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Experiments with moduli of quadrilaterals II

Antti Rasila and Matti Vuorinen

Abstract. The numerical performance of the AFEM method of K. Samuelsson is studied in the computation of moduli of quadrilaterals.

Keywords. conformal modulus, quadrilateral modulus.

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1. Introduction

The moduli of quadrilaterals and rings are some of the fundamental tools in geometric function theory, see [Ahl], [AVV], [Küh], [LV]. The purpose of this paper is to report on our experimental work on the numerical computation of the moduli of quadrilaterals, based on the algorithms and software of [BSV] and motivated by the geometric considerations in [HVV] and [DV]. The methods considered here may be classified into two classes:

(1) methods based on the definition of the modulus and on conformal mapping of the quadrilateral onto a canonical rectangle,
(2) methods based on the solution of the Dirichlet-Neumann problem for the Laplace equation.

With the exception of a few special cases both methods lead to extensive numerical computation. For both classes of methods there are several options, see [Gai], [Hen], [Pap]. Among other things, historical remarks are given in [Por].

We study the case of a polygonal quadrilateral and the way its modulus depends on the shape of the quadrilateral. Following the approach of [BSV] our main method is the adaptive finite element method AFEM of Klas Samuelsson and it belongs to class (2). We compare this method to a method of class (1), the Schwarz-Christoffel method of L.N. Trefethen [DrTr] and its MATLAB implementation, the SC Toolbox written by T. Driscoll [Dri].

In the two test cases we have used, the performance of the SC Toolbox was superior to AFEM. On the other hand, the AFEM software applies also to computation of moduli of polygonal ring domains as shown in [BSV]. AFEM also has an advantage in the problems where the quadrilateral has a large number of vertices. This situation arises when approximating nonpolygonal quadrilaterals (e.g. Example 3.7). We will report our results also in [RV2].

2. Preliminaries

A Jordan domain D in $\mathbb{C}$ with marked (positively ordered) points $z_1, z_2, z_3, z_4 \in \partial D$ is a quadrilateral and denoted by $(D; z_1, z_2, z_3, z_4)$. We use the canonical map of this quadrilateral onto a rectangle $(D'; 1+ih, ih, 0, 1)$, with the vertices corresponding, to define the modulus h of a quadrilateral $(D; z_1, z_2, z_3, z_4)$. The modulus of $(D; z_2, z_3, z_4, z_1)$ is $1/h$.

We mainly study the situation where the boundary of D consists of the polygonal line segments through z_1, z_2, z_3, z_4 (always positively oriented). In this case, the modulus is denoted by $\mathrm{QM}(D; z_1, z_2, z_3, z_4)$. If the boundary of D consists of straight lines connecting the given boundary points, we omit the domain D and denote the corresponding modulus simply by $\mathrm{QM}(z_1, z_2, z_3, z_4)$.

The following problem is known as the *Dirichlet-Neumann problem*. Let D be a region in the complex plane whose boundary ∂D consists of a finite number of regular Jordan curves, so that at every point, except possibly at finitely many points of the boundary, a normal is defined. Let ψ to be a real-valued continuous function defined on ∂D. Let $\partial D = A \cup B$ where A, B both are unions of Jordan arcs. Find a function u satisfying the following conditions:

1. u is continuous and differentiable in $\overline{D}$.
2. $u(t) = \psi(t), \qquad t \in A$.
3. If $\partial/\partial n$ denotes differentiation in the direction of the exterior normal, then

$$\frac{\partial}{\partial n} u(t) = \psi(t), \qquad t \in B.$$

One can express the modulus of a quadrilateral $(D; z_1, z_2, z_3, z_4)$ in terms of the solution of the Dirichlet-Neumann problem as follows. Let $\gamma_j, j = 1, 2, 3, 4$ be the arcs of ∂D between (z_1, z_2), (z_2, z_3), (z_3, z_4), (z_4, z_1), respectively. If u is the (unique) harmonic solution of the Dirichlet-Neumann problem with boundary values of u equal to 0 on γ_2, equal to 1 on γ_4 and with $\partial u/\partial n = 0$ on $\gamma_1 \cup \gamma_3$, then by [Ahl, p. 65/Thm 4.5]:

$$\mathrm{QM}(D; z_1, z_2, z_3, z_4) = \int_D |\nabla u|^2 \, dm. \tag{2.1}$$

We also have the following connection to the modulus curve family (see e.g. [AVV, pp. 158–165]): $\mathrm{QM}(D; z_1, z_2, z_3, z_4) = \mathsf{M}(\Gamma)$, where Γ is the family of all curves joining γ_2 and γ_4 in D.

