry of the hyperbolic metric. The hyperbolic distance d_Ω on a simply connected proper subregion Ω of $\mathbb{C}$ can be defined in two equivalent ways. First, one can pull-back the hyperbolic distance on $\mathbb{D}$ to Ω by setting $d_\Omega(z, w) = d_{\mathbb{D}}(f(z), f(w))$ for any conformal map $f : \Omega \to \mathbb{D}$ and verifying that this is independent of the choice of the conformal mapping onto the unit disk. Alternatively, the hyperbolic length of a path γ in Ω is

$$\ell_\Omega(\gamma) = \int_\gamma \lambda_\Omega(z)|dz|,$$

and one can define

$$d_\Omega(z,w) = \inf \ell_\Omega(\gamma),$$

where the infimum is taken over all piecewise smooth curves γ in Ω that join z and w. These two definitions of the hyperbolic distance are equivalent. The hyperbolic distance d_Ω on Ω is complete. Moreover, a path γ in Ω connecting z and w is a hyperbolic geodesic in Ω if and only if $f \circ \gamma$ is a hyperbolic geodesic in $\mathbb{D}$. Also, for any $a \in \Omega$ and $r > 0$, $f(D_\Omega(a,r)) = D_\mathbb{D}(f(a), r)$.

In fact, the essence of Definition 6.2 is that the entire body of geometric facts about the Poincaré model $\mathbb{D}$ of the hyperbolic plane transfers, without any essential change, to an arbitrary simply connected proper subregion of $\mathbb{C}$ with its own hyperbolic metric. If $f : \Omega \to \mathbb{D}$ is any conformal mapping, then f is an isometry relative to the hyperbolic metrics and hyperbolic distances on Ω and $\mathbb{D}$. The next result is an immediate consequence of Definition 6.2 and we omit its proof; it asserts that all conformal maps of simply connected proper regions are isometries relative to the hyperbolic metrics and hyperbolic distances of the regions.

Theorem 6.3 (Conformal Invariance)**.** *Suppose that Ω_1 and Ω_2 are simply connected proper subregions of $\mathbb{C}$, and that f is a conformal map of Ω_1 onto Ω_2. Then f is a hyperbolic isometry, so that for any z in Ω_1,*

$$\lambda_{\Omega_2}\big(f(z)\big)|f'(z)| = \lambda_{\Omega_1}(z), \tag{6.2}$$

and for all z, $w \in \Omega_1$

$$d_{\Omega_2}(f(z), f(w)) = d_{\Omega_1}(z,w).$$

Note that if γ is a smooth curve in Ω_1, then (6.2) implies

$$\ell_{\Omega_2}(f \circ \gamma) = \ell_{\Omega_1}(\gamma).$$

Theorem 6.3 implies that each element of $\mathcal{A}(\Omega)$, the group of conformal automorphisms of Ω, is a hyperbolic isometry.

Theorem 6.4 (Schwarz-Pick Lemma for Simply Connected Regions)**.** *Suppose that Ω_1 and Ω_2 are simply connected proper subregions of $\mathbb{C}$, and that f is a holomorphic map of Ω_1 into Ω_2. Then either*
(a) *f is a hyperbolic contraction; that is, for all z and w in Ω_1,*

$$d_{\Omega_2}\big(f(z), f(w)\big) < d_{\Omega_1}(z,w), \quad \lambda_{\Omega_2}\big(f(z)\big)|f'(z)| < \lambda_{\Omega_1}(z),$$

or
(b) *f is a hyperbolic isometry; that is, f is a conformal map of Ω_1 onto Ω_2 and for all z and w in Ω_1,*

$$d_{\Omega_2}\big(f(z), f(w)\big) = d_{\Omega_1}(z,w), \quad \lambda_{\Omega_2}\big(f(z)\big)|f'(z)| = \lambda_{\Omega_1}(z).$$

Proof. Because of Theorem 6.3 we only need verify (a) when the holomorphic map $f : \Omega_1 \to \Omega_2$ is not a holomorphic bijection. Choose any point z_0 in Ω_1, and let $w_0 = f(z_0)$. Next, construct a holomorphic bijection h of $\mathbb{D}$, and a holomorphic bijection g of Ω_1 onto Ω_2; these can be constructed so that $h(0) = z_0$ and $g(z_0) = w_0 = f(z_0)$. Now let $k = (gh)^{-1} fh$. Then k is a holomorphic map of $\mathbb{D}$ into itself and $k(0) = 0$. Moreover, k is not a conformal automorphism of $\mathbb{D}$ or else f would be a holomorphic bijection. Thus $|k'(0)| < 1$ and, using the Chain Rule, this gives $|f'(z_0)| < |g'(z_0)|$. With this,

$$\lambda_{\Omega_2}\big(f(z_0)\big)|f'(z_0)| < \lambda_{\Omega_1}(z_0)$$

follows as (6.2) holds (with f replaced by g).

This establishes the second strict inequality in (a); the first strict inequality for hyperbolic distances follows by integrating the strict inequality for hyperbolic metrics. ■

This version of the Schwarz-Pick Lemma can be stated in the following equivalent form. If $f : \Omega_1 \to \Omega_2$ is holomorphic, then for all z and w in Ω_1,

$$d_{\Omega_2}(f(z), f(w)) \le d_{\Omega_1}(z, w), \tag{6.3}$$

and

$$\lambda_{\Omega_2}\big(f(z)\big)|f'(z)| \le \lambda_{\Omega_1}(z). \tag{6.4}$$

Further, if either equality holds in (6.3) for a pair of distinct points or at one point z in (6.4) , then f is a conformal bijection of Ω_1 onto Ω_2.

Corollary 6.5 (Schwarz's Lemma for Simply Connected Regions)**.** *Suppose Ω is a simply connected proper subregion of Ω and $a \in \Omega$. If f is a holomorphic self-map of Ω that fixes a, then $|f'(a)| \le 1$ and equality holds if and only if $f \in \mathcal{A}(\Omega, a)$, the group of conformal automorphisms of Ω that fix a. Moreover, $f'(a) = 1$ if and only if f is the identity.*

Theorem 6.4 is the fundamental reason for the existence of many distortion theorems in complex analysis. Consider the class of holomorphic maps of Ω_1 *into* Ω_2. Then any such map f will have to satisfy the *universal constraints* (6.3) and (6.4) where the metrics λ_{Ω_1} and λ_{Ω_2} are uniquely determined (albeit implicitly) by the regions Ω_1 and Ω_2. Thus (6.3) and (6.4) are, in some sense, the generic distortion theorems for holomorphic maps.

This is the appropriate place to point out that neither the complex plane $\mathbb{C}$ nor the extended complex plane $\mathbb{C}_\infty$ has a metric analogous to the hyperbolic metric in the sense that the metric is invariant under the group of conformal automorphisms. Recall that $\mathcal{A}(\mathbb{C})$ is the set of all maps $z \mapsto az + b$, $a, b \in \mathbb{C}$ and $a \neq 0$, and $\mathcal{A}(\mathbb{C}_\infty)$ is the group $\mathcal{M}$ of Möbius transformations. The group $\mathcal{A}(\mathbb{C})$ acts doubly transitively on $\mathbb{C}$; that is, given two pairs z_1, z_2 and w_1, w_2 of distinct points in $\mathbb{C}$ there is a conformal automorphism f of $\mathbb{C}$ with $f(z_j) = w_j$, $j = 1, 2$. Similarly, $\mathcal{M}$ acts triply transitively on $\mathbb{C}_\infty$. If there were a conformal metric on either $\mathbb{C}$ or $\mathbb{C}_\infty$ invariant under the full conformal automorphism group, then the distance function induced from this metric would also be invariant under the action of the full group of conformal automorphisms. The following result shows that only trivial distance functions are invariant under $\mathcal{A}(\mathbb{C})$ or $\mathcal{A}(\mathbb{C}_\infty)$.

Theorem 6.6. *If d is a distance function on $\mathbb{C}$ or $\mathbb{C}_\infty$ that is invariant under the full group of conformal automorphisms, then there exists $t > 0$ such that $d(z, w) = 0$ if $z = w$ and $d(z, w) = t$ otherwise.*

Proof. Let d be a distance function on $\mathbb{C}$ that is invariant under $\mathcal{A}(\mathbb{C})$. Set $t = d(0, 1)$. Consider any distinct z, $w \in \mathbb{C}$. Because $\mathcal{A}(\mathbb{C})$ acts doubly transitively on $\mathbb{C}$, there exists $f \in \mathcal{A}(\mathbb{C})$ with $f(0) = z$ and $f(1) = w$. The invariance of d under $\mathcal{A}(\mathbb{C})$ implies $d(z, w) = d(f(0), f(1)) = d(0, 1) = t$. The same argument applies to $\mathbb{C}_\infty$. ■

The Euclidean metric $|dz|$ on $\mathbb{C}$ is invariant under the proper subgroup of $\mathcal{A}(\mathbb{C})$ given by $z \mapsto az + b$, where $|a| = 1$ and $b \in \mathbb{C}$. The spherical metric $2|dz|/(1 + |z|^2)$ on $\mathbb{C}_\infty$ is invariant under the group of rotations of $\mathbb{C}_\infty$, that is, Möbius maps of the form

$$z \mapsto \frac{az - \bar{c}}{cz + \bar{a}}, \quad a, c \in \mathbb{C}, \quad |a|^2 + |c|^2 = 1,$$

or of the equivalent form

$$z \mapsto e^{i\theta} \frac{z - a}{1 + \bar{a}z}, \quad \theta \in \mathbb{R}, \quad a \in \mathbb{C}_\infty.$$

The group of rotations of $\mathbb{C}_\infty$ is a proper subgroup of $\mathcal{M}$.

Exercises.

1. Suppose Ω is a simply connected proper subregion of $\mathbb{C}$ and $a \in \Omega$. Let $\mathcal{F}$ denote the family of all holomorphic functions f defined on $\mathbb{D}$ such that $f(\mathbb{D}) \subseteq \Omega$ and $f(0) = a$. Set $M = \sup\{|f'(0)| : f \in \mathcal{F}\}$. Prove $M < +\infty$ and that $|f'(0)| = M$ if and only if f is a conformal map of $\mathbb{D}$ onto Ω with $f(0) = a$. Show $M = 2/\lambda_\Omega(a)$.
2. Suppose Ω is a simply connected proper subregion of $\mathbb{C}$ and $a \in \Omega$. Let $\mathcal{G}$ denote the family of all holomorphic functions f defined on Ω such that $f(\Omega) \subseteq \mathbb{D}$. Set $N = \sup\{|f'(a)| : f \in \mathcal{G}\}$. Prove $N < +\infty$ and that $|f'(a)| = N$ if and only if f is a conformal map of Ω onto $\mathbb{D}$ with $f(a) = 0$. Show $N = \lambda_\Omega(a)/2$.
3. Suppose Ω is a simply connected proper subregion of $\mathbb{C}$ and $a \in \Omega$. Let $\mathcal{H}(\Omega, a)$ denote the family of all holomorphic self-maps of Ω that fix a. Prove that $\{f'(a) : f \in \mathcal{H}(\Omega, a)\}$ equals the closed unit disk.

7. Examples of the hyperbolic metric

We give examples of simply connected regions and their hyperbolic metrics. These metrics are computed by using (6.2) in the following way: one finds an explicit conformal map f *from the region* Ω_1 *whose metric is sought* onto a region

Ω_2 *whose metric is known.* Then (6.2) enables one to find an explicit expression for $\lambda_{\Omega_1}(z)$ for z in Ω_1. We omit almost all of the computations.

The simplest instance of the Riemann Mapping Theorem is the fact that any disk or half-plane is Möbius equivalent to the unit disk. Because hyperbolic circles (disks) in $\mathbb{D}$ are Euclidean circle (disks) in $\mathbb{D}$, we deduce that an analogous result holds for any disk or half-plane. Also, in any disk or half-plane hyperbolic geodesics are arcs of circles orthogonal to the boundary; in the case of a half-planes we allow half-lines orthogonal to the edge of the half-plane.

