Proceedings of the International Conference on Geometric Function Theory,
Special Functions and Applications (ICGFT)
Editors: R. W. Barnard and S. Ponnusamy
J. Analysis
Volume 15 (2007), 1–15

Some Results on Spaces of Packable Riemann Surfaces

Roger W. Barnard, Eric M. Murphy, and G. Brock Williams

Abstract. In this paper, we describe the basic theory of circle packings on Riemann surfaces. We also state and prove two propositions regarding the interplay between the patterns of tangencies in combinatorial strictures defining Riemann surfaces which admit circle packings and the geometry of the circle packings realized on those surfaces.

Keywords. Circle Packing, Riemann Surfaces, Conformal Maps, Teichmüller Theory.

2000 MSC. 30F60, 52C26.

1. Introduction

A circle packing is a collection of circles with a prescribed pattern of tangencies. Of course, given a collection of circles, there is no guarantee that they will "fit together" in a particular pattern; in fact, some patterns may not be possible. For example, given that all the circles have the same radius, one circle cannot be tangent to more than six other Euclidean circles. The "prescribed pattern" is a strictly combinatorial structure with no inherent geometry. As the circles adjust their radii, trying to meet the constraints of the combinatorial pattern prescribed, a rigid geometry is realized. The interplay between the combinatorial structure and the rigid constraints inherent in the geometry of the circles provides a deep link to geometric function theory and the structure of Riemann surfaces.

Imposing the geometry of a circle implies the existence of a metric; thus we can speak about circle packings on any surface with a metric. In particular, we may discuss circle packing on any Riemann surface. In fact, it is known that given any reasonable pattern of tangencies there exists a unique Riemann surface which supports a circle packing having that pattern of tangencies [2]. Not all surfaces, though, support a circle packing. However, it has been shown by Brooks [7], Bowers and Stephenson [5, 4], Barnard and Williams [1], Williams [29], and

The views expressed in this article are those of the authors and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the U.S. Government. Supported by NSF grant #0536665.

Murphy [19], that these packable Riemann surfaces are dense in Teichmüller space. To go beyond this notion of density and enter into a deterministic discussion of how the pattern of tangencies affects the geometry of the surfaces and their circle packings can be very difficult [2, 4, 5, 21].

Once we outline the essential background material in the theory of circle packing, we then prove two results which speak specifically to the relationships between the prescribed patterns of tangencies and the geometries of surfaces and their circle packings. First, we prove a proposition regarding the interaction between the pattern of tangencies, the geometry of a surface and the unique packing on that surface. We then prove a second proposition regarding the density of a subclass of packable surfaces with constrained geometries. Specifically, we show that we may approximate any Riemann surface with a sequence of packable surfaces such that the radii of the circles in the packings admitted on the surfaces tend toward zero.

2. Riemann Surfaces and Circle Packing

2.1. Teichmüller Spaces of Riemann Surfaces. Here we review some of the important definitions and properties of Riemann surfaces and their Teichmüller spaces. Excellent references giving detailed development of the descriptions which follow are available in [8, 9, 12, 14, 15, 16, 25].

Definition 2.1. A **Riemann surface** is a one complex-dimensional manifold with charts whose overlap maps are analytic. The maximal collection of charts on a Riemann surface is the **conformal structure** for that surface.

Two Riemann surfaces R_1 and R_2 are said to be conformally equivalent if and only if there exists a conformal homeomorphism $f : R_1 \to R_2$. The set of equivalence classes of surfaces of the same topological type as R_1 under this equivalence relation is the **moduli space** of R_1. The equivalence relation defined by conformal equivalence which determines the moduli space of R_1 is not adequate for our purposes, however. We will require that for surfaces R_1 and R_2 to be considered equivalent they must first be conformally equivalent (the same point in moduli space) and we will further require that the generators of their fundamental groups correspond.

Definition 2.2. Let R be a Riemann surface, and let Σ be a collection of canonical generators for $\pi_1(R)$. The collection Σ is called a **marking** for R. Two markings on R are equivalent if they differ only by the choice of their base point.

This gives us a new equivalence relation which we use to describe another space of surfaces.