Another approach is to use the *Schwarz-Christoffel formula* to approximate the conformal mapping f onto the canonical rectangle. This formula gives an expression for a conformal map from the upper half-plane onto the interior of a n-gon. Its vertices are denoted $w_1, \ldots, w_n$, and $\alpha_1\pi, \ldots, \alpha_n\pi$ are the corresponding interior angles. The preimages of the vertices (prevertices) are denoted by $z_1 < z_2 < \ldots < z_n$. The Schwarz-Christoffel formula for the map f is

$$f(z) = f(z_0) + c\int_{z_0}^{z} \prod_{j=1}^{n-1} (\zeta - z_j)^{\alpha_j - 1} d\zeta, \tag{2.2}$$

where c is a (complex) constant. The main difficulty in applying this formula is that the prevertices z_j cannot, in general, be solved for analytically. By using a Möbius transformation, one may choose three of the prevertices arbitrarily. The remaining $n-3$ prevertices are then obtained by solving a system of nonlinear equations. Several methods for solving this problem are discussed in [Bis], [DrTr], and [DrVa].

The MATLAB toolbox by T. Driscoll [Dri] contains a collection of algorithms for constructing Schwarz-Christoffel maps and computing the moduli of polygonal quadrilaterals. The toolbox also gives an estimate for the accuracy of the numerical approximation of the modulus.

3. Experiments

The solutions of the Dirichlet and the Dirichlet-Neumann problems can be approximated by the method of finite elements, see [Hen, pp. 305–314], [Pap]. Hence, this method can also be used to approximate the modulus of quadrilaterals and rings. The Dirichlet-Neumann problem can be numerically solved with AFEM (Adaptive FEM) numerical PDE analysis package by Klas Samuelsson. This software applies, e.g., to compute the modulus (capacity) of a bounded ring whose boundary components are broken lines. Examples and applications for this software are given in [BSV].

In [HVV] a theoretical formula for computing $\mathrm{QM}(A, B, 0, 1)$ was given with its implementation with Mathematica. This led to a study of the modulus of quadrilateral in [DV]. In the course of the work on [DV], the variation of the modulus was studied when one of the vertices varies and others are kept fixed, and several conjectures were formulated. For these purposes, neither the theoretical algorithm in [HVV] nor the implemented Mathematica program based on it was no longer adequate and we started to look for a robust program to compute $\mathrm{QM}(A, B, 0, 1)$. It seems that the AFEM software of Samuelsson is very efficient for this purpose. As in [RV1] we use the AFEM software of Samuelsson for computations involving moduli of polygonal quadrilaterals.

3.1. Example. Let $f(x,y) = \mathrm{QM}(x+iy, i, 0, 1) - 1/\mathrm{QM}(y+ix, i, 0, 1)$. Then by [Hen, p. 433] we see that $f(x,y) \equiv 0$. Therefore we may use this function as a measure of the accuracy of AFEM software and SC Toolbox.

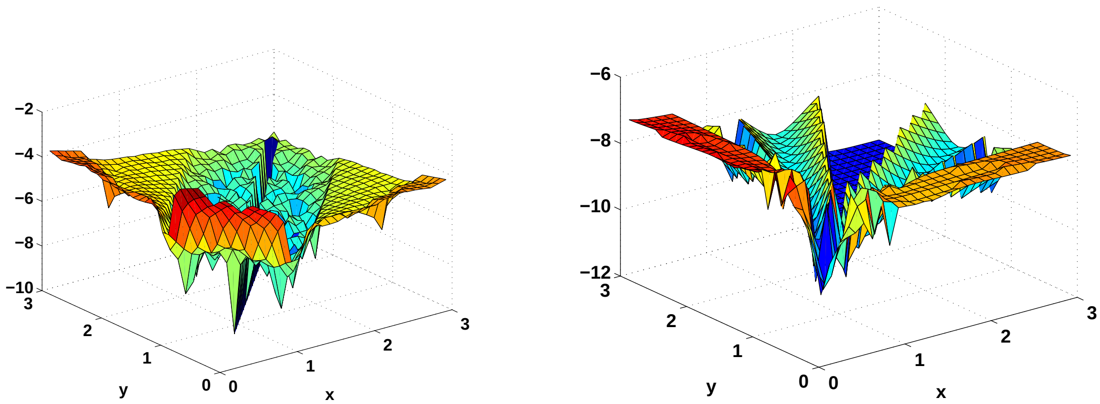

FIGURE 1: Function $\log_{10}(|f(x,y)| + 10^{-10})$ for $x \in (0,3]$, $y \in (0,3]$ with AFEM (left) and SC Toolbox (right).