Example 7.1 (disk)**.** As $f(z) = (z - z_0)/R$ is a conformal map of the disk $D = \{z : |z - z_0| < R\}$ onto $\mathbb{D}$, we find

$$\lambda_D(z)|dz| = \frac{2R\,|dz|}{R^2 - |z - z_0|^2}. \qquad \square$$

In particular,

$$\lambda_D(z_0) = \frac{2}{R}.$$

Example 7.2 (half-plane)**.** Let $\mathbb{H}$ be the upper half-plane $\{x+iy : y > 0\}$. Then $g(\mathbb{H}) = \mathbb{D}$, where $g(z) = (z - i)/(z + i)$, so $\mathbb{H} = \{x + iy : y > 0\}$ has hyperbolic metric

$$\lambda_{\mathbb{H}}(z)|dz| = \frac{|dz|}{y} = \frac{|dz|}{\operatorname{Im} z}.$$

Similarly, the hyperbolic metric of the *right half-plane* $\mathbb{K} = \{x + iy : x > 0\}$ is $|dz|/x$. More generally, if H is any open half-plane, then

$$\lambda_H(z)|dz| = \frac{|dz|}{d(z, \partial H)},$$

where $d(z, \partial H)$ denotes the Euclidean distance from z to ∂H.

Theorem 7.3. *If $f : \mathbb{D} \to \mathbb{K}$ is holomorphic and $f(0) = 1$, then*

$$\frac{1 - |z|}{1 + |z|} \le \operatorname{Re} f(z) \le \frac{1 + |z|}{1 - |z|} \tag{7.1}$$

and

$$|\operatorname{Im} f(z)| \le \frac{2|z|}{1 - |z|^2} \tag{7.2}$$

Proof. This is an immediate consequence of the Schwarz-Pick Lemma after converting the conclusion into weaker Euclidean terms. Fix $z \in \mathbb{D}$ and set

$$r = d_{\mathbb{D}}(0, z) = 2\tanh^{-1}|z| = \log\frac{1 + |z|}{1 - |z|}.$$

The Schwarz-Pick Lemma implies that $f(z)$ lies in the closed hyperbolic disk $\bar{D}_{\mathbb{K}}(1, r)$. The closed hyperbolic disk $\bar{D}_{\mathbb{K}}(1, r)$ has Euclidean center $\cosh r$, Euclidean radius $\sinh r$ and the bounding circle meets the real axis at e^{-r} and e^r;

see the exercises. Therefore, $f(z)$ lies in the closed Euclidean square $\{z = x+iy : e^{-r} \le x \le e^r, |y| \le \sinh r\}$. Since

$$e^{-r} = \frac{1-|z|}{1+|z|} \quad \text{and} \quad e^r = \frac{1+|z|}{1-|z|},$$

(7.1) is established. Finally,

$$\sinh r = \frac{2|z|}{1-|z|^2}$$

demonstrates (7.2). ∎

Theorem 7.4. *Suppose that $\mathcal{H}$ is any disk or half-plane. Then for all z and w in $\mathcal{H}$,*

$$\sinh^2 \frac{1}{2} d_{\mathcal{H}}(z,w) = \tfrac{1}{4}|z-w|^2 \lambda_{\mathcal{H}}(z)\lambda_{\mathcal{H}}(w).$$

Proof. It is easy to verify that for any Möbius map g we have

$$\big(g(z)-g(w)\big)^2 = (z-w)^2\, g'(z)\, g'(w). \tag{7.3}$$

Now take any Möbius map g that maps $\mathcal{H}$ onto $\mathbb{D}$, and recall that g is an isometry from $\mathcal{H}$ to $\mathbb{D}$ if both are given their hyperbolic metrics. Then, using (2.8) and (7.3)

$$\begin{aligned}
\tfrac{1}{4}|z-w|^2\lambda_{\mathcal{H}}(z)\lambda_{\mathcal{H}}(w) &= \tfrac{1}{4}|z-w|^2\lambda_{\mathbb{D}}\big(g(z)\big)\lambda_{\mathbb{D}}\big(g(w)\big)|g'(z)|\,|g'(w)| \\
&= \tfrac{1}{4}|g(z)-g(w)|^2\lambda_{\mathbb{D}}(g(z))\lambda_{\mathbb{D}}(g(w)) \\
&= \sinh^2 \frac{1}{2} d_{\mathbb{D}}(g(z),g(w)) \\
&= \sinh^2 \frac{1}{2} d_{\mathcal{H}}(z,w)
\end{aligned}$$

∎

There is another, less well known, version of the Schwarz-Pick Theorem available which is an immediate consequence of Theorem 7.4, and which we state in a form that is valid for all disks and half-planes.

Theorem 7.5 (Modified Schwarz-Pick Lemma for Disks and Half-Planes)**.** *Suppose that $\mathcal{H}_j$ is any disk or half-plane, $j = 1, 2$, and that $f : \mathcal{H}_1 \to \mathcal{H}_2$ is holomorphic. Then, for all z and w in $\mathcal{H}_1$,*

$$\frac{|f(z)-f(w)|^2}{|z-w|^2} \le \frac{\lambda_{\mathcal{H}_1}(z)\lambda_{\mathcal{H}_1}(w)}{\lambda_{\mathcal{H}_2}\big(f(z)\big)\lambda_{\mathcal{H}_2}\big(f(w)\big)}.$$

Proof. By Theorem 7.4 and the Schwarz-Pick Lemma

$$\begin{aligned}
\tfrac{1}{4}|f(z)-f(w)|^2\lambda_{\mathcal{H}_2}(f(z))\lambda_{\mathcal{H}_2}(f(w)) &= \sinh^2 \frac{1}{2} d_{\mathcal{H}_2}(f(z),f(w)) \\
&\le \sinh^2 \frac{1}{2} d_{\mathcal{H}_1}(z,w) \\
&= \tfrac{1}{4}|z-w)|^2\lambda_{\mathcal{H}_1}(z)\lambda_{\mathcal{H}_1}(w).
\end{aligned}$$

■

Observe that if $w \to z$ in Theorem 7.5, then we obtain (6.4) in the special case of disks and half-planes. We give an application of Theorem 7.5 to holomorphic functions.

Example 7.6. Suppose that f is holomorphic in the open unit disk and that f has positive real part. Then f maps $\mathbb{D}$ into $\mathbb{K}$, and we have

$$\frac{|f(z)-f(w)|^2}{|z-w|^2} \leq \frac{4\,\mathrm{Re}\,[f(z)]\,\mathrm{Re}\,[f(w)]}{(1-|z|^2)(1-|w|^2)}.$$

This implies, for example, that if we also have $f(0)=1$ then $|f'(0)| \leq 2$.

Example 7.7 (slit plane)**.** Since $f(z)=\sqrt{z}$ maps $P=\mathbb{C}\backslash\{x \in \mathbb{R} : x \leq 0\}$ onto $\mathbb{K}=\{x+iy : x>0\}$, the hyperbolic metric on P is

$$\lambda_P(z)\,|dz| = \frac{|dz|}{2|\sqrt{z}|\,\mathrm{Re}\,[\sqrt{z}]}.$$

This gives

$$\lambda_P(z) = \frac{1}{2r\cos(\theta/2)} \geq \frac{1}{2|z|},$$

where $z=re^{i\theta}$.

Example 7.8 (sector)**.** Let $S(\alpha)=\{z : 0<\arg(z)<\alpha\pi\}$, where $0<\alpha\leq 2$. Here, $f(z)=z^{1/\alpha}=\exp\left(\alpha^{-1}\log z\right)$ is a conformal map of $S(\alpha)$ onto $\mathbb{H}$, so $S(\alpha)$ has hyperbolic metric

$$\lambda_{S(\alpha)}(z)\,|dz| = \frac{|z|^{1/\alpha}}{\alpha|z|\,\mathrm{Im}[z^{1/\alpha}]}\,|dz|.$$

Note that this formula for the hyperbolic metric agrees (as it must) with the formula for $\lambda_{\mathbb{H}}$ in Example 7.2 (which is the case $\alpha=1$). The special case $\alpha=2$ is the preceding example.

Example 7.9 (doubly infinite strip)**.** $S=\{x+iy : |y|<\pi/2\}$ has hyperbolic metric

$$\lambda_S(z)\,|dz| = \frac{|dz|}{\cos y}.$$

In this case we use the fact that e^z maps S conformally onto $\mathbb{K}=\{x+iy : x>0\}$. Notice that $\lambda_S(z)\geq 1$ with equality if and only if z lies on the real axis. In particular, the hyperbolic distance between points on $\mathbb{R}$ is the same as the Euclidean distance between the points.

Theorem 7.10. *Let $S=\{z : |\mathrm{Im}(z)|<\pi/2\}$. Then for any $a\in\mathbb{R}$ and any holomorphic self-map f of S, $|f'(a)|\leq 1$. Moreover, $f'(a)=1$ if and only if $f(z)=z+c$ for some $c\in\mathbb{R}$ and $f'(a)=-1$ if and only if $f(z)=-z+c$ for some $c\in\mathbb{R}$. In particular, for any interval $[a,b]$ in $\mathbb{R}$, the Euclidean length of the image $f([a,b])$ is at most $b-a$.*

Proof. From Example 7.9 for $z \in S$

$$\lambda_S(z) = \frac{1}{\cos y} \geq 1$$

and equality holds if and only if $\mathrm{Im}(z) = 0$. This observation together with the Schwarz-Pick Lemma gives

$$|f'(a)| \leq \lambda_S(f(a))|f'(a)| \leq \lambda_S(a) = 1$$

and equality implies $f(a) \in \mathbb{R}$. In this case, $f - f(a) + a$ is a holomorphic self-map of S that fixes a and has derivative 1 at a, so it is the identity by the general form of Schwarz's Lemma. Thus, from Corollary 6.5 $f'(a) = 1$ implies $f(z) = z + c$ for some $c \in \mathbb{R}$; the converse is trivial. If $f'(a) = -1$, then $-f$ is a holomorphic self-map of S with derivative 1 at a, and so $f(z) = -z - c$ for some $c \in \mathbb{R}$. ∎

For a simply connected proper subregion Ω of $\mathbb{C}$ and $a \in \Omega$, each hyperbolic disk $D_\Omega(a, r) = \{z \in \Omega : d_\Omega(a, z) < r\}$ is simply connected and the closed disk $\bar{D}_\Omega(a, r)$ is compact. When Ω is a disk or half-plane, hyperbolic disks are Euclidean disks since any conformal map of the unit disk onto a disk or half-plane is a Möbius transformation. Of course, this is no longer true when Ω is simply connected and not a disk or half-plane. For particular types of simply connected regions, more can be said about hyperbolic disks than just the fact that they are simply connected.

Theorem 7.11. *Suppose Ω is a convex hyperbolic region. Then for any $a \in \Omega$ and all $r > 0$ the hyperbolic disc $D_\Omega(a, r)$ is Euclidean convex.*

Proof. Fix $a \in \Omega$. Let $h : \mathbb{D} \to \Omega$ be a conformal mapping with $h(0) = a$. Since $h(D_\mathbb{D}(0, r)) = D_\Omega(a, r)$, it suffices to show that h maps each disc $D_\mathbb{D}(0, r) = D(0, \tanh(r/2))$ onto a convex set. Set $R = \tanh(r/2)$. Given b, $c \in D(0, R)$ we must show $(1 - t)h(b) + th(c)$ lies in $h(D(0, R))$ for $t \in I$. Choose S so that $|b|, |c| < S < R$ and fix $t \in I$. The function

$$g(z) = (1 - t)h\left(\frac{bz}{S}\right) + th\left(\frac{cz}{S}\right)$$

is holomorphic in $\mathbb{D}$, $g(0) = a$ and maps into Ω because Ω is convex. Therefore, $f = h^{-1} \circ g$ is a holomorphic self-map of $\mathbb{D}$ that fixes the origin and so $f(D(0, R)) \subseteq D(0, R)$. Then $(1 - t)h(b) + th(c) = g(S) = h(f(S))$ lies in $h(D(0, S))$ because $f(S) \in D(0, S)$. Therefore, $h(D(0, S)) = D_\Omega(a, r)$ is Euclidean convex. ∎

This result is effectively due to Study who proved that if f is convex univalent in $\mathbb{D}$, then for any Euclidean disk D contained in $\mathbb{D}$, $f(D)$ is Euclidean convex, see [13]. The converse of Theorem 7.11 is elementary: If Ω is a simply connected proper subregion of $\mathbb{C}$ and there exists $a \in \Omega$ such that every hyperbolic disk $D_\Omega(a, r)$ is Euclidean convex, then Ω is Euclidean convex since $\Omega = \cup\{D_\Omega(a, r) : r > 0\}$, an increasing union of Euclidean convex sets. The radius of convexity for a univalent function on $\mathbb{D}$ is $2 - \sqrt{3}$; see [13]. This implies that if Ω is simply connected, then for each $a \in \Omega$ and $0 < r < (1/2)\log 3$ the hyperbolic disk $D_\Omega(a, r)$ is Euclidean convex.