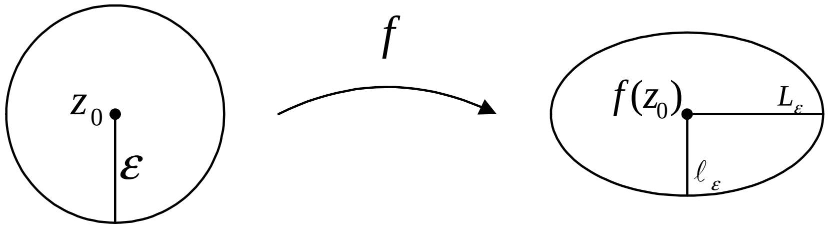


Figure 1: Action of a Map f with Dilatation $D_f(z_0) > 1$

Definition 2.3. Two marked Riemann surfaces (R, Σ) and $(\widehat{R}, \widehat{\Sigma})$ are said to be equivalent if and only if there exists a conformal map $f : R \to \widehat{R}$ for which the marking $f(\Sigma)$ is equivalent to $\widehat{\Sigma}$. The **Teichmüller space of R** is the set of these equivalence classes.

Another useful description of the Teichmüller space of a surface involves equivalence classes of maps from a reference or base surface. First we require a generalization of the concept of a conformal map.

Definition 2.4. Let f be a continuous, orientation-preserving map from a domain $\Omega \subset \mathbb{C}$ into the complex plane, and fix $z_0 \in \Omega$. Define L_ε and ℓ_ε by

$$L_\varepsilon = max\left\{|f(z) - f(z_0)| : |z - z_0| = \varepsilon\right\},$$

and

$$\ell_\varepsilon = min\left\{|f(z) - f(z_0)| : |z - z_0| = \varepsilon\right\},$$

as shown in Figure 1. The **dilatation** of f at z_0 is

$$D_f(z_0) = \limsup_{\varepsilon \to 0^+} \left(\frac{L_\varepsilon}{\ell_\varepsilon}\right).$$

If $\sup_{z\in\Omega}\{D_f(z)\} \le K$, then we say that $f : \Omega \to \mathbb{C}$ is K**-quasiconformal**.

Now, the class of quasiconformal maps allows us to define a new equivalence relation of the set of Riemann surfaces.

Definition 2.5. Quasiconformal maps f_1 and f_2 defined on a Riemann surface R are Teichmüller equivalent if and only if $f_2 \circ f_1^{-1}$ is homotopic to a conformal map.

Proposition 2.6 establishes the equivalence of these two characterizations of points in Teichmüller space [12, 16].

Proposition 2.6. *Fix a Riemann surface R, and suppose f_1 and f_2 are maps from R to Riemann surfaces R_1 and R_2, respectively. R_1 and R_2 are equivalent in the Teichmüller space of R if and only if f_1 and f_2 are Teichmüller equivalent.*

Thus, points in the Teichmüller space of a Riemann surface R may be considered as either points or as functions. There is a natural metric on Teichmüller space as a function of how close to conformal (or how quasiconformal) maps which preserve the markings might be. If we fix a Riemann surface R and a marking Σ on R, the distance between two points $R_1 = f_1(R)$ and $R_2 = f_2(R)$ in the Teichmüller space of R is given by

$$d(R_1, R_2) = \frac{1}{2}\log(K^*),$$

where K^* is the infimum of the dilatation of $g_2 \circ g_1^{-1}$ where g_1 and g_2 are equivalent to f_1 and f_2, respectively. This infimum is attained, by definition, by the unique **Teichmüller map**.

2.2. Introduction to Circle Packing. A circle packing is a configuration of circle with a prescribed pattern of tangencies. William Thurston conjectured in 1985 that these circle packings might be used to approximate the action of conformal maps [27]. These circle packings have since been widely studied, with applications in many different areas of mathematics. We begin here with some basic definitions and a general discussion of circle packing. Several excellent resources are available with much greater detail [11, 21, 23, 24, 31].

Definition 2.7. A **bounded degree abstract triangulation** $\mathcal{K}$ is an abstract simplicial 2-complex which triangulates an orientable topological surface such that

1. the set of interior vertices (vertices such that every incident edge belongs to two faces) is non-empty and edge-connected;
2. no interior edge (an edge belonging to two faces) in $\mathcal{K}$ has both vertices on the boundary;
3. no vertex in $\mathcal{K}$ belongs to more than two boundary edges;
4. there is an upper bound on the degree of vertices in $\mathcal{K}$.