3.2. Example. We study the function

$$g(t,h) = \mathrm{QM}(1 + he^{it}, he^{it}, 0, 1).$$

An analytic expression for this function has been given in [AQVV, 2.3]:

$$g(t,h) = \mathcal{K}'(r_{t/\pi})/\mathcal{K}(r_{t/\pi}), \tag{3.3}$$

where

$$r_a = \mu_a^{-1}\left(\frac{\pi h}{2\sin(\pi a)}\right), \text{ for } 0 < a \leq 1/2, \tag{3.4}$$

and the decreasing homeomorphism $\mu_a\colon (0,1) \to (0,\infty)$ is defined by

$$\mu_a(r) \equiv \frac{\pi}{2\sin(\pi a)} \frac{F(a, 1-a; 1; 1-r^2)}{F(a, 1-a; 1; r^2)}. \tag{3.5}$$

Here

$$F(a,b;c;z) = {}_2F_1(a,b;c;z) \equiv \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{z^n}{n!}, \qquad |z| < 1,$$

is the *Gaussian hypergeometric function*,

$$(a,n) \equiv a(a+1)(a+2)\ldots(a+n-1), \qquad (a,0) = 1 \text{ for } a \neq 0,$$

is the *shifted factorial function*, and the *elliptic integrals* $\mathcal{K}(r), \mathcal{K}'(r)$ are defined by

$$\mathcal{K}(r) = \frac{\pi}{2} F(1/2, 1/2; 1; r^2), \qquad \mathcal{K}'(r) = \mathcal{K}(r'), \text{ and } r' = \sqrt{1 - r^2}.$$

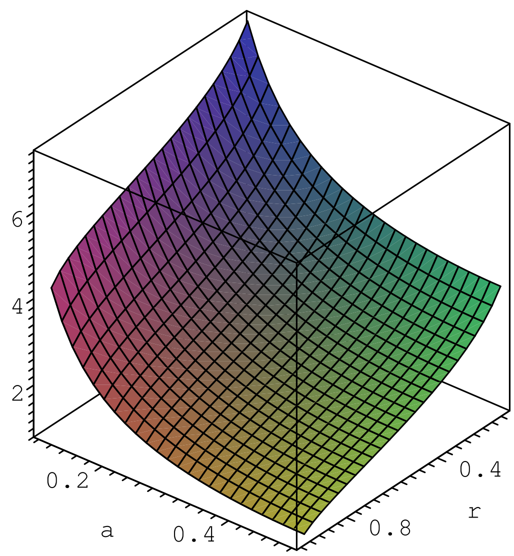

FIGURE 2: Function $\mu_a(r)$.

The function $g(t,h)$ is the modulus of the parallelogram with opposite sides 1 and h, respectively, and we see that there are three cases $h \in (0,1)$, $h = 1$ and $h > 1$. In the first case the function is monotone increasing with respect to $t \in (0, \pi/2)$, in the second case the function $g(t,1) \equiv 1$ is constant and in the third case decreasing.

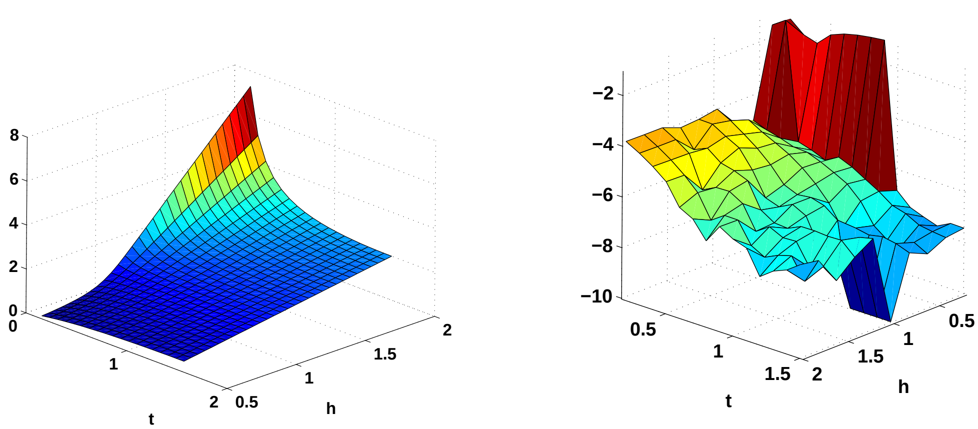

FIGURE 3: Function $g(t,h)$ for $t \in (0,\pi/2)$ and $h \in [1/2,2]$ (left), and the error estimate $\log_{10}(|g_{\mathrm{exact}}(t,h) - g_{\mathrm{numer}}(t,h)| + 10^{-10})$ for the function $g(t,h)$ (right).