In a general simply connected region hyperbolic geodesics are no longer arcs of circles or segments of lines. It is possible to give a simple geometric property of hyperbolic geodesics in Euclidean convex regions that characterize convex regions, see [19] and [20].

Exercises.

1. Let $\mathbb{K} = \{z = x + iy : x > 0\}$. For $a > 0$ and $r > 0$ verify that the closed hyperbolic disk $\bar{D}_{\mathbb{K}}(1, r)$ is the Euclidean disk with Euclidean center $c = \cosh r$ and Euclidean radius $R = \sinh r$. This Euclidean disk meets the real axis at e^{-r} and e^r.
2. Suppose $f : \mathbb{D} \to \mathbb{K}$ is holomorphic. Prove that
$$(1 - |z|^2)|f'(z)| \leq 2 \operatorname{Re} f(z)$$
for all $z \in \mathbb{D}$. When does equality hold?
3. Suppose $f : \mathbb{K} \to \mathbb{D}$ is holomorphic. Prove that
$$2|f'(z)| \operatorname{Re} z \leq 1 - |f(z)|^2$$
for all $z \in \mathbb{K}$. When does equality hold?
4. Suppose Ω is a simply connected proper subregion of $\mathbb{C}$ that is (Euclidean) starlike with respect to $a \in \Omega$. This means that for each $z \in \Omega$ the Euclidean segment $[a, z]$ is contained in Ω. For any $r > 0$ prove that the hyperbolic disk $D_\Omega(a, r)$ is starlike with respect to a.

8. The Comparison Principle

There is a powerful, and very general, Comparison Principle for hyperbolic metrics, which we state here only for simply connected plane regions. This Principle allows us to estimate the hyperbolic metric of a region in terms of other hyperbolic metrics which are known, or which can be more easily estimated. In general it is not possible to explicitly calculate the density of the hyperbolic metric, so estimates are useful.

Theorem 8.1 (Comparison Principle)**.** *Suppose that Ω_1 and Ω_2 are simply connected proper subregions of $\mathbb{C}$. If $\Omega_1 \subseteq \Omega_2$ then $\lambda_{\Omega_2} \leq \lambda_{\Omega_1}$ on Ω_1. Further, if $\lambda_{\Omega_1}(z) = \lambda_{\Omega_2}(z)$ at any point z of Ω_2, then $\Omega_1 = \Omega_2$ and $\lambda_{\Omega_1} = \lambda_{\Omega_2}$.*

Proof. Let $f(z) = z$ be the inclusion map of Ω_1 into Ω_2. Then the Schwarz-Pick Lemma gives $\lambda_{\Omega_2}(z) \leq \lambda_{\Omega_1}(z)$. If equality holds at a point, then f is a conformal bijection of Ω_1 onto Ω_2, that is, $\Omega_1 = \Omega_2$. ∎

In other words, the Comparison Principle asserts that the hyperbolic metric on a simply connected region decreases as the region increases. The hyperbolic metric on the disk $D_r = \{z : |z| < r\}$ is $2r|dz|/(r^2 - |z|^2)$ which decreases to zero as r increases to $+\infty$.

The Comparison Principle is used in the following way. Suppose that we want to estimate the hyperbolic metric λ_Ω of a region Ω. We attempt to find regions Ω_j with known hyperbolic metrics (or metrics that can be easily estimated) such that $\Omega_1 \subseteq \Omega \subseteq \Omega_2$; then $\lambda_{\Omega_2} \leq \lambda_\Omega \leq \lambda_{\Omega_1}$. The next result is probably the simplest application of the Comparison Principle, and it gives an upper bound of the hyperbolic metric λ_Ω of a region Ω in terms of the Euclidean distance

$$d(z, \partial\Omega) = \inf\{|z - w| : w \in \partial\Omega\}$$

of z to the boundary of Ω. The geometric significance of this quantity is that $d(z, \partial\Omega)$ is the radius of the largest open disk with center z that lies in Ω. Note, however, that $d(z, \partial\Omega)$ (which is sometimes denoted by $\delta_\Omega(z)$ in the literature) is not conformally invariant. The metric

$$\frac{|dz|}{d(z, \partial\Omega)} = \frac{|dz|}{\delta_\Omega(z)}$$

is called the *quasihyperbolic metric* on Ω. Example 7.2 shows that the quasihyperbolic metric for a half-plane is the hyperbolic metric.

Theorem 8.2. *Suppose that Ω is a simply connected proper subregion of $\mathbb{C}$. Then for all $z \in \Omega$*

$$\lambda_\Omega(z) \leq \frac{2}{d(z, \partial\Omega)}, \tag{8.1}$$

and equality holds if and only if Ω is a disk with center z.

Proof. Take any z_0 in Ω, and let $R = d(z_0, \partial\Omega)$ and $D = \{z : |z - z_0| < R\}$. As $D \subseteq \Omega$ the Comparison Principle and Example 7.1 yield

$$\lambda_\Omega(z_0) \leq \lambda_D(z_0) = \frac{2}{R} = \frac{2}{d(z_0, \partial\Omega)},$$

which is (8.1). If $\lambda_\Omega(z_0) = 2/d(z_0, \partial\Omega)$ then $\lambda_\Omega(z_0) = \lambda_D(z_0)$ so, by the Comparison Principle, $\Omega = D$. The converse is trivial. ∎

Theorem 8.2 gives an upper bound on the hyperbolic metric of Ω in terms of the Euclidean quantity $d(z, \partial\Omega)$. It is usually more difficult to obtain a lower bound on the hyperbolic metric. For convex regions it is easy to use geometric methods to obtain a good lower bound on the hyperbolic metric.

Theorem 8.3. *Suppose that Ω is convex proper subregion of $\mathbb{C}$. Then for all $z \in \Omega$*

$$\frac{1}{d(z, \partial\Omega)} \leq \lambda_\Omega(z) \tag{8.2}$$

and equality holds if and only if Ω is a half-plane.

Proof. We suppose that Ω is convex. Take any z in Ω and let ζ be one of the points on $\partial\Omega$ that is nearest to z. Let H be the supporting half-plane of Ω at ζ;

thus $\Omega \subseteq H$, and the Euclidean line that bounds H is orthogonal to the segment from z to ζ. Thus, from the Comparison Principle, for any $z \in \Omega$

$$\lambda_\Omega(z) \geq \lambda_H(z) = \frac{1}{|z-\zeta|} = \frac{1}{d(z, \partial\Omega)}$$

which is (8.2). The equality statement follows from the Comparison Principle and Example 7.2. ■

Theorems 8.2 and 8.3 show that the hyperbolic and quasihyperbolic metrics are bi-Lipschitz equivalent on convex regions:

$$\frac{1}{d(z, \partial\Omega)} \leq \lambda_\Omega(z) \leq \frac{2}{d(z, \partial\Omega)}.$$

Lower bounds for the hyperbolic metric in terms of the quasihyperbolic metric are equivalent to covering theorems for univalent functions.

Theorem 8.4. *Suppose that f is holomorphic and univalent in $\mathbb{D}$, and that $f(\mathbb{D})$ is a convex region. Then $f(\mathbb{D})$ contains the Euclidean disk with center $f(0)$ and radius $|f'(0)|/2$.*

Proof. From Theorem 8.3 we have

$$\begin{aligned} 2 &= \lambda_{\mathbb{D}}(0) \\ &= \lambda_{f(\mathbb{D})}\big(f(0)\big)|f'(0)| \\ &\geq \frac{|f'(0)|}{d\big(f(0), \partial f(\mathbb{D})\big)}. \end{aligned}$$

We deduce that

$$d\big(f(0), \partial f(\mathbb{D})\big) \geq |f'(0)|/2,$$

so that $f(\mathbb{D})$ contains the Euclidean disk with center $f(0)$ and radius $|f'(0)|/2$. ■

There is an analogous covering theorem for general univalent functions on the unit disk, see [13].

Theorem 8.5 (Koebe 1/4–Theorem)**.** *Suppose that f is holomorphic and univalent in $\mathbb{D}$. Then the region $f(\mathbb{D})$ contains the open Euclidean disk with center $f(0)$ and radius $|f'(0)|/4$.*

The Koebe 1/4–Theorem gives a lower bound on the hyperbolic metric in terms of the quasihyperbolic metric on a simply connected proper subregion of $\mathbb{C}$.

Theorem 8.6. *Suppose that Ω is a simply connected proper subregion of $\mathbb{C}$. Then for all $z \in \Omega$*

$$\frac{1}{2d(z, \partial\Omega)} \leq \lambda_\Omega(z) \tag{8.3}$$

and equality holds if and only if Ω is a slit-plane.

Proof. Fix $z \in \Omega$ and let $f : \mathbb{D} \to \Omega$ be a conformal map with $f(0) = z$. Koebe's 1/4–Theorem implies $d(z, \partial\Omega) \geq |f'(0)|/4$. Now

$$\begin{aligned} 2 &= \lambda_\Omega(f(0))|f'(0)| \\ &\leq 4\lambda_\Omega(z)d(z, \partial\Omega) \end{aligned}$$

which establishes (8.3). Sharpness follows from the sharp form of the Koebe 1/4–Theorem and Example 7.7. ■

Theorems 8.2 and 8.6 show that the hyperbolic and quasihyperbolic metrics are bi-Lipschitz equivalent on simply connected regions:

$$\frac{1}{2d(z, \partial\Omega)} \leq \lambda_\Omega(z) \leq \frac{2}{d(z, \partial\Omega)}. \tag{8.4}$$

Exercises.

1. (a) Suppose Ω is a simply connected proper subregion of $\mathbb{C}$. Prove that $\lim_{z\to\zeta} \lambda_\Omega(z) = +\infty$ for each boundary point ζ of Ω that lies in $\mathbb{C}$.
(b) Given an example of a simply connected proper subregion Ω of $\mathbb{C}$ that has ∞ as a boundary point and $\lambda_\Omega(z)$ does not tend to infinity as $z \to \infty$.
2. Suppose Ω is starlike with respect to the origin; that is, for each $z \in \Omega$ the Euclidean segment $[0, z]$ is contained in Ω. Use the Comparison Theorem to prove that (8.3) holds; do not use Theorem 8.6.

9. Curvature and the Ahlfors Lemma

A *conformal semimetric* on a region Ω in $\mathbb{C}$ is $\rho(z)|dz|$, where $\rho : \Omega \to [0, +\infty)$ is a continuous function and $\{z : \rho(z) = 0\}$ is a discrete subset of Ω. A conformal semimetric $\rho(z)|dz|$ is a *conformal metric* if $\rho(z) > 0$ for all $z \in \Omega$. The curvature of a conformal semimetric $\rho(z)|dz|$ can be defined at any point where ρ is positive and of class C^2.

Definition 9.1. Suppose $\rho(z)|dz|$ is a conformal metric on a region Ω. If $a \in \Omega$, $\rho(a) > 0$ and $\rho(z)$ is of class C^2 at a, then the *Gaussian curvature* of $\rho(z)|dz|$ at a is

$$K_\rho(a) = -\frac{\triangle \log \rho(a)}{\rho^2(a)},$$

where $\triangle$ is the usual (Euclidean) Laplacian,

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Typically, we shall speak of the curvature of a conformal metric rather than Gaussian curvature. In computing the Laplacian it is often convenient to use

$$\triangle = 4\frac{\partial^2}{\partial z \partial \bar{z}} = 4\frac{\partial^2}{\partial \bar{z} \partial z},$$

where the complex partial derivatives are defined by

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$

$$\bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Alternatively the Laplacian expressed can be expressed in terms of polar coordinates, namely

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

One reason semimetrics play an important role in complex analysis is that they transform simply under holomorphic functions.

