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A Note on a Minimum Area Problem for Non-Vanishing Functions

Roger W. Barnard, Clint Richardson, Alex Yu. Solynin

Abstract. We find the minimal area covered by the image of the unit disk for nonvanishing univalent functions normalized by the conditions $f(0) = 1, f'(0) = \alpha$. We discuss two different approaches, each of which contributes to the complete solution of the problem. The first approach reduces the problem, via symmetrization, to the class of typically real functions, where we can employ the well known integral representation to obtain the solution upon prior knowledge about the extremal function. The second approach, requiring smoothness assumptions, leads, via some variational formulas, to a boundary value problem for analytic functions, which admits an explicit solution.

Keywords. Symmetrization, Minimal Area Problem.

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Contents

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ and $A^p = \left\{ f \text{ analytic in } \mathbb{D} : \int_{\mathbb{D}} |f(z)|^p \, dA = ||f||_{A^p}^p < \infty \right\}$, the Bergman space of analytic functions in $\mathbb{D}$.

Recently, Aharanov, Beneteau, Khavinson, and Shapiro [2] considered a general minimization problem on A^p

$$\inf\{||f||_{A^p} : f \in A^p, \ell_i(f) = c_i, i = 1, \ldots, n\}$$

where ℓ_i are bounded linear functionals on $A^p, p > 1$. They proved several general results about this problem.

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As we know, in recent years tremendous progress has been achieved in the study of Bergman spaces. For a detailed account of this progress, we refer to the recent monograph by Peter Duren and Alex Schuster, *Bergman Spaces*, [7].

Aharanov, Beneteau, Khavinson, and Shapiro [2] also mentioned that to obtain a complete solution of a particular problem, one often needs additional information which does not follow from their methods.

A particular example is the following open problem:

$$\inf \left\{ \int_{\mathbb{D}} |f|^2 \, dA : f \neq 0 \text{ in } \mathbb{D}, f(0) = 1, f'(0) = \alpha \right\}$$

This is a “typical” extremal problem on the class of **non-vanishing** analytic functions. The nonlinearity of the class is the obvious obstacle here.

But, we have a method which allows us to solve some problems similar to this one.

Let

$$\begin{aligned} N_\alpha = \{ f :\ & f \text{ is univalent, and non-vanishing on } \mathbb{D}, \\ & f(z) = 1 + a_1(f) z + \dots, \\ & \text{normalized by } a_1(f) = \alpha \ \} \end{aligned}$$

The area of the image $f(\mathbb{D})$ is given by

$$D(f) = \int_{\mathbb{D}} |f'|^2 \, dA = \pi \sum_{n=1}^{\infty} n |a_n(f)|^2.$$

Thus

$$D(f) \geq \pi \alpha^2,$$

with equality iff $f(z) = 1 + \alpha z$.

Since this map f is in N_α, $0 < \alpha \leq 1$, Koebe's 1/4 Theorem implies $N_\alpha = \emptyset$ for $\alpha > 4$. So the nontrivial range is $1 < \alpha < 4$.

For the non-trivial range, the minimal area problem for N_α is solved by

Theorem 1.1. *For $1 < \alpha < 4$, let $f \in N_\alpha$. Then*

$$D(f) \geq \pi \alpha a^2 \left(a + \sqrt{a^2 - 1} \right)^2 \left(\alpha a^2 - 2\sqrt{a^2 - 1} \left(a + \sqrt{a^2 - 1} \right) \right) \tag{1.1}$$

where $a = a(\alpha)$ is the solution to

$$\frac{1}{\alpha} = a^2 \left[1 - \sqrt{a^2 - a} (a + \sqrt{a^2 - a})^3 \log \left((a + \sqrt{a^2 - 1})^4 / 16 a^2 (a^2 - 1) \right) \right], \tag{1.2}$$

which is unique in the interval $1 < \alpha < \infty$.