It is this combinatorial object, the abstract triangulation, which encapsulates the "prescribed pattern of tangencies" in our circle packing. We refer to these triangulations as abstract to emphasize the fact that in the definition we have implied no concrete geometric realization. A 2-complex and, by extension, the associated abstract triangulation are purely combinatorial objects; they have no inherent geometric structure until they are realized as a circle packing. An example of a simple valid abstract triangulation is shown in Figure 2. This triangulation comprises 27 vertices, with edges between vertices indicating tangencies.

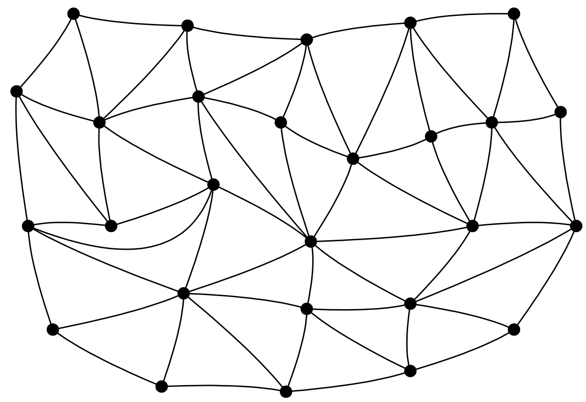


Figure 2: An Abstract Triangulation $\mathcal{K}$

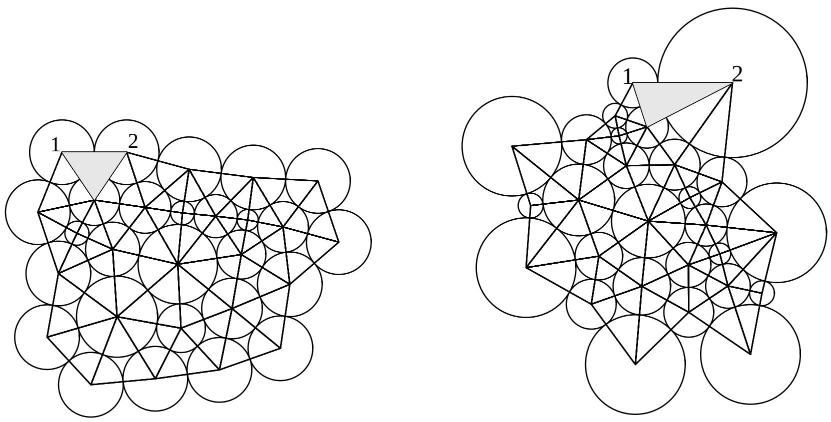
Definition 2.8. A **circle packing** is a configuration of circles with a specified pattern of tangencies. In particular, if $\mathcal{K}$ is an abstract triangulation of a topological surface, then a circle packing P for $\mathcal{K}$ is a configuration of circles such that

1. P contains a circle C_v for every vertex $v \in \mathcal{K}$;
2. if $[u, v]$ is an edge of $\mathcal{K}$, then C_v is externally tangent to C_u;
3. if $\langle v, u, w \rangle$ is a positively oriented face of $\mathcal{K}$, then $\langle C_v, C_u, C_w \rangle$ forms a positively oriented mutually tangent triple of circles in P.

Realizing such a configuration is, at its heart, a problem in computing the necessary centers and radii of each circle in the packing. Examples of such algorithms are given by Collins and Stephenson [10, 21] and Mohar [18]. A circle packing is called **univalent** if the circles in the packing have mutually disjoint interiors. This univalent circle packing represents a geometric realization of the underlying abstract triangulation $\mathcal{K}$. Vertices in the triangulation may be realized in this packing as the centers (in some particular geometry, hyperbolic, Euclidean, or spherical) of the circles, and the edges as geodesic segments connecting the centers. This embedding is called the **carrier**, written carr P, of the circle packing P.

If $\mathcal{K}$ is embedded in $\mathbb{C}$ in two different ways (e.g., by giving two different sets of values for the radii of the boundary circles), there is a natural piecewise map from the carrier associated with one packing to the other achieved by sending triangles of one packing to their counterparts in the other using affine maps. These piecewise affine maps are referred to as **discrete conformal maps**.