Definition 9.2. Suppose $\rho(w)|dw|$ is a semimetric on a region Ω and $f : \Delta \to \Omega$ is a holomorphic function. The *pull-back* of $\rho(w)|dw|$ by f is

$$f^*(\rho(w)|dw|) = \rho(f(z))|f'(z)||dz|. \tag{9.1}$$

Since $(\rho \circ f)|f'|$ is a continuous nonnegative function defined on Δ, the pull-back $f^*(\rho(w)|dw|)$ of $\rho(w)|dw|$ is a semimetric on Δ provided f is nonconstant. Sometimes we write simply $f^*(\rho)$ to denote the pull-back. However, the notation (9.1) is precise and indicates that the formal substitution $w = f(z)$ converts $\rho(w)|dw|$ into $f^*(\rho(w)|dw|)$. The pull-back operation has several useful properties:

$$(f \circ g)^*(\rho(w)|dw|) = g^*\left(f^*(\rho(w)|dw|)\right)$$

and

$$(f^{-1})^* = (f^*)^{-1},$$

when f is a conformal mapping. If $f : \Omega_1 \to \Omega_2$ is a conformal mapping of simply connected proper subregions of $\mathbb{C}$, then the conclusion of Theorem 6.3 in the pull-back notation is: $f^*(\lambda_{\Omega_2}) = \lambda_{\Omega_1}$.

In the context of complex analysis, a fundamental property of the curvature is its conformal invariance. More generally, curvature is invariant under the pull-back operation.

Theorem 9.3. *Suppose Ω and Δ are regions in $\mathbb{C}$, $\rho(w)|dw|$ is a metric on Ω and $f : \Delta \to \Omega$ is a holomorphic function. Suppose $a \in \Delta$, $f'(a) \neq 0$, $\rho(f(a)) > 0$ and ρ is of class C^2 at $f(a)$. Then $K_{f^*(\rho)}(a) = K_\rho(f(a))$.*

Proof. Recall that $f^*(\rho(w)|dw|) = \rho(f(z))|f'(z)||dz|$. Now,

$$\begin{aligned}\log\left(\rho(f(z))|f'(z)|\right) &= \log\rho(f(z)) + \log|f'(z)| \\ &= \log\rho(f(z)) + \frac{1}{2}\log f'(z) + \frac{1}{2}\log\overline{f'(z)},\end{aligned}$$

so that

$$\frac{\partial}{\partial z}\log\left(\rho(f(z))|f'(z)|\right) = \frac{\partial\log\rho}{\partial w}(f(z))f'(z) + \frac{1}{2}\frac{f''(z)}{f'(z)}.$$

Then

$$\begin{aligned}\frac{\partial^2}{\partial\bar{z}\partial z}\log\left(\rho(f(z))|f'(z)|\right) &= \frac{\partial^2\log\rho}{\partial\bar{w}\partial w}(f(z))f'(z)\overline{f'(z)}\\ &= \frac{\partial^2\log\rho}{\partial\bar{w}\partial w}(f(z))|f'(z)|^2\end{aligned}$$

gives

$$\triangle_z[\log\left(\rho(f(z))|f'(z)|\right)] = \left(\triangle_w\log\rho\right)(f(z))|f'(z)|^2.$$

This is the transformation law for the Laplacian under a holomorphic function. Consequently,

$$\begin{aligned}K_{f^*(\rho)}(a) &= -\frac{\triangle_z\log(\rho(f(a))|f'(a)|)}{\rho^2(f(a))|f'(a)|^2}\\ &= -\frac{\left(\triangle_w\log\rho\right)(f(a))|f'(a)|^2}{\rho^2(f(a))|f'(a)|^2}\\ &= K_\rho(f(a)).\end{aligned}$$

■

Theorem 9.4. *The hyperbolic metric on a simply connected proper subregion of $\mathbb{C}$ has constant curvature -1.*

Proof. First, we establish the result for the unit disk. From

$$\lambda_{\mathbb{D}}(z) = \frac{2}{1-|z|^2} = \frac{2}{1-z\bar{z}}$$

we obtain

$$\begin{aligned}\frac{\partial^2}{\partial\bar{z}\partial z}\log\frac{2}{1-z\bar{z}} &= -\frac{\partial^2}{\partial\bar{z}\partial z}\log(1-z\bar{z})\\ &= \frac{\partial}{\partial\bar{z}}\frac{\bar{z}}{1-z\bar{z}}\\ &= \frac{1}{(1-z\bar{z})^2}.\end{aligned}$$

Consequently, $K_{\lambda_{\mathbb{D}}}(z) = -1$.

The general case of the hyperbolic metric on a simply connected proper subregion Ω of $\mathbb{C}$ follows from Theorem 9.3 since $f^*(\lambda_{\mathbb{D}}(w)|dw|) = \lambda_\Omega(z)|dz|$ for any conformal map $f:\Omega\to\mathbb{D}$. ■

Ahlfors recognized that the Schwarz-Pick Lemma was a consequence of an extremely important maximality property of the hyperbolic metric in $\mathbb{D}$.

Theorem 9.5 (Maximality of the hyperbolic metric)**.** *Suppose $\rho(z)|dz|$ is a C^2 semimetric on a simply connected proper subregion Ω of $\mathbb{C}$ such that $K_\rho(z)\le -1$ whenever $\rho(z)>0$. Then $\rho\le\lambda_\Omega$ on Ω.*

Proof. First, we assume $\Omega = \mathbb{D}$. Given z_0 in $\mathbb{D}$ choose any r satisfying $|z_0| < r < 1$. The hyperbolic metric on the disk $D_r = \{z : |z| < r\}$ is

$$\lambda_r(z) = \frac{2r}{r^2 - |z|^2}.$$

Consider the function

$$v(z) = \frac{\rho(z)}{\lambda_r(z)}$$

which is defined on the disk D_r. Then $v(z) \geq 0$ when $|z| < r$, and $v(z) \to 0$ as $|z| \to r$, so that v attains its maximum at some point a in D_r. It suffices to show that $v(a) \leq 1$ for then $v(z) \leq 1$ on D_r and we have

$$\rho(z_0) \leq \frac{2r}{r^2 - |z_0|^2}.$$

By letting $r \to 1$ we find that $\rho(z_0) \leq \lambda_{\mathbb{D}}(z_0)$.

We now show that $v(a) \leq 1$. If $\rho(a) = 0$, then $v(a) = 0 < 1$. Otherwise, $\rho(a) > 0$ and $K_\rho(a) \leq -1$. As a is a local maximum of v, it is also a local maximum of $\log v$ so that

$$\frac{\partial^2 \log v}{\partial x^2}(a) \leq 0, \quad \frac{\partial^2 \log v}{\partial y^2}(a) \leq 0.$$

We deduce that

$$\begin{aligned} 0 &\geq \big(\triangle \log v\big)(a) \\ &= \big(\triangle \log \rho\big)(a) - \big(\triangle \log \lambda_r\big)(a) \\ &= -K_\rho(a)\rho(a)^2 + K_{\lambda_r}(a)\lambda_r(a)^2 \\ &\geq \rho(a)^2 - \lambda_r(a)^2. \end{aligned} \tag{9.2}$$

This implies that $v(a) \leq 1$, and completes the proof in the special case $\Omega = \mathbb{D}$.

We now turn to the general case. Let $h : \mathbb{D} \to \Omega$ be a conformal mapping. Then $h^*(\rho(w)|dw|) := \tau(z)|dz|$ is a C^2 semimetric on $\mathbb{D}$ such that $K_\tau(z) \leq -1$ whenever $\tau(z) > 0$. Hence,

$$\rho(h(z))|h'(z)| \leq \lambda_{\mathbb{D}}(z) = \lambda_\Omega(h(z))|h'(z)|,$$

and so $\rho \leq \lambda_\Omega$ on Ω. ∎

In fact, Ahlfors actually established a more general result (see [1] and [2]). The stronger conclusion that either $\rho < \lambda_\Omega$ or else $\rho = \lambda_\Omega$ is valid but less elementary. This sharp result was established by Heins [15]. Simpler proofs of the stronger conclusion are due to Chen [12], Minda [28] and Royden [32].

The Schwarz-Pick Lemma is a special case of Theorem 9.5. If $f : \Omega_1 \to \Omega_2$ is a nonconstant holomorphic function, then $f^*(\lambda_{\Omega_2}(w)|dw|)$ is a semimetric on Ω_1 with curvature -1 at each point where f' is nonvanishing, so is dominated by the hyperbolic metric $\lambda_{\Omega_1}(z)|dz|$, or equivalently, (6.4) holds. The equality statement associated with (6.4) follows from the sharp version of Theorem 9.5.

Theorem 9.6. *There does not exist a* C^2 *semimetric* $\rho(z)|dz|$ *on* $\mathbb{C}$ *such that* $K_\rho(z) \le -1$ *whenever* $\rho(z) > 0$.

Proof. Suppose there existed a semimetric $\rho(z)|dz|$ on $\mathbb{C}$ such that $K_\rho(z) \le -1$ whenever $\rho(z) > 0$. Theorem 9.5 applied to the restriction of this metric to the disk $\{z : |z| < r\}$ gives

$$\rho(z) \le \lambda_r(z) = \frac{2r}{r^2 - |z|^2} \tag{9.3}$$

for $|z| < r$. If we fix z and let $r \to +\infty$, (9.3) gives $\rho(z) = 0$ for all $z \in \mathbb{C}$. This contradicts the fact that a semimetric vanishes only on a discrete set. ■

Corollary 9.7 (Liouville's Theorem)**.** *A bounded entire function is constant.*

Proof. Suppose f is a bounded entire function. There is no harm in assuming that $|f(z)| < 1$ for all $z \in \mathbb{C}$. If f were nonconstant, then $f^*(\lambda_{\mathbb{D}}(z)|dz|)$ would be a semimetric on $\mathbb{C}$ with curvature at most -1, a contradiction. ■

Theorem 9.3 provides a method to produce metrics with constant curvature -1. Loosely speaking, bounded holomorphic functions correspond to metrics with curvature -1. If $f : \Omega \to \mathbb{D}$ is holomorphic and locally univalent (f' does not vanish), then $f^*(\lambda_{\mathbb{D}}(z)|dz|)$ has curvature -1 on Ω. In fact, on a simply connected proper subregion of $\mathbb{C}$ *every* metric with curvature -1 has this form; see [36]. This reference also contains a stronger result that represents certain semimetrics with curvature -1 at points where the semimetric is nonvanishing by holomorphic (not necessarily locally univalent) maps of Ω into $\mathbb{D}$.

Theorem 9.8 (Representation of Negatively Curved Metrics)**.** *Let* $\rho(z)|dz|$ *be a* C^3 *conformal metric on a simply connected proper subregion* Ω *of* $\mathbb{C}$ *with constant curvature* -1. *Then* $\rho(z)|dz| = f^*(\lambda_{\mathbb{D}}(w)|dw|)$ *for some locally univalent holomorphic function* $f : \Omega \to \mathbb{D}$. *The function* f *is unique up to post-composition with an isometry of the hyperbolic metric. Given* $a \in \Omega$ *the function* f *representing the metric is unique if* f *is normalized by* $f(a) = 0$ *and* $f'(a) > 0$.

Moreover, $\rho(z)|dz| = f^*(\lambda_{\mathbb{D}}(w)|dw|)$ is complete if and only if f is a conformal bijection; that is, the hyperbolic metric is the only conformal metric on Ω that has curvature -1 and is complete.

Exercises.

1. Determine the curvature of the Euclidean metric $|dz|$ and of the spherical metric $\sigma(z)|dz| = 2|dz|/(1 + |z|^2)$.
2. Show that $(1 + |z|^2)|dz|$ has negative curvature on $\mathbb{C}$.
3. Determine the curvature of $e^x|dz|$ on $\mathbb{C}$.
4. Determine the curvature of $|dz|/|z|$ on $\mathbb{C} \setminus \{0\}$.
5. Prove there does not exist a semimetric on $\mathbb{C} \setminus \{0\}$ with curvature at most -1.
6. Prove the following extension of Liouville's Theorem: If f is an entire function and $f(\mathbb{C}) \subseteq \Omega$, where Ω is a simply connected proper subregion of $\mathbb{C}$, then f is constant.