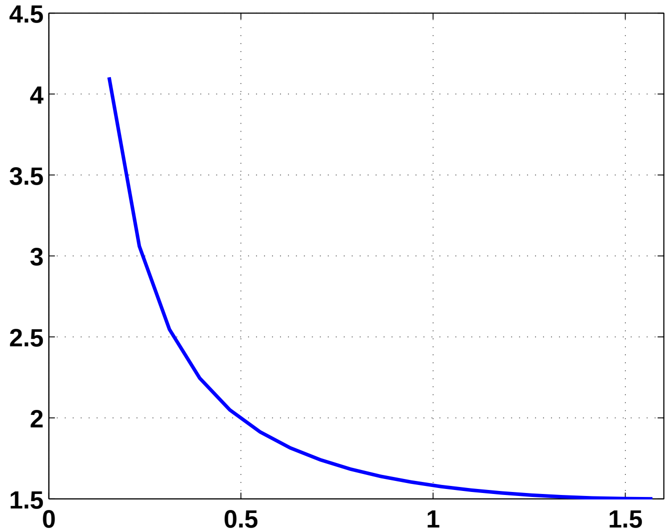

FIGURE 4: Function $g(t,1.5)$ for $t \in (0,\pi/2)$.

3.6. Example. The modulus $\mathrm{QM}(1+i|a-1|, i|b|, 0, 1)$ has an analytic expression if $|a-1| = h = |b|+1$. Bowman [Bow, pp. 103-104] (see also [Bur, p.197]) gives a formula for the conformal modulus of the quadrilateral with vertices $1+hi$, $(h-1)i$, 0, and 1 when $h>1$ as $M(h) \equiv \mathcal{K}(r)/\mathcal{K}(r')$ where

$$r = \left(\frac{t_1 - t_2}{t_1 + t_2}\right)^2, \quad t_1 = \mu_{1/2}^{-1}\left(\frac{\pi}{2c}\right), \quad t_2 = \mu_{1/2}^{-1}\left(\frac{\pi c}{2}\right), \quad c = 2h-1\,.$$

Therefore, the quadrilateral can be conformally mapped onto the rectangle $1 + iM(h)$, $iM(h)$, 0, 1, with the vertices corresponding to each other. It is clear that $h-1 \le M(h) \le h$. The formula

$$M(h) = h + c + O(e^{-\pi h}), \qquad c = -1/2 - \log 2/\pi \approx -0.720636\,,$$

is given in [PS]. As far as we know there is neither an explicit nor asymptotic formula for the case when the angle $\pi/4$ of the trapezoid is equal to $\alpha \in (0,\pi/2)$. We compute the modulus $\mathrm{QM}(ih, i(h-1), 0, 1)$ by using the square frame capacity formula, AFEM and Schwarz-Christoffel Toolbox.