Equality in (1.1) holds iff $f = f_\alpha$ defined by

$$f_\alpha(z) = \int_{-1}^{z} \frac{-\beta \sqrt{\xi^2 - a^2}}{\left(\xi + \sqrt{\xi^2 + 1} \right)^2 \left(a \sqrt{\xi^2 - 1} + \xi \sqrt{a^2 - 1} \right)} \frac{dz}{z}$$

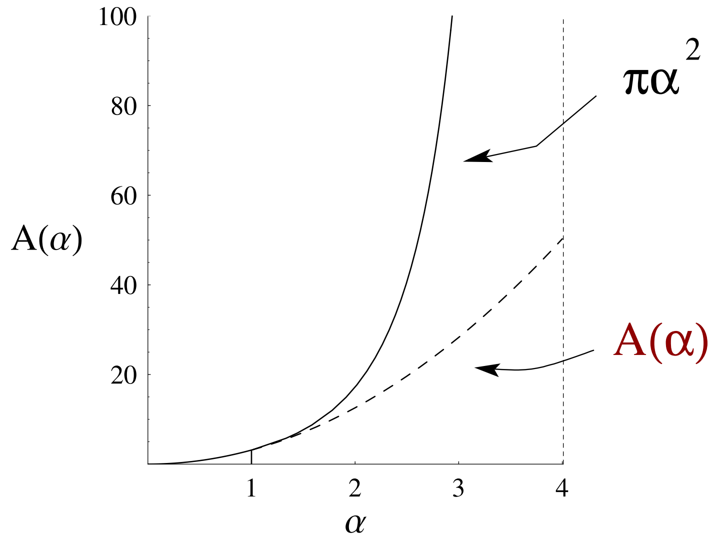

FIGURE 1. The graph of $A(\alpha)$.

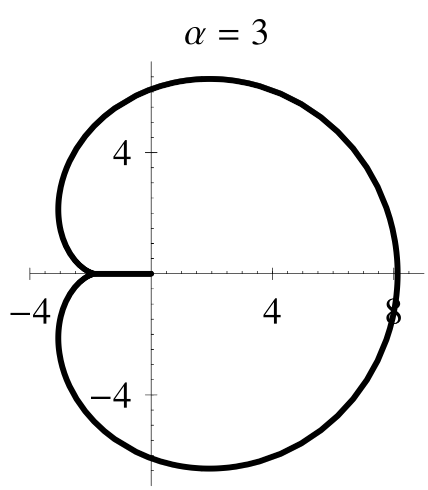

FIGURE 2. The extremal domain $D_\alpha = f_\alpha(\mathbb{D})$ for $\alpha = 3$.

with $\xi = \frac{ia}{2}\frac{1-z}{\sqrt{z}}$ *and* $\beta = \alpha a^2\left(a + \sqrt{a^2+1}\right)$.

For $0 < \alpha < 4$, let

$$A(\alpha) = \min_{f \in N_\alpha} D(f)$$

denote the minimal area covered by the images of functions in the class N_α. Note $A(\alpha)$ is convex and increasing. This can be proven from the formulas, geometry, and variational arguments. See Figure 1.

2. Outline of Our Method

First consider the minimal area problem on T_α, the typically real nonvanishing functions (not necessarily univalent). Use the linear structure of T_α and reformulate to show uniqueness and get simple "sufficient conditions" for extremality corresponding to linearized functions. This gives

Theorem 2.1. *For $1 < \alpha < 4$, let $f \in T_\alpha$. Then* (1.1) *holds with the same cases of equality.*

The technique of this proof was developed earlier in [1]. What is missing is how to construct the extremal function!

Next. Assuming sufficient smoothness, we can apply a variant of Julia's Variational Formula in [5]. This leads to boundary conditions for an extremal analytic function. To obtain this "conditional" solution requires *a priori* smoothness.

Next, to achieve the "required smoothness," we exploit geometric control of the mapping radius and apply standard symmetrization techniques to obtain the sufficient initial Jordan rectifiability as in [4].

Then we can apply earlier smoothing variations developed by Barnard and Solynin in [5] giving "required smoothness."

Thus the "conditional" proof becomes a true proof. We then verify that the function recovered from the first step satisfies the sufficient conditions of extremality which also leads then to a complete solution of the problem.