We can produce a geometric realization (as a circle packing) of the combinatorial structure given in Figure 2 in any number of ways, by defining the radii of the boundary vertices. Two different circle packings, obtained through applying different values to the boundary vertices, are shown in Figure 3. In each of Figure 3a and Figure 3b, a triangle is shaded for reference; the triangle in each case corresponds to a face determined by the same three vertices from the triangulation shown in Figure 2.



(a) Euclidean Carrier for P_1 (b) Euclidean Carrier for P_2

Figure 3: The Carriers of Two Different Circle Packings of $\mathcal{K}$

2.3. Discrete Function Theory. The important characteristic of the discrete conformal maps induced by circle packings is not that they are quasiconformal; the key fact, suggested by Thurston [28] and proven by Rodin and Sullivan [20], is that these maps are “nearly conformal.” This is the result given in Theorem 2.15, the Rodin-Sullivan Theorem. Before we state this theorem, however, we first state some geometric results associated with circle packing.

Definition 2.9. A **chain of circles** in a packing P for an abstract triangulation $\mathcal{K}$ is a collection of circles $(C_{v_1}, C_{v_2}, \ldots, C_{v_n})$ in P such that v_i and v_{i+1} share an edge in $\mathcal{K}$ for $i = 1, 2, \ldots, n-1$, and $v_i \neq v_j$, if $i \neq j$. Thus, a chain of circles describes a non-self-intersecting edge path in $\mathcal{K}$. Similarly, a **closed chain** is a collection of circles corresponding to a closed non-self-intersecting edge path in $\mathcal{K}$. An examples of a closed chain of circles in the packing P_2 is shown (as shaded circles) in Figure 4.

Lemma 2.10 (Length-Area Lemma)**.** *Let P be a univalent packing in $\mathbb{D}$ and C_v a circle in P with Euclidian radius r. Assume there exist m disjoint chains of circles in P having combinatorial lengths $n_1, n_2, \ldots, n_m$, such that each chain*

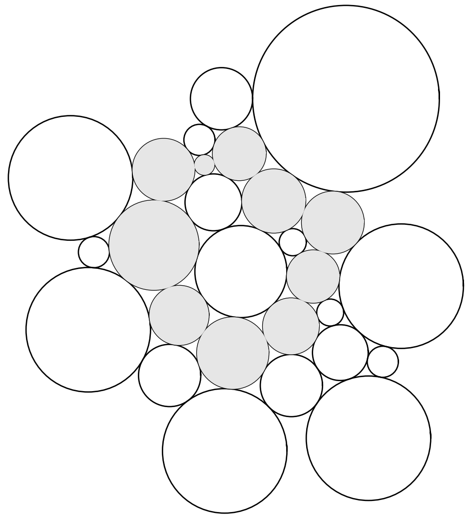


Figure 4: A Chain of Circles

separates C_v from 0 and a point on $\partial\mathbb{D}$. Then

$$r < \frac{4}{\sqrt{\sum_{i=1}^{n} \frac{1}{n_i}}}. \tag{2.11}$$

As the number of generations separating a circle in a packing from the boundary increases Lemma 2.10, the Length-Area Lemma, has the effect of forcing the radius of this circle to zero (in the limit). The Length Area Lemma can be extended to packings on the Riemann sphere [30] and other surfaces as well. For a more detailed discussion of Lemma 2.10, the Length-Area Lemma, see [20, 21].

Lemma 2.12 (Ring Lemma)**.** *Given a univalent packing in $\mathbb{C}$ there is a lower bound C_n, depending only on n, on the ratio of the radius r_0 of any interior circle to the radius r_i of any of its neighbors.*

Lemma 2.12, the Ring Lemma, guarantees that central angles in the carrier on a flower are bounded away from zero and π. That is, suppose we are given a complex $\mathcal{K}$ in which the **degree** of each vertex, the number of adjacent vertices, is bounded; also suppose we have two (different) packings P_1 and P_2 associated with $\mathcal{K}$. The Ring Lemma guarantees that the quasiconformality of the induced conformal map from P_1 to P_2 is bounded.

Lemma 2.13 (Hexagonal Packing Lemma)**.** *There is a sequence $\{s_n\}_n \in \mathbb{N}$, decreasing to zero, with the following property. Let c_1 be a circle in a univalent*

Euclidean circle packing P, and suppose the first n generations of circles about c_1 have degree 6. Then for any circle $c \in P$ tangent to c_1,

$$\left|1 - \frac{r_c}{r_{c_1}}\right| \leq s_n, \tag{2.14}$$

where r_c is the radius of the circle c in P and r_{c_1} is the radius of the circle c_1 in P.