10. The hyperbolic metric on a hyperbolic region

In order to transfer the hyperbolic metric from the unit disk to nonsimply connected regions, a substitute for the Riemann Mapping Theorem is needed. For this reason we must understand holomorphic coverings. For the general theory of topological covering spaces the reader should consult [22]. A holomorphic function $f : \Delta \to \Omega$ is called a *covering* if each point $b \in \Omega$ has an open neighborhood V such that $f^{-1}(V) = \cup\{U_\alpha : \alpha \in A\}$ is a disjoint union of open sets U_α such that $f|U_\alpha$, the restriction of f to U_α, is a conformal map of U_α onto V. Trivially, a conformal mapping $f : \Delta \to \Omega$ is a holomorphic covering. If Ω is simply connected, then the only holomorphic coverings $f : \Delta \to \Omega$ are conformal maps of a simply connected region Δ onto Ω.

Example 10.1. The complex exponential function $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is a holomorphic covering. Consider any $w \in \mathbb{C} \setminus \{0\}$ and let $\theta = \arg w$ be any argument for w. Let $V = \mathbb{C} \setminus \{-re^{i\theta} : r \geq 0\}$ be the complex plane slit from the origin along the ray opposite from w. Then $\exp^{-1}(V) = \bigcup\{S_n : n \in \mathbb{Z}\}$, where $S_n = \{z : \theta - n\pi < \operatorname{Im} z < \theta + n\pi\}$. Note that exp maps each horizontal strip S_n of width 2π conformally onto V.

A region Ω is called *hyperbolic* provided $\mathbb{C}_\infty \setminus \Omega$ contains at least three points. The unit disk covers every hyperbolic plane region; that is, there is a holomorphic covering $h : \mathbb{D} \to \Omega$ for any hyperbolic region Ω. As a consequence we demonstrate that every hyperbolic region has a hyperbolic metric that is real-analytic with constant curvature -1.

Theorem 10.2 (Planar Uniformization Theorem)**.** *Suppose Ω is a region in $\mathbb{C}$. There exists a holomorphic covering $f : \mathbb{D} \to \Omega$ if and only if Ω is a hyperbolic region. Moreover, if $a \in \Omega$, then there is a unique holomorphic universal covering $h : \mathbb{D} \to \Omega$ with $h(0) = a$ and $h'(0) > 0$.*

For a proof of the Planar Uniformization see [14] or [34]. If Ω is a simply connected hyperbolic region, then any holomorphic universal covering $h : \mathbb{D} \to \Omega$ is a conformal mapping. Therefore, the Riemann Mapping Theorem is a consequence of the Planar Uniformization Theorem. When Ω is a nonsimply connected hyperbolic region, then a holomorphic covering $h : \mathbb{D} \to \Omega$ is never injective. In fact, for each $a \in \Omega$, $h^{-1}(a)$ is an infinite discrete subset of $\mathbb{D}$. If $h : \mathbb{D} \to \Omega$ is one holomorphic universal covering, then $\{h \circ g : g \in \mathcal{A}(\mathbb{D})\}$ is the set of all holomorphic universal coverings of $\mathbb{D}$ onto Ω. The Planar Uniformization Theorem enables us to project the hyperbolic metric from the unit disk to any hyperbolic region.

Theorem 10.3. *Given a hyperbolic region Ω there is a unique metric $\lambda_\Omega(w)|dw|$ on Ω such that $h^*(\lambda_\Omega(w)|dw|) = \lambda_{\mathbb{D}}(z)|dz|$ for any holomorphic universal covering $h : \mathbb{D} \to \Omega$. The metric $\lambda_\Omega(w)|dw|$ is real-analytic with curvature -1.*

Proof. We construct a metric with curvature -1 on any hyperbolic region. First we define the metric locally. For a hyperbolic region Ω, let $h : \mathbb{D} \to \Omega$ be a holomorphic covering. A metric is defined on Ω as follows. For any simply connected subregion U of Ω let $H = h^{-1}$ denote a branch of the inverse that is holomorphic on U. Set $\lambda_\Omega(z) = \lambda_\mathbb{D}(H(z))|H'(z)|$. This defines a metric with curvature -1 on U. In fact, this defines a metric on Ω. Suppose U_1 and U_2 are simply connected subregions of Ω and $U_1 \cap U_2$ is nonempty. Let H_j be a single-valued holomorphic branch of h^{-1} defined on U_j. Then there is a $g \in \mathcal{A}(\mathbb{D})$ such that $H_2 = g \circ H_1$ locally on $U_1 \cap U_2$. Hence,

$$\begin{aligned} H_2^*(\lambda_\mathbb{D}(z)|dz|) &= (g \circ H_1)^*(\lambda_\mathbb{D}(z)|dz|) \\ &= H_1^*(g^*(\lambda_\mathbb{D}(z)|dz|)) \\ &= H_1^*(\lambda_\mathbb{D}(z)|dz|) \end{aligned}$$

since each conformal automorphism of $\mathbb{D}$ is an isometry of the hyperbolic metric $\lambda_\mathbb{D}(z)|dz|$. Therefore, $\lambda_\Omega(z)$ is defined independently of the branch of h^{-1} that is used and $h^*(\lambda_\Omega) = \lambda_\mathbb{D}$.

Moreover, this metric is independent of the covering. Suppose $k : \mathbb{D} \to \Omega$ is another covering. Then $k = h \circ g$ for some $g \in \mathcal{A}(\mathbb{D})$, and so

$$\begin{aligned} k^*(\lambda_\Omega) &= (h \circ g)^*(\lambda_\Omega) \\ &= g^*(h^*(\lambda_\Omega)) \\ &= g^*(\lambda_\mathbb{D}) \\ &= \lambda_\mathbb{D}. \end{aligned}$$

That λ_Ω is real-analytic is clear from its construction. That the curvature is -1 follows from $h^*(\lambda_\Omega) = \lambda_\mathbb{D}$ and Theorems 9.3 and 9.4. ■

The unique metric $\lambda_\Omega(w)|dw|$ on a hyperbolic region Ω given by Theorem 10.3 is called the *hyperbolic metric* on Ω. The hyperbolic distance on a hyperbolic region is complete. The hyperbolic distance d_Ω is defined by

$$d_\Omega(z, w) = \inf \ell_\Omega(\gamma),$$

where the infimum is taken over all piecewise smooth paths γ in Ω that joining z and w. Unlike the case of simply connected regions, a holomorphic covering $f : \mathbb{D} \to \Omega$ onto a multiply connected hyperbolic region is not an isometry, but only a *local isometry*. That is, each point $a \in \Omega$ has a neighborhood U such that $f|U$ is an isometry. In general, one can only assert that $d_\Omega(f(z), f(w)) \le d_\mathbb{D}(z, w)$ for $z, w \in \mathbb{D}$. When Ω is multiply connected, then f is not injective, so there exist distinct $z, w \in \mathbb{D}$ with $f(z) = f(w)$. In this situation, $d_\Omega(f(z), f(w)) = 0 < d_\mathbb{D}(z, w)$.

In general, the hyperbolic metric is not just invariant under conformal mappings, but is invariant under holomorphic coverings.

Theorem 10.4 (Covering Invariance)**.** *If $f : \Delta \to \Omega$ is a holomorphic covering of hyperbolic regions, then $f^*(\lambda_\Omega(w)|dw|) = \lambda_\Delta(z)|dz|$. In other words, every holomorphic covering of hyperbolic regions is a local isometry.*

Proof. Let $h : \mathbb{D} \to \Delta$ be a holomorphic covering. Then $k = f \circ h : \mathbb{D} \to \Omega$ is also a holomorphic covering, so

$$\begin{aligned}\lambda_{\mathbb{D}} &= k^*(\lambda_\Omega) \\ &= h^*(f^*(\lambda_\Omega)).\end{aligned}$$

Thus, $f^*(\lambda_\Omega)$ is a conformal metric on Δ whose pull-back to the unit disk by a covering projection is $\lambda_{\mathbb{D}}$, so $f^*(\lambda_\Omega)$ is the hyperbolic metric on Δ by Theorem 10.3. ■

Theorem 10.4 implies that every $h \in \mathcal{A}(\Omega)$ is an isometry of the hyperbolic metric, and more generally, each holomorphic self-covering h of Ω is a local isometry of the hyperbolic metric. A hyperbolic region can have self-coverings that are not conformal automorphisms, see Section 12.

The maximality property of the hyperbolic metric given in Theorem 9.5 remains valid for hyperbolic regions. As noted after the proof of Theorem 9.5 this means that a version of the Schwarz-Pick Lemma holds for holomorphic maps between hyperbolic regions. In order to establish a sharp result, we provide an independent proof.

Theorem 10.5 (Schwarz-Pick Lemma - general version)**.** *Suppose Δ and Ω are hyperbolic regions and $f : \Delta \to \Omega$ is holomorphic. Then for all $z \in \Delta$,*

$$\lambda_\Omega(f(z))|f'(z)| \le \lambda_\Delta(z). \tag{10.1}$$

If $f : \Delta \to \Omega$ is a covering projection, then $\lambda_\Omega(f(z))|f'(z)| = \lambda_\Delta(z)$ for all $z \in \Delta$. If there exists a point in Δ such that equality holds in (10.1)*, then f is a covering.*

Proof. Let $k : \mathbb{D} \to \Delta$ and $h : \mathbb{D} \to \Omega$ be holomorphic coverings. The function $f \circ k : \mathbb{D} \to \Omega$ lifts relative to h to a holomorphic function $F : \mathbb{D} \to \mathbb{D}$. Then $f \circ k = h \circ F$ and the Schwarz-Pick Lemma for the unit disk give

$$\begin{aligned}k^*(f^*(\lambda_\Omega)) &= (f \circ k)^*(\lambda_\Omega) \\ &= (h \circ F)(\lambda_\Omega) \\ &= F^*(h^*(\lambda_\Omega)) \\ &= F^*(\lambda_{\mathbb{D}}) \\ &\le \lambda_{\mathbb{D}} \\ &= k^*(\lambda_\Delta).\end{aligned}$$

Because k is a surjective local homeomorphism, the inequality $k^*(f^*(\lambda_\Omega)) \le k^*(\lambda_\Delta)$ gives the inequality (10.1). Because h and k are coverings, f is a covering if and only if F is a covering. This observation establishes the sharpness. ■

We need to establish a result about holomorphic self-coverings of a hyperbolic region that have a fixed point in order to obtain a good analog of Schwarz's Lemma for hyperbolic regions.

Theorem 10.6. *A self-covering of a hyperbolic region that fixes a point is a conformal automorphism. In particular, if a hyperbolic region Ω is not simply connected and $a \in \Omega$, then $\mathcal{A}(\Omega, a)$ is isomorphic to the group of nth roots of unity for some positive integer n.*

Proof. Suppose Ω is a hyperbolic region, $a \in \Omega$ and f is a self-covering of Ω that fixes a. We prove f is a conformal automorphism. The result is trivial if Ω is simply connected since every covering of a simply connected region is a homeomorphism. Let $h : \mathbb{D} \to \Omega$ be a holomorphic universal covering with $h(0) = a$ and $h'(0) > 0$. Because a covering is surjective, it suffices to prove f is injective. Let $\tilde{f}$ be the lift of $f \circ h$ relative to h that satisfies $\tilde{f}(0) = 0$. Since h and $f \circ h$ are coverings, so is $\tilde{f}$. Because $\mathbb{D}$ is simply connected, $\tilde{f}$ is a conformal automorphism of $\mathbb{D}$. Then $\tilde{f}(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$. Because Ω is not simply connected, the fiber $h^{-1}(a)$ contains infinitely many points besides 0. As this fiber is a discrete subset of $\mathbb{D}$, the nonzero elements of $h^{-1}(a)$ have a minimum positive modulus r; say $h^{-1}(a) \cap \{z : |z| = r\} = \{a_j : j = 1, \ldots, m\}$. From $\tilde{f}(h^{-1}(a)) = h^{-1}(a)$, we conclude that $\tilde{f}$ induces a permutation of the set $\{a_j : j = 1, \ldots, m\}$. Therefore, there exists $n \leq m!$ such that $\tilde{f}^n$ is the identity. Then $f^n \circ h = h \circ \tilde{f}^n = h$ and so f^n is the identity. If $n = 1$, then f is the identity. If $n \geq 2$, then $f^n = I$, the identity, implies f is a conformal automorphism of Ω with inverse f^{n-1}.