For a first step on T_α, we renormalize so that

$$f(0) = 1, f'(0) = \alpha.$$

Subordination implies $0 < \alpha \leq 4$. Since T_α is compact and convex, the minimizer exists and is unique. The uniqueness follows by letting f_1 and f_2 be two minimizers. Then

$$\begin{aligned} D((f_1 + f_2)/2) &= \frac{1}{4}\int_{\mathbb{D}} |f_1' + f_2'|^2 \, d\sigma \\ &\leq \frac{1}{2}\left(\int_{\mathbb{D}} |f_1'|^2 \, d\sigma + \int_{\mathbb{D}} |f_2'|^2 \, d\sigma\right) \\ &= \frac{1}{2}\left(D(f_1) + D(f_2)\right), \end{aligned} \tag{2.1}$$

with equality iff $f_1' \equiv f_2'$.

We note here that the uniqueness obtained here is fortunate, since uniqueness is in general not obtained when variational and approximation methods are used.

Reformulating the problem using the linearity of T_α, we use the following lemma from [1, 3]

Lemma 2.2. *For f_α' continuous on $\overline{\mathbb{D}}$, f_α minimizes $D(f)$ on T_α iff f_α minimizes*

$$L(f) = \Re \int_{\mathbb{D}} \overline{f_\alpha'(z)} f'(z) \, d\sigma$$

on T_α.

Proof. See Lemma 1 of [3]. ■

Lemma 2.3. *If f'_α is continuous on $\overline{\mathbb{D}}$, then*

$$L(f) = \int_0^\pi K_\alpha(t) d\mu_f(t),$$

where

$$K_\alpha(t) = \frac{2\pi\alpha}{\sin t} \Im \left\{ e^{it} f'(e^{it}) \right\}.$$

Proof. See [3]. ■

Proof of Theorem 2.1 for T_α. For $D_\alpha = f_\alpha(\mathbb{D})$, first show $(0,1] \subset D_\alpha$ by considering

$$\widetilde{f}(z) = 1 - \frac{1}{\tau} + \frac{f_\alpha(\tau z)}{\tau}$$

for $\tau < 1$ and compare $D(\widetilde{f})$ with $D(f_\alpha)$. Then $f_\alpha(-1) = 0$ since f_α is not identically 0 and $f_\alpha \in T_\alpha$.

Thus with Lemmas 2.2 and 2.3, f_α minimizes $D(f)$ on T_α iff f_α minimizes $L(f)$ under the constraints

$$2 \int_0^\pi d\mu_f = 1$$

$$\int_0^\pi \sec^2\left(\frac{t}{2}\right) d\mu_f = \frac{2}{\alpha}$$

Now we can use well known results to show f_α is extremal iff K_α satisfies

$$K_\alpha(t) = \lambda_0 + \lambda_1 \sec^2\left(\frac{t}{2}\right) \qquad \forall\, t \in \text{ Supp } (\mu_{f_\alpha})$$

$$K_\alpha(t) \geq \lambda_0 + \lambda_1 \sec^2\left(\frac{t}{2}\right) \qquad \forall\, t \notin \text{ Supp } (\mu_{f_\alpha}),$$

where λ_0, λ_1 are real constants.

Long computations, see [3], show our f_α gives K_α that satisfies these conditions! ■

Next we characterize the geometry of extremal domains for N_α.

Lemma 2.4. 1. *$\forall\, \alpha, 1 < \alpha < 4$, an extremal f_α minimizing $D(f)$ exists in N_α.*

2. *If f_α is extremal, then $f_\alpha(\mathbb{D})$ is bounded, starlike with respect to 1, and circularly symmetric with respect to rays*

$$\ell_\tau = \{z = x + iy : y = 0, x \geq \tau, \forall\, \tau, 0 \leq \tau \leq 1\}.$$

3. *The min area $A(\alpha) := D(f_\alpha)$ for $1 < \alpha < 4$.*

Proof. Apply circular and radial symmetrizations, then polarizations similar to arguments in [5]. ■