The immediate value of Lemma 2.10, Lemma 2.12, and Lemma 2.13 is their use in proving Theorem 2.15, the Rodin-Sullivan Theorem, one of the fundamental results in the study of circle packing.

Theorem 2.15 (Rodin-Sullivan Theorem)**.** *Fix a simply connected domain $\Omega \subsetneq \mathbb{C}$ and points $p, q \in \Omega$. Let P_k be the portion lying in Ω of the infinite regular hexagonal packing whose circles all have radius $\frac{1}{k}$, and let $\mathcal{K}_k$ be the underlying complex for the packing P_k. Suppose $\widetilde{P}_k$ is a packing in $\mathbb{D}$ for $\mathcal{K}_k$ with all boundary circles tangent to $\partial\mathbb{D}$, and let $f_k : carr(P_k) \to carr(\widetilde{P}_k)$ be the induced discrete conformal map. If each $\widetilde{P}_k$ has been normalized so that $f_k(p) = 0$ and $f_k(q) > 0$, then $\{f_k\}$ converges locally uniformly to the unique Riemann map $f : \Omega \to \mathbb{D}$ satisfying $f(p) = 0$ and $f(q) > 0$.*

The requirement that the packings used in Theorem 2.15, the Rodin-Sullivan Theorem, all be of uniformly degree 6 is quite restrictive. Since the initial proof of Theorem 2.15, however, Stephenson [22] relaxed the degree 6 condition using techniques of random walks, and He and Rodin [13] showed that only a uniform bound on the degree is necessary. To thus relax the requirement on the combinatorics of the packing, we require Lemma 2.16, sometimes referred to as the Packing Lemma [21].

Lemma 2.16 (Packing Lemma)**.** *Let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of combinatorial closed disks such that*

1. *there exists a uniform bound on the degree of the vertices in K_n for each $n \in \mathbb{N}$, and*
2. *the sequence $\{K_n\}_{n\in\mathbb{N}}$ is either a nested sequence which exhausts a parabolic combinatorial disk or is asymptotically parabolic.*

There exists a sequence $\{s_m\}_{m\in\mathbb{N}} \subset \mathbb{R}$, decreasing to zero, with the following property. Suppose that for some n, u and v are adjacent interior vertices of K_n whose combinatorial distance from ∂K_n are both at least m, and suppose that P_n and $\widetilde{P}_n$ are two univalent, Euclidean circle packings for K_n. Then

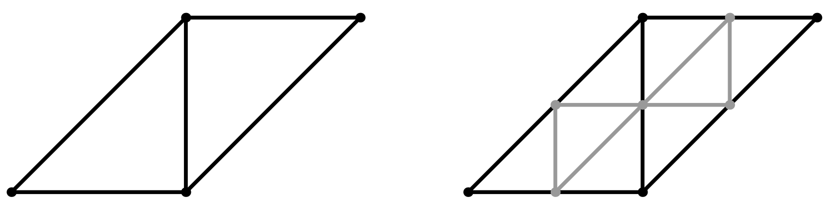
$$\left|\frac{\tilde{r}_u}{\tilde{r}_v} - \frac{r_u}{r_v}\right| \leq s_m, \tag{2.17}$$

where r_u and r_v are the radii of the circle in P_n corresponding to u and v, and $\tilde{r}_u$ and $\tilde{r}_v$ are the radii of the circle in $\tilde{P}_n$ corresponding to u and v.

Essentially, Lemma 2.16, the Packing Lemma, states that for a circle "deep" in a packing, the ratio of its radius to any given neighbor is nearly the same in the packings P_n and $\widetilde{P}_n$; in other words, the triangles in P_n and $\widetilde{P}_n$ are nearly similar triangles. This means that away from the boundary, the induced discrete conformal map between P_n and $\widetilde{P}_n$ is nearly conformal. This fact will play an important role in the proofs of many results related to circle packing.

2.4. Hex Refinement. In order to obtain the various approximation results for circle packing and discrete analytic function theory, we need a method to refine given circle packings. The primary requirement in any such refinement is that we maintain some uniform control over the degree of the complexes generated by the refinement algorithm since we require that Lemma 2.12, the Ring Lemma, applies at each successive level of refinement. The hex refinement method developed by Bowers and Stephenson [6] is especially nice.