This argument shows that if Ω is not simply connected, then there is a non-negative integer m such that for all $f \in \mathcal{A}(\Omega, a)$, f^m is the identity. Therefore, $f'(a)^m = 1$, or $f'(a)$ is an mth root of unity. Thus, $f \mapsto f'(a)$ defines a homomorphism of $\mathcal{A}(\Omega, a)$ into the unit circle $\mathbb{T}$ and the image is a subgroup of the mth roots of unity. Hence, $\mathcal{A}(\Omega, a)$ is a finite group isomorphic to the group of nth roots of unity for some positive integer n. ∎

Example 10.7. Let $A_R = \{z : 1/R < |z| < R\}$, where $R > 1$. The group $\mathcal{A}(A_R, 1)$ has order two; the only conformal automorphisms of A_R that fix 1 are the identity map and $f(z) = 1/z$.

Corollary 10.8 (Schwarz's Lemma - General Version)**.** *Suppose Ω is a hyperbolic region, $a \in \Omega$ and f is a holomorphic self-map of Ω that fixes a. Then $|f'(a)| \leq 1$ and equality holds if and only if $f \in \mathcal{A}(\Omega, a)$, the group of conformal automorphisms of Ω that fix a. Moreover, $f'(a) = 1$ if and only if f is the identity.*

Proof. The Schwarz-Pick Lemma implies $|f'(a)| \leq 1$ with equality if and only if f is a self-covering of Ω that fixes a. Each $f \in \mathcal{A}(\Omega, a)$ is a covering, so $|f'(a)| = 1$. If f is a self-covering of Ω that fixes a, then $f \in \mathcal{A}(\Omega, a)$ by Theorem 10.6. We use the proof of Theorem 10.6 to verify that $f'(a) = 1$ implies f is the identity. Let $\tilde{f}$ be the lift of $f \circ h$ as in the proof of Theorem 10.6. Then $h \circ f = f \circ h$ gives $1 = f'(a) = \tilde{f}'(0)$, so $\tilde{f}$ is the identity. This implies f is the identity. ∎

Picard established a vast generalization of Liouville's Theorem.

Theorem 10.9 (Picard's Small Theorem). *If an entire function omits two finite complex values, then f is constant.*

Proof. Suppose f is an entire function and $f(\mathbb{C}) \subseteq \mathbb{C}\backslash\{a,b\} := \mathbb{C}_{a,b}$, where a and b are distinct complex numbers. We derive a contradiction if f were nonconstant. The region $\mathbb{C}_{a,b}$ is hyperbolic; let $\lambda_{a,b}(z)|dz|$ denote the hyperbolic metric on $\mathbb{C}_{a,b}$. If f were nonconstant, then $f^*(\lambda_{a,b}(z)|dz|)$ would be a semi-metric on $\mathbb{C}$ with curvature at most -1; this contradicts Theorem 9.6. ■

Exercises.

1. Verify that $f(z) = \exp(iz)$ is a covering of the upper half-plane $\mathbb{H}$ onto the punctured disk $\mathbb{D} \setminus \{0\}$.
2. Verify that for each nonzero integer n the function $p_n(z) = z^n$ defines a holomorphic covering of the punctured plane $\mathbb{C} \setminus \{0\}$ onto itself.
3. Verify that for each positive integer n the function $p_n(z) = z^n$ defines a holomorphic covering of the punctured disk $\mathbb{D} \setminus \{0\}$ onto itself.
4. Suppose Ω is a hyperbolic region and $a \in \Omega$. Let $\mathcal{F}$ denote the family of all holomorphic functions $f : \mathbb{D} \to \Omega$ such that $f(0) = a$ and set $M = \sup\{|f'(0)| : f \in \mathcal{F}\}$. Prove M is finite and for $f \in \mathcal{F}$, $|f'(0)| = M$ if and only if f is a holomorphic covering of $\mathbb{D}$ onto Ω. Conclude that $M = 2/\lambda_\Omega(a)$.
5. Suppose Ω is a hyperbolic region in $\mathbb{C}$ and $a, b \in \Omega$ are distinct points. If f is a holomorphic self-map of Ω that fixes a and b, prove f is a conformal automorphism of Ω with finite order. Give an example to show that f need not be the identity when Ω is not simply connected.

11. Hyperbolic distortion

In Section 5 the hyperbolic distortion of a holomorphic self-map of the unit disk was introduced. We now define an analogous concept for holomorphic maps of hyperbolic regions.

Definition 11.1. Suppose Ω and Δ are hyperbolic regions and $f : \Delta \to \Omega$ is holomorphic. The *(local) hyperbolic distortion factor* for f at z is

$$f^{\Delta,\Omega}(z) := \frac{\lambda_\Omega(f(z))|f'(z)|}{\lambda_\Delta(z)} = \lim_{w \to z} \frac{d_\Omega(f(z), f(w))}{d_\Delta(z, w)}.$$

If $\Omega = \Delta$, write f^Δ in place of $f^{\Delta,\Omega}$.

The hyperbolic distortion factor defines a mapping of Δ into the closed unit disk by the Schwarz-Pick Lemma. If f is not a covering, then the hyperbolic distortion factor gives a map of Δ into the unit disk. There is a Schwarz-Pick type of result for the hyperbolic distortion factor which extends Corollary 5.7 to holomorphic maps between hyperbolic regions.

Theorem 11.2 (Schwarz-Pick Lemma for Hyperbolic Distortion)**.** *Suppose Δ and Ω are hyperbolic regions and $f : \Delta \to \Omega$ is holomorphic. If f is not a covering, then*

$$d_{\mathbb{D}}(f^{\Delta,\Omega}(z), f^{\Delta,\Omega}(w)) \le 2d_{\Delta}(z,w) \tag{11.1}$$

for all z, $w \in \Delta$.

Proof. Fix $w \in \Omega$. Let $h : \mathbb{D} \to \Delta$ and $k : \mathbb{D} \to \Omega$ be holomorphic coverings with $h(0) = w$ and $k(0) = f(w)$. Then there is a lift of f to a self-map $\tilde{f}$ of $\mathbb{D}$ such that $k \circ \tilde{f} = f \circ h$. $\tilde{f}$ is not a conformal automorphism of $\mathbb{D}$ because f is not a covering of Δ onto Ω. We begin by showing that $\tilde{f}^{\mathbb{D}}(\tilde{z}) = f^{\Delta,\Omega}(h(\tilde{z}))$ for all $\tilde{z}$ in $\mathbb{D}$. From $k \circ \tilde{f} = f \circ h$ and $\lambda_{\mathbb{D}}(\tilde{z}) = \lambda_{\Delta}(h(\tilde{z}))|h'(\tilde{z})|$ we obtain

$$\begin{aligned} f^{\Delta,\Omega}(h(\tilde{z})) &= \frac{\lambda_{\Omega}(f(h(\tilde{z})))|f'(h(\tilde{z}))|}{\lambda_{\Delta}(h(\tilde{z}))} \\ &= \frac{\lambda_{\Omega}(k(\tilde{f}(\tilde{z})))|k'(\tilde{f}(\tilde{z}))||\tilde{f}'(\tilde{z})|}{\lambda_{\Delta}(h(\tilde{z}))|h'(\tilde{z})|} \\ &= \frac{\lambda_{\mathbb{D}}(\tilde{f}(\tilde{z}))|\tilde{f}'(\tilde{z})|}{\lambda_{\mathbb{D}}(z)} \\ &= \tilde{f}^{\mathbb{D}}(\tilde{z}). \end{aligned}$$

Now we establish (11.1). For $z \in \Omega$ there exists $\tilde{z} \in h^{-1}(z)$ with $d_{\mathbb{D}}(0,\tilde{z}) = d_{\Omega}(w,z)$. Then

$$\begin{aligned} d_{\mathbb{D}}(f^{\Delta,\Omega}(z), f^{\Delta,\Omega}(w)) &= d_{\mathbb{D}}(f^{\Delta,\Omega}(h(\tilde{z})), f^{\Delta,\Omega}(h(0))) \\ &= d_{\mathbb{D}}(\tilde{f}^{\mathbb{D}}(\tilde{z}), \tilde{f}^{\mathbb{D}}(0)) \\ &\le 2d_{\mathbb{D}}(\tilde{z},0) \\ &= 2d_{\Delta}(z,w). \end{aligned}$$

■

Corollary 11.3. *Suppose Δ and Ω are hyperbolic regions. Then for any holomorphic function $f : \Delta \to \Omega$,*

$$f^{\Delta,\Omega}(z) \le \frac{f^{\Delta,\Omega}(w) + \tanh d_{\Delta}(z,w)}{1 + f^{\Delta,\Omega}(w)\tanh d_{\Delta}(z,w)}. \tag{11.2}$$

for all z, $w \in \Delta$.

Proof. Inequality (11.2) is trivial when f is a covering since both sides are identically one, Thus, it suffices to establish the inequality when f is not a covering of Δ onto Ω. Then

$$\begin{aligned} d_{\mathbb{D}}(0, f^{\Delta,\Omega}(z)) &\le d_{\mathbb{D}}(0, f^{\Delta,\Omega}(w)) + d_{\mathbb{D}}(f^{\Delta,\Omega}(z), f^{\Delta,\Omega}(w)) \\ &\le d_{\mathbb{D}}(0, f^{\Delta,\Omega}(w)) + 2d_{\Delta}(z,w) \end{aligned}$$

gives

$$\begin{aligned} f^{\Delta,\Omega}(z) &= \tanh\left(\frac{1}{2}d_{\mathbb{D}}(0, f^{\Delta,\Omega}(z))\right) \\ &\le \tanh\left(\frac{1}{2}d_{\mathbb{D}}(0, f^{\Delta,\Omega}(w)) + d_{\Delta}(z,w)\right) \\ &= \frac{f^{\Delta,\Omega}(w) + \tanh d_{\Delta}(z,w)}{1 + f^{\Delta,\Omega}(w)\tanh d_{\Delta}(z,w)}\,. \end{aligned}$$

■

Exercises.

1. For a holomorphic function $f : \mathbb{D} \to \mathbb{K}$, explicitly calculate $f^{\mathbb{D},\mathbb{K}}(z)$.
2. Suppose Ω is a simply connected proper subregion of $\mathbb{C}$ and $a \in \Omega$. Let $\mathcal{H}(\Omega, a)$ denote the set of holomorphic self-maps of Ω that fix a. Prove that $\{f^{\Omega}(a) : f \in \mathcal{H}(\Omega, a)\}$ is the closed unit interval $[0, 1]$.

12. The hyperbolic metric on a doubly connected region

There is a simple conformal classification of doubly connected regions in $\mathbb{C}_\infty$. If Ω is a doubly connected region in $\mathbb{C}_\infty$, then Ω is conformally equivalent to exactly one of:

(a) $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$,
(b) $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$, or
(c) an annulus $A(r, R) = \{z : r < |z| < R\}$, where $0 < r < R$.

In the first case Ω itself is the extended plane $\mathbb{C}_\infty$ punctured at two points and so is not hyperbolic. In this section we calculate the hyperbolic metric for the punctured unit disk $\mathbb{D}^*$ and for the annulus $A_R = \{z : 1/R < |z| < R\}$, where $R > 1$.

12.1. Hyperbolic metric on the punctured unit disk. To determine the hyperbolic metric on $\mathbb{D}^*$ we make use of a holomorphic covering from $\mathbb{H}$ onto $\mathbb{D}^*$ and Theorem 10.4. The function $h(z) = \exp(iz)$ is a holomorphic covering from $\mathbb{H}$ onto $\mathbb{D}^*$. Therefore, the density of the hyperbolic metric on $\mathbb{D}^*$ is

$$\lambda_{\mathbb{D}^*}(z) = \frac{1}{|z|\log(1/|z|)}.$$

For simply connected hyperbolic regions the only hyperbolic isometries of the hyperbolic metric are conformal automorphisms. For multiply connected regions there can be self-coverings that leave the hyperbolic metric invariant. For instance, the maps $z \mapsto z^n$, $n = 2, 3, \ldots$, are self-coverings of $\mathbb{D}^*$ that leave $\lambda_{\mathbb{D}^*}(z)|dz|$ invariant. Up to composition with a rotation about the origin these are the only self-coverings of $\mathbb{D}^*$ that are not automorphisms.