h	**AFEM**	**SC**	**Accuracy /SC**	**Bowman**	**Error /AFEM**	**Error/SC**
1.1	0.3403159	0.3403135	$1.787 \cdot 10^{-8}$	0.3403135	$2.41655 \cdot 10^{-6}$	$1.57002 \cdot 10^{-9}$
1.2	0.4614938	0.4614926	$1.734 \cdot 10^{-8}$	0.4614926	$1.20727 \cdot 10^{-6}$	$2.98441 \cdot 10^{-9}$
1.3	0.5704380	0.5704374	$5.310 \cdot 10^{-8}$	0.5704374	$5.83493 \cdot 10^{-7}$	$4.59896 \cdot 10^{-9}$
1.4	0.6747519	0.6747518	$1.046 \cdot 10^{-7}$	0.6747518	$8.83554 \cdot 10^{-8}$	$6.24458 \cdot 10^{-9}$
1.5	0.7769433	0.7769434	$2.408 \cdot 10^{-8}$	0.7769434	$1.10607 \cdot 10^{-7}$	$3.39673 \cdot 10^{-9}$
1.6	0.8780836	0.8780838	$1.920 \cdot 10^{-9}$	0.8780838	$1.53305 \cdot 10^{-7}$	$8.10543 \cdot 10^{-10}$
1.7	0.9786840	0.9786842	$5.439 \cdot 10^{-10}$	0.9786842	$2.41392 \cdot 10^{-7}$	$2.02109 \cdot 10^{-10}$
1.8	1.0790020	1.0790024	$2.102 \cdot 10^{-10}$	1.0790024	$4.03325 \cdot 10^{-7}$	$4.94438 \cdot 10^{-11}$
1.9	1.1791710	1.1791715	$6.225 \cdot 10^{-11}$	1.1791715	$5.22481 \cdot 10^{-7}$	$1.20739 \cdot 10^{-11}$
2.0	1.2792610	1.2792616	$1.536 \cdot 10^{-11}$	1.2792616	$5.71171 \cdot 10^{-7}$	$2.97451 \cdot 10^{-12}$

TABLE 1: Error estimates for AFEM and SC Toolbox with the square frame capacity formula. The accuracy of the estimate given by SC Toolbox is also consistent with the experiment.

3.7. Example. Let Q be the quadrilateral whose sides are defined by two circular arcs in the upper and lower half plane, perpendicular to the unit circle at the points $e^{i\theta}$, $e^{i(\pi-\theta)}$, $e^{i(\theta-\pi)}$, $e^{-i\theta}$ as well as by the two circular arcs through r, i, $-i$ and $-r$, i, $-i$, see Figure 6. If a, b, c, d are the points of intersection of these four circular arcs in the II, III, IV and I quadrants respectively, then $Q = (Q; a, b, c, d)$ defines a quadrilateral in the unit disk with

$$\mathrm{QM}(Q; a, b, c, d) = (\pi - 2\beta)/\rho,$$

where

$$\rho = 2 \log \frac{1+u}{1-u}, \quad \beta = \operatorname{arccot} \frac{2r}{1-r^2}, \text{ and } u = \tan(\theta/2).$$

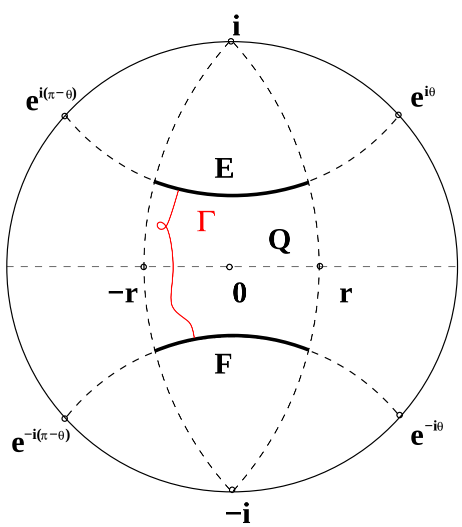

FIGURE 5: The circular quadrilateral Q.

θ	**AFEM**	**Exact**	**Error**
0.10	7.592357	7.597433	$5.076 \cdot 10^{-3}$
0.15	5.056044	5.054357	$1.687 \cdot 10^{-3}$
0.20	3.784480	3.779611	$4.869 \cdot 10^{-3}$
0.25	3.010221	3.012175	$1.954 \cdot 10^{-3}$
0.30	2.497983	2.498368	$3.849 \cdot 10^{-4}$
0.35	2.130426	2.129465	$9.616 \cdot 10^{-4}$
...	...	...	...
1.00	0.620785	0.620631	$1.531 \cdot 10^{-4}$
1.05	0.575335	0.575402	$6.645 \cdot 10^{-5}$
1.10	0.533529	0.533010	$5.185 \cdot 10^{-4}$
1.15	0.493114	0.492934	$1.805 \cdot 10^{-4}$
1.20	0.454678	0.454689	$1.114 \cdot 10^{-5}$

TABLE 2: The modulus of the quadrilateral Q for $r = 0.4$.

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Antti Rasila E-MAIL: `antti.rasila@tkk.fi`
ADDRESS:
Institute of Mathematics,
P.O.Box 1100, FIN-02015,
Helsinki University of Technology,
FINLAND

Matti Vuorinen E-MAIL: `vuorinen@utu.fi`
ADDRESS:
Department of Mathematics,
FIN-20014 University of Turku,
FINLAND