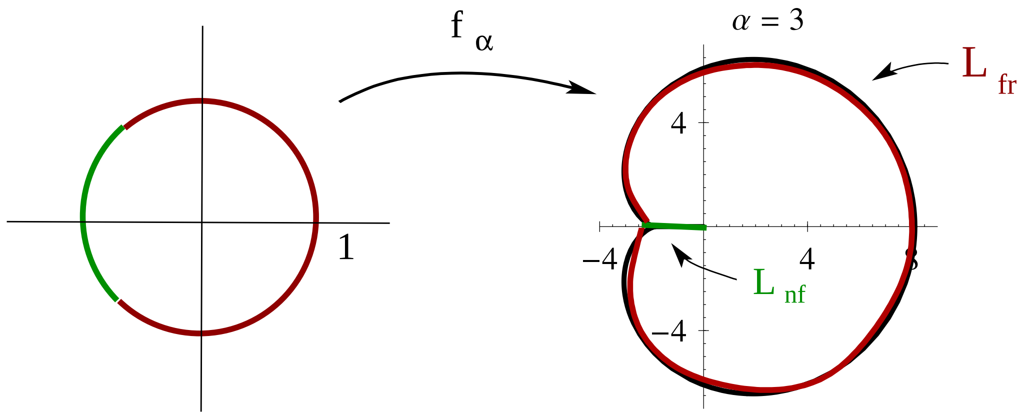

FIGURE 3. The free (L_{fr}) and non-free (L_{nf}) portions of the boundary.

Now combine Theorem 2.1 and Lemma 2.2 to see that if f_α is extremal in N_α, then since $f_\alpha \in T_\alpha$, Theorem 2.1 implies Theorem 1.1.

Next we show how the extremal f_α in N_α can be recovered from its boundary values.

Lemma 2.5. *Let f_α be extremal for N_α. Then f' is continuous on $\overline{\mathbb{D}}$ and $|f'| \equiv \beta \geq \alpha \, \forall \, z \in \ell_{fr}$. See Figure 2.*

Proof. Apply the deep "two point variation techniques" from [5] twice giving f' these properties on ℓ_{fr}. Then use the Julia-Wolff Theorem and boundary behavior properties from Pommerenke [6], giving f' these properties everywhere. ∎

Lemma 2.6. *If f_α is extremal, $\varphi(z) = \log(zf'_\alpha(z))$ maps as described in Figure 2, with*

$$q_1(z) = \frac{i(1-z)}{2\sin(\frac{\varphi_0}{2})\sqrt{z}}$$

$$\varphi_2(\xi) = ci\int_0^\xi \frac{t^2 - b^2}{(t^2-a^2)\sqrt{1-t^2}}\, dt + s.$$

Long computations are used to show monotonicity, then we use line integral formulae to compute the area as in [4].

3. The Iceberg Problem

A related problem is known as the Iceberg Problem: Given a fixed volume above the water, how deep can the iceberg go? See Figure 3.

This problem can be modeled by supposing a slice $\boldsymbol{I}$ is a continuum in $\mathbb{C}$ and $\mathcal{E} = \{\boldsymbol{I} : \text{cap}\ \boldsymbol{I} = 1,\ \text{area}\ [\boldsymbol{I} \cap \text{UHP}] = \alpha\}$.