Because each hyperbolic geodesic in $\mathbb{D}^*$ is the image of a hyperbolic geodesic in $\mathbb{H}$ under h, every radial segment $[re^{i\theta}, Re^{i\theta}]$, where $0 < r < R < 1$, is part of a hyperbolic geodesic. Since the density $\lambda_{\mathbb{D}^*}$ is independent of θ, the hyperbolic length of a geodesic segment $\sigma_{r,R} = [re^{i\theta}, Re^{i\theta}]$ is independent of θ. Direct calculation gives

$$\ell_{\mathbb{D}^*}(\sigma_{r,R}) = \int_{[r,R]} \lambda_{\mathbb{D}^*}(z)|dz| = \int_r^R \frac{dt}{t\log t} = \log\left|\frac{\log R}{\log r}\right|.$$

As the formula shows, this length tends to infinity if either $r \to 0$ or $R \to 1$ which also follows from the completeness of the hyperbolic metric. The Euclidean circle $C_r = \{z : |z| = r\}$, where $0 < r < 1$, is not a hyperbolic geodesic; it has hyperbolic length

$$\ell_{\mathbb{D}^*}(C_r) = \int_{|z|=r} \frac{|dz|}{|z|\log(1/|z|)} = \frac{2\pi}{\log(1/r)}.$$

The hyperbolic length of C_r approaches 0 when $r \to 0$ and ∞ when $r \to 1$. The hyperbolic area of the annulus $A(r,R) = \{z : r < |z| < R\} \subset \mathbb{D}^*$ is

$$\begin{aligned} a_{\mathbb{D}^*}(A(r,R)) &= \int\int_{A(r,R)} \frac{1}{|z|^2\log^2|z|}\,dx\,dy \\ &= 2\pi\int_r^R \frac{dt}{t\log^2 t} \\ &= 2\pi\left(\frac{1}{\log(1/R)} - \frac{1}{\log(1/r)}\right). \end{aligned}$$

The hyperbolic area of $A(r,R)$ tends to infinity when $R \to 1$ and has the finite limit $2\pi/\log(1/R)$ when $r \to 0$.

There is a Euclidean surface in $\mathbb{R}^3$ that is isometric to $\{z : 0 < |z| < 1/e\}$ with the restriction of the hyperbolic metric on $\mathbb{D}^*$ and makes it easy to see these curious results about length and area in a neighborhood of the puncture. If a tractrix is rotated about the y-axis and the resulting surface is given the geometry induced from the Euclidean metric on $\mathbb{R}^3$, then this surface has constant curvature -1 and is isometric to $\{z : 0 < |z| < 1/e\}$ with the restriction of the hyperbolic metric on $\mathbb{D}^*$. This picture provides a simple isometric embedding of a portion of $\mathbb{D}^*$ into $\mathbb{R}^3$. Radial segments correspond to rotated copies of the tractrix and these have infinite Euclidean length. At the same time the surface has finite Euclidean area.

Recall that for a simply connected region Ω, the hyperbolic density λ_Ω and the quasihyperbolic density $1/\delta_\Omega$ are bi-Lipschitz equivalent; see (8.4). These two metrics are not bi-Lipschitz equivalent on $\mathbb{D}^*$ because the behavior of $\lambda_{\mathbb{D}^*}$ near the unit circle differs from its behavior near the origin. For $1/2 < |z| < 1$, $\delta_{\mathbb{D}^*}(z) = 1 - |z|$ and so

$$\lim_{|z|\to 1} \delta_{\mathbb{D}^*}(z)\lambda_{\mathbb{D}^*}(z) = 1.$$

For $0 < |z| < 1/2$, $\delta(z) = |z|$ and so

$$\lim_{|z|\to 1} \delta_{\mathbb{D}^*}(z)\lambda_{\mathbb{D}^*}(z) = 0.$$

12.2. Hyperbolic metric on an annulus. Now we determine the hyperbolic metric on an annulus by using a holomorphic covering from a strip onto an annulus. In each conformal equivalence class of annuli we choose the unique representative that is symmetric about the unit circle.

For $0 < r < R$ let $A(r,R) = \{z : r < |z| < R\}$. The number $\operatorname{mod}(A(r,R)) = \log(R/r)$ is called the *modulus* of $A(r,R)$. Two annuli $A(r_j, R_j)$, $j = 1, 2$, are conformally (actually Möbius) equivalent if and only if $R_1/r_1 = R_2/r_2$; that is, if and only if they have equal moduli. If $S = \sqrt{(R/r)}$, then $A_S = \{z : 1/S < |z| < S\}$ is the unique annulus conformally equivalent to $A(r,R)$ that is symmetric about the unit circle. Here symmetry means A_S is invariant under $z \mapsto 1/\bar{z}$, reflection about the unit circle. Note that $\operatorname{mod}(A_S) = 2\log S$.

The function $k(z) = \exp(z)$ is a holomorphic universal covering from the vertical strip $S_b = \{z : |\operatorname{Im} z| < b\}$, where $b = \log R$, onto the annulus $A_R = \{z : 1/R < |z| < R\}$. Therefore, the density of the hyperbolic metric of the annulus A_R is

$$\lambda_R(z) = \frac{\pi}{2\log R}\frac{1}{|z|\cos\left(\frac{\pi\log|z|}{2\log R}\right)}.$$

Example 12.1. We investigate the hyperbolic lengths of the Euclidean circles $C_r = \{z : |z| = r\}$ in A_R. The hyperbolic length of C_r is

$$\ell_R(C_r) = \int_{|z|=r} \frac{\pi}{2\log R}\frac{|dz|}{|z|\cos\left(\frac{\pi\log|z|}{2\log R}\right)} = \frac{\pi^2}{(\log R)\cos\left(\frac{\pi\log r}{2\log R}\right)}.$$

The symmetry of A_R about the unit circle is reflected by the fact that two circles symmetric about the unit circle have the same hyperbolic length. Also, the hyperbolic length of C_r increases from $\pi^2/\log R = 2\pi^2/\operatorname{mod} A_R$ to ∞ as r increases from 1 to R. Hence, the hyperbolic lengths of the Euclidean circles C_r in A_R have a positive minimum hyperbolic length. The Euclidean circle C_1 is a hyperbolic geodesic; C_r is not a hyperbolic geodesic when $r \neq 1$.

If $\gamma_n(t) = \exp(2\pi int)$, then $I(\gamma_n, 0) = n$, where $I(\delta, 0)$ denotes the *index* or *winding number* of a closed path δ about the origin, and

$$\ell_R(\gamma_n) = \frac{2\pi^2|n|}{\operatorname{mod} A(R)}.$$

We now show that γ_n has minimal hyperbolic length among all closed paths in A_R that wind n times about the origin.

Theorem 12.2. *Suppose γ is a piecewise smooth closed path in A_R and $I(\gamma, 0) = n \neq 0$. Then*

$$\frac{2\pi^2|n|}{\operatorname{mod} A(R)} \leq \ell_R(\gamma), \tag{12.1}$$

where $\ell_R(\gamma)$ denotes the hyperbolic length of γ. Equality holds in (12.1) *if and only if γ is a monotonic parametrization of the unit circle traversed n times.*

Proof. Suppose $\gamma : [0,1] \to A_R$ is a closed path with $I(\gamma, 0) = n \neq 0$. Then

$$\begin{aligned}
\ell_R(\gamma) &= \int_\gamma \lambda_R(z)\,|dz| \\
&= \frac{\pi}{2\log R}\int_0^1 \frac{|\gamma'(t)|\,dt}{|\gamma(t)|\cos\left(\frac{\pi\log|\gamma(t)|}{2\log R}\right)} \\
&\geq \frac{\pi}{2\log R}\int_0^1 \frac{|\gamma'(t)|}{|\gamma(t)|}
\end{aligned}$$

since

$$0 < \cos\left(\frac{\pi\log|\gamma(t)|}{2\log R}\right) \leq 1 \tag{12.2}$$

and equality holds if and only if $|\gamma(t)| = 1$ for $t \in [0,1]$. Next,

$$\begin{aligned}
\int_0^1 \frac{|\gamma'(t)|}{|\gamma(t)|} &\geq \left|\int_0^1 \frac{\gamma'(t)}{\gamma(t)}\,dt\right| \\
&= \left|\int_\gamma \frac{dz}{z}\right| \\
&= |2\pi i I(\gamma,0)| \\
&= 2\pi|n|.
\end{aligned} \tag{12.3}$$

Hence,

$$\ell_R(\gamma) \geq \frac{\pi^2|n|}{\log R} = \frac{2\pi^2|n|}{\operatorname{mod} A_R}.$$

It is straightforward to verify that if $\gamma_n(t) = \exp(2\pi int)$, $t \in [0,1]$, then equality holds in (12.1). Conversely, suppose γ is a path for which equality holds. Then equality holds in (12.2), so $|\gamma(t)| = 1$ for $t \in [0,1]$. Let $\delta : [0,1] \to \mathbb{C}$ be a lift of γ relative to the covering $\exp : \mathbb{C} \to \mathbb{C}^*$. From $I(\gamma,0) = n$, we obtain $\delta(1) - \delta(0) = 2\pi ni$. The condition $|\gamma(t)| = 1$ implies $\delta(t) \in i\mathbb{R}$ for $t \in [0,1]$. The function $h(t) = (\delta(t) - \delta(0))/2\pi i$ is real-valued, $h(0) = 0$ and $h(1) = n$. Also, $\gamma(t) = \exp(2\pi i h(t) + \delta(0))$. Equality must hold in (12.3) and this means $\gamma'(t)/\gamma(t) = 2\pi i h'(t)$ has constant argument. Hence, $h'(t)$ is either positive or negative, so $t \mapsto \exp(2\pi i h(t) + \delta(0)$ is a parametrization of the unit circle starting at $\gamma(0) = \exp\delta(0)$ and the unit circle is traversed either clockwise or counterclockwise. ■

Exercises.

1. Consider the metric $|dz|/|z|$ on $\mathbb{C} \setminus \{0\} := \mathbb{C}^*$ and let $\ell_{\mathbb{C}^*}(\gamma)$ denote the length of a path γ in $\mathbb{C}^*$ relative to this metric. If γ is a closed path in $\mathbb{C}^*$, prove
$$\ell_{\mathbb{C}^*}(\gamma) \geq 2\pi|I(\gamma,0)|.$$
When does equality hold?
2. Suppose f is holomorphic on $\mathbb{D}$ and $f(\mathbb{D}) \subseteq \mathbb{D} \setminus \{0\}$. Prove that $|f'(0)| \leq 2/e$. Determine when equality holds.

13. Rigidity theorems

If a holomorphic mapping of hyperbolic regions is not a covering, then strict inequality holds in the general version of the Schwarz-Pick Lemma. It is often possible to provide a quantitative version of this strict inequality that is independent of the holomorphic mapping for multiply connected regions. We begin by establishing a refinement of Schwarz's Lemma.