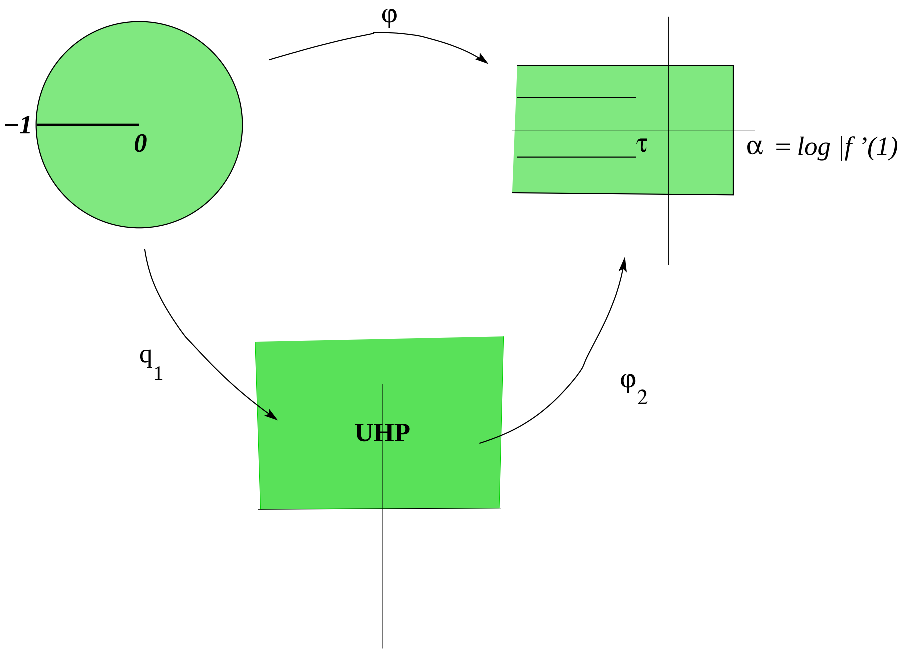

FIGURE 4. The mapping $\varphi(z) = \log(z f'_\alpha(z))$.

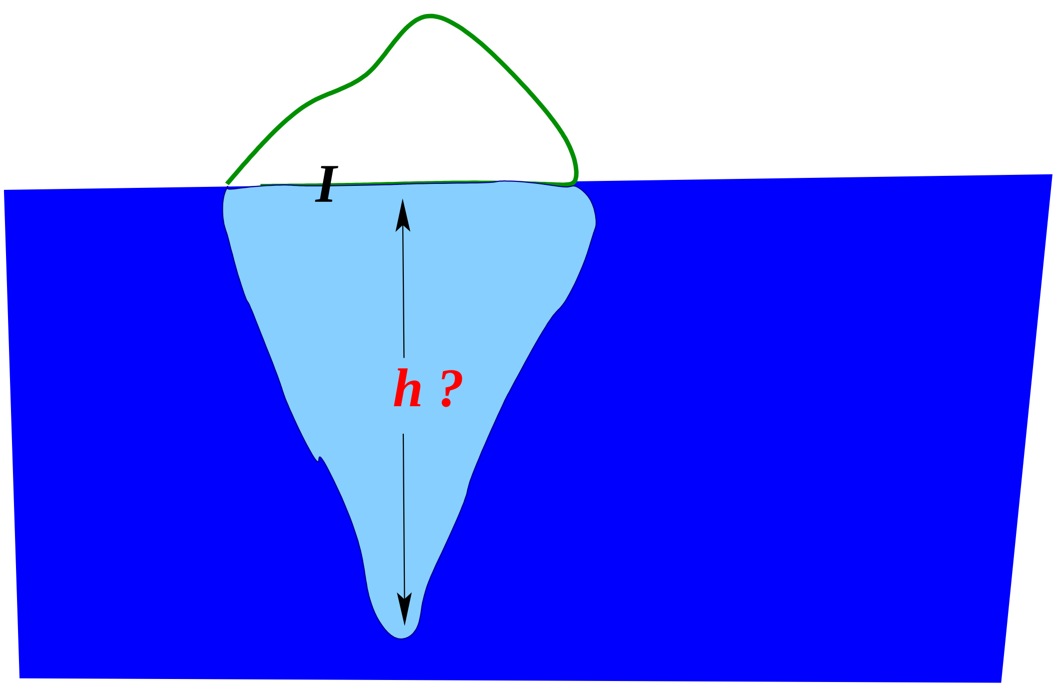

FIGURE 5. The Iceberg Problem.

We anticipate using similar arguments to those in this paper to find

$$h = \min_{\boldsymbol{I} \in \mathcal{E}} \min\{\Im z : z \in \boldsymbol{I}\}.$$

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Roger W. Barnard, Clint Richardson, Alex Yu. Solynin ADDRESS: *Department of Mathematics and Statistics, Texas Tech University, Lubbock, Texas 79409*

ADDRESS: *Department of Mathematics and Statistics, Stephen F. Austin University, Nacogdoches, Texas 75962*

ADDRESS: *Department of Mathematics and Statistics, Texas Tech University, Lubbock, Texas 79409*

E-MAIL: roger.w.barnard@ttu.edu

E-MAIL: crichardson@sfasu.edu

E-MAIL: alex.solynin@ttu.edu