Lemma 13.1. *Suppose $0 \neq a \in \mathbb{D}$ and $0 < t < 1$. If f is a holomorphic self-map of $\mathbb{D}$, $f(0) = 0$ and $|f(a)| \leq t|a|$, then*

$$|f'(0)| \leq \frac{t + |a|}{1 + t|a|} < 1. \tag{13.1}$$

Proof. The Three-point Schwarz-Pick Lemma (Theorem 4.4) with $z = 0 = v$ and $w = a$ gives

$$\begin{aligned} d_{\mathbb{D}}(f'(0), f(a)/a) &= d_{\mathbb{D}}(f^*(0,0), f^*(a,0)) \\ &\leq d_{\mathbb{D}}(0,a). \end{aligned}$$

Hence,

$$\begin{aligned} d_{\mathbb{D}}(0, |f'(0)|) &= d_{\mathbb{D}}(0, f'(0)) \\ &\leq d_{\mathbb{D}}(0.f(a)/a) + d_{\mathbb{D}}(0,a) \\ &\leq d_{\mathbb{D}}(0,t) + d_{\mathbb{D}}(0,-|a|) \\ &= d(-|a|, t), \end{aligned}$$

which is equivalent to (13.1). ■

Theorem 13.2. *Suppose Δ and Ω are hyperbolic regions with $a \in \Delta$, $b \in \Omega$ and Δ is not simply connected. There is a constant $\alpha = \alpha(a, \Delta; b, \Omega) \in [0,1)$ such that if $f : \Delta \to \Omega$ is any holomorphic mapping with $f(a) = b$ that is not a covering, then*

$$\lambda_\Omega(f(a))|f'(a)| \leq \alpha \lambda_\Delta(a); \tag{13.2}$$

or equivalently, $f^{\Delta,\Omega}(a) \leq \alpha$. Moreover, for all $z \in \Delta$

$$f^{\Delta,\Omega}(z) \leq \frac{\alpha + \tanh d_\Delta(a,z)}{1 + \alpha \tanh d_\Delta(a,z)} < 1. \tag{13.3}$$

Proof. Let $h : \mathbb{D} \to \Delta$ and $k : \mathbb{D} \to \Omega$ be holomorphic coverings with $h(0) = a$ and $k(0) = b$. Because Δ is not simply connected, the fiber $h^{-1}(a)$ is a discrete subset of $\mathbb{D}$ and contains infinitely many points in addition to 0. Let $0 < r = \min\{|z| : z \in h^{-1}(a), z \neq 0\} < 1$. The set $h^{-1}(a) \cap \{z : |z| = r\}$ is finite, say $\tilde{a}_j$, $1 \leq j \leq m$. Next, $\{|z| : z \in k^{-1}(b)\}$ is a discrete subset of $[0,1)$, so this set contains finitely many values in the interval $[0,r)$. Let s be the maximum value of the finite set $\{|z| : z \in k^{-1}(b)\} \cap [0,r)$. Suppose $f : \Delta \to \Omega$ is any holomorphic mapping with $f(a) = b$ and f is not a covering. Let $\tilde{f}$ be the unique lift of $f \circ h$ relative to k that satisfies $\tilde{f}(0) = 0$. Then $|\tilde{f}'(0)| = f^{\Delta,\Omega}(a)$. Because f is not a covering, $\tilde{f}$ is not a rotation about the origin. From $f \circ h = k \circ \tilde{f}$ we deduce that $\tilde{f}$ maps $h^{-1}(a)$ into $k^{-1}(b)$. In particular, $|\tilde{f}(\tilde{a})| \le s = t|\tilde{a}|$, where $\tilde{a} = \tilde{a}_1$ and $t = s/r < 1$. Lemma 13.1 gives

$$|\tilde{f}'(0)| \le \frac{(s/r) + |\tilde{a}|}{1 + (s/r)|\tilde{a}|} = \frac{(s/r) + r}{1 + s} = \alpha < 1.$$

Since $|\tilde{f}'(0)| = f^{\Delta,\Omega}(a)$, this establishes (13.2). Inequality (13.3) follows immediately from Corollary 11.3. ■

The pointwise result (13.2) is due to Minda [24] and was motivated by the Aumann-Carathéodory Rigidity Theorem [4] which is the special case when $\Omega = \Delta$ and $a = b$. The global result (13.3) is due to the authors [9]. The Aumann-Carathéodory Rigidity Theorem asserts there is a constant $\alpha = \alpha(a, \Omega) \in [0, 1)$ such that $|f'(a)| \le \alpha$ for all holomorphic self-maps of Ω that fix a and are not conformal automorphisms. The exact value of the Aumann-Carathéodory rigidity constant for an annulus was determined in [23]. The following extension of the Aumann-Carathéodory Rigidity Theorem to a local result is due to the authors [9]. The corollary is given in Euclidean terms and asserts that holomorphic self-maps with a fixed point are locally strict Euclidean contractions if they are not conformal automorphisms.

Corollary 13.3 (Aumann-Carathéodory Rigidity Theorem - Local Version)**.** *Suppose Ω is a hyperbolic region, $a \in \Omega$ and Ω is not simply connected. There is a constant $\beta = \beta(a, \Omega) \in [0, 1)$ and a neighborhood N of a such that if f is a holomorphic self-map of Ω that fixes a and is not a conformal automorphism of Ω, then $|f'(z)| \le \beta$ for all $z \in N$.*

Proof. From the theorem

$$|f'(z)| \le \frac{\lambda_\Omega(z)}{\lambda_\Omega(f(z))} \frac{\alpha + \tanh d_\Omega(a, z)}{1 + \alpha \tanh d_\Omega(a, z)}.$$

Set $M(r) = \max\{\lambda_\Omega(z) : d_\Omega(a, z) \le r\}$ and $m(r) = \min\{\lambda_\Omega(z) : d_\Omega(a, z) \le r\}$. Since $f(D_\Omega(a, r)) \subseteq D_\Omega(a, r)$, we have

$$|f'(z)| \le \frac{M(r)}{m(r)} \frac{\alpha + \tanh d_\Omega(a, z)}{1 + \alpha \tanh d_\Omega(a, z)}.$$

The right-hand side of the preceding equality is independent of f and tends to α as $r \to 1$. Therefore, for $\alpha < \beta < 1$ there exists $r > 0$ such that

$$\frac{M(r)}{m(r)} \frac{\alpha + \tanh d_\Omega(a, z)}{1 + \alpha \tanh d_\Omega(a, z)} \le \beta$$

for $d_\Omega(a, z) \le r$. Then $|f'(a)| \le \beta$ holds in $D_\Omega(a, r)$. ■

Our final topic is a rigidity theorem for holomorphic maps between annuli. The original results of this type are due to Huber [17]. Marden, Richards and Rodin [21] presented an extensive generalization of Huber's work to holomorphic self-maps of hyperbolic Riemann surfaces.

Definition 13.4. Suppose $1 < R, S \le +\infty$ and $f : A_R \to A_S$ is a continuous function. The *degree* of f is the integer $\deg f := I(f \circ \gamma, 0)$, where $\gamma(t) = \exp(2\pi i t)$.

The positively oriented unit circle γ generates the fundamental group of both A_R and A_S. Therefore, for any continuous map $f : A_R \to A_S$, $f \circ \gamma \approx \gamma^n$ for the unique integer $n = \deg f$, where $\approx$ denotes free homotopy. Algebraically, f induces a homomorphism f_* from $\pi(A_R, a) \cong \mathbb{Z}$ to $\pi(A_S, f(a)) \cong \mathbb{Z}$ and the image of 1 is an integer n; here $\pi(A_R, a)$ denotes the fundamental group of A_R with base point a. The reader should verify that

$$\deg(f \circ g) = (\deg f)\,(\deg g).$$

Since the degree of the identity map is one, this implies that $\deg f = \pm 1$ for any homeomorphism f. If f is holomorphic, then

$$\deg f = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz.$$

Suppose $f, g : A_R \to A_S$ are continuous functions. Then $\deg f = \deg g$ if and only if f and g are homotopic maps of A_R into A_S.

Example 13.5. For any integer n the holomorphic self-map $p_n(z) = z^n$ of $\mathbb{C}^*$ has degree n. Given annuli A_R and A_S with $1 < R, S < +\infty$, it is easy to construct a continuous map of A_R into A_S with degree n; for example, the function $z \mapsto (z/|z|)^n$ has degree n. For $R = S$ each conformal automorphism has degree ± 1. In fact, the rotations $z \mapsto e^{i\theta} z$ have degree 1 and the maps $z \mapsto e^{i\theta}/z$ have degree -1. Constant self-maps of A_R have degree 0. Can you find a holomorphic self-map of A_R with degree $n \neq 0, \pm 1$? Surprisingly, the answer is negative! Holomorphic mappings of proper annuli are very rigid. The moduli of the annuli provide sharp bounds for the possible degrees of a holomorphic mapping of one annulus into another.

Theorem 13.6. *If $f : A_R \to A_S$ is a holomorphic mapping, then*

$$|\deg f| \le \frac{\operatorname{mod} A_S}{\operatorname{mod} A_R}. \tag{13.4}$$

For $n = \deg f \neq 0$ equality holds if and only if $S = R^{|n|}$ and $f(z) = e^{i\theta} z^n$ for some $\theta \in \mathbb{R}$.

Proof. Let $\gamma(t) = \exp(2\pi i t)$ for $t \in [0,1]$. If $n = \deg f$, then $I(f \circ \gamma, 0) = n$, so Theorem 12.2 gives

$$\frac{2\pi^2 |n|}{\operatorname{mod} A_S} \le \ell_S(f \circ \gamma).$$

Since holomorphic functions are distance decreasing relative to the hyperbolic metric,

$$\ell_S(f \circ \gamma) \le \ell_R(\gamma) = \frac{2\pi^2}{\operatorname{mod} A_R}.$$

The preceding two inequalities imply (13.4). Suppose equality holds. Then f is a covering of A_R onto A_S that maps the unit circle onto itself. By post-composing

f with a rotation about the origin, we may assume that f fixes 1. Equality in (13.4) implies $S = R^{|n|}$, where $n = \deg f$. The covering f lifts relative to $p_n(z) = z^n$ to a holomorphic self-covering $\tilde{f}$ of A_R that fixes 1. Theorem 10.6 implies that $\tilde{f}$ is the identity and so $f(z) = z^n$. ∎

Corollary 13.7 (Annulus Theorem). *Suppose f is a holomorphic self-map of A_R. Then $|\deg f| \leq 1$ and equality holds if and only if $f \in \mathcal{A}(A_R)$.*

Proof. The inequality follows immediately from Theorem 13.6. If $f \in \mathcal{A}(A_R)$, then $|\deg f| = 1$ since this holds for any homeomorphism. It remains to show that if $|\deg f| = 1$, then $f \in \mathcal{A}(A_R)$. Equality implies f maps the unit circle into itself. By post-composing f with a rotation about the origin, we may assume f fixes 1. By Theorem 10.6 if a self-covering of a hyperbolic region has a fixed point, it is a conformal automorphism. ∎

Note that if f is any holomorphic self-map of A_R that is not a conformal automorphism, then $\deg f = 0$. A result analogous to Corollary 13.7 is not valid for a punctured disk. For each integer $n \geq 0$ the function $z \mapsto z^n$ is a holomoprhic self-map of $\mathbb{D}^*$ with degree n.

Exercises.

1. Show that Theorem 13.2 is false when Δ is simply connected. *Hint*: Suppose $\Delta = \mathbb{D}$ and $a = 0$. For any number $r \in [0, 1)$ show there exists a holomorphic function $f : \mathbb{D} \to \Omega$ that is not a covering and $f^{\mathbb{D},\Omega}(0) = r$.
2. Suppose f is a holomorphic self-map of $\mathbb{C} \setminus \{0\}$. Prove that $\deg f = 0$ if and only if $f = \exp \circ g$ for some holomorphic function g defined on $\mathbb{C} \setminus \{0\}$.

14. Further reading

There are numerous topics involving the hyperbolic metric and geometric function theory that are not discussed in these notes. The subject is too extensive to include even a reasonably complete bibliography. We mention selected books and papers that the reader might find interesting. Anderson [3] gives an elementary introduction to hyperbolic geometry in two dimensions. Krantz [18] provides an elementary introduction to certain aspects of the hyperbolic metric in complex analysis.

Ahlfors introduced the powerful method of ultrahyperbolic metrics [1]. A discussion of this method and several applications to geometric function theory, including a lower bound for the Bloch constant, can be found in [2]. Ahlfors' method can be used to estimate various types of Bloch constants, see [25], [26], [27]. In a long paper Heins [15] treates the general topic of conformal metrics on Riemann surfaces. He gives a detailed treatment of SK-metrics, a generalization of ultrahyperbolic metrics. Roughly speaking, SK-metrics are to metrics with curvature -1 as subharmonic functions are to harmonic functions.

The circle of ideas surrounding the theorems of Picard, Landau and Schottky and Montel's normality criterion all involve three omitted values. Theorems of this type follow immediately from the existence of the hyperbolic metric on $\mathbb{C}_\infty$ punctured at three points. Interestingly, only a metric with curvature at most -1 on a thrice punctured sphere is needed to establish these results. An elementary construction of such a metric, based on earlier work of R. M. Robinson [31], is given in [30].

Hejhal [16] obtained a Carathéodory kernel-type of theorem for coverings of the unit disk onto hyperbolic regions. This result implies that the hyperbolic metric depends continuously on the region.

The method of polarization was extended by Solynin to apply to the hyperbolic metric, see [10]. It include earlier work of Weitsman [35] on symmetrization and Minda [29] on a reflection principle for the hyperbolic metric.

For the role of hyperbolic geometry in the study of discrete groups of Möbius transformations, see [5]. This reference includes a brief treatment of hyperbolic trigonometry.

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