Definition 2.18. If $\mathcal{K}$ is a 2-complex, the **hex refinement** of $\mathcal{K}$ is the complex formed by adding a vertex to each edge and adding an edge between any two vertices lying on the same face, as shown in Figure 5.



(a) Before Hex Refinement (b) After Hex Refinement

Figure 5: Hex Refinement of Triangles

Note that hex refinement, and refinement in general, is really a combinatorial process, refining the combinatorics of the complex $\mathcal{K}$; one must repack the new complex obtained by refining $\mathcal{K}$ in order to realize the effect of the refinement in a circle packing. Hex refinement has a number useful characteristics described by Bowers and Stephenson [6] and summarized in Proposition 2.19.

Proposition 2.19 (Bowers and Stephenson). *Any new interior vertices added to $\mathcal{K}$ by hex refinement have degree 6, while the degrees of the original vertices remain unchanged. If $\mathcal{K}$ is embedded in $\mathbb{C}$ in such a way that the edges correspond* *to Euclidean line segments, then its hex refinement may be realized by adding line segments joining the midpoint of each edge within every face. In this case, each face in $\mathcal{K}$ is subdivided into four new faces, each similar to the original face in which it is contained and having edges one-half as long.*

Notice that refining only one edge in a complex is not permitted, since this would result in a complex that is not a triangulation; the faces bordering the refined face will have an extra vertex along the common edge they share with the refined face, giving combinatorial quadrilaterals rather than triangles. We can, however, locally refine only those triangles in the complex which present some difficulty with respect to desired characteristics of the complex, then correct the introduced problems on adjacent faces by adding a single edge from a vertex to the midpoint of the opposite side, as shown in Figure 6.

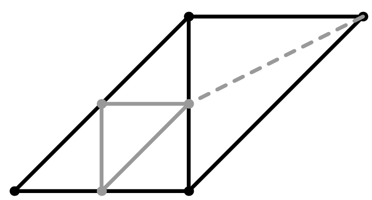


Figure 6: Hex Refinement and Correction on Adjacent Triangle

This process of hex refinement of individual faces and the addition of edges to absorb extra vertices can be used to locally refine an abstract triangulation in order to improve the discrete conformal approximation in troublesome areas. Local refinement will play a key role in the proofs to follow in Section 3.

3. Density and Geometry of Packable Surfaces

Demonstrations of the existence of circle packings have been given variously by Thurston [28], Minda and Rodin [17], Beardon and Stephenson [2], and Schramm [26]. Brooks [7] showed that compact packable surfaces are dense in moduli space, and Bowers and Stephenson [4, 5] extended this result to include surfaces of finite analytic type. Other results on the density of packable surfaces have been given by Barnard and Williams [1], Murphy [19], and Williams [29]. The results thus demonstrated which are germane to this research are summarized in Theorem 3.1.

Theorem 3.1. *Let $\mathcal{K}$ be an abstract triangulation of an orientable surface. Then there exists a unique surface in moduli space which supports a packing for $\mathcal{K}$.*

A complex, along with a choice of marking, then determines a unique point in Teichmüller space. Moreover, the collection of all packable surfaces is dense in Teichmüller space.

The density of packable surfaces as given in Theorem 3.1 is extremely useful and powerful. Of equal interest is the density of other subsets of packable surfaces and the characteristics of convergent sequences of packable surfaces.

Proposition 3.2. *Let R be a compact Riemann surface that is not packable. If $\{R_n\}_{n=1}^{\infty}$ is any sequence of compact, packable Riemann surfaces such that $R_n \to R$ in the Teichmüller metric where each admits a packing P_n with degree uniformly bounded through the sequence, then the radii in the circle packings $\{P_n\}_{n=1}^{\infty}$ tend to zero as $n \to \infty$.*

Proof. First note that since the circle packings $\{P_n\}_{n=1}^{\infty}$ are assumed to be of bounded degree, Lemma 2.10 guarantees that the radius of one circle in a packing is arbitrarily small if and only if the radii of all circles in that packing are small. Proceeding by way of contraposition, suppose that the radii of circles in P_n are bounded away from zero by some fixed positive constant for each $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, R_n is a compact Riemann surface, and therefore has finite area. Since $R_n \to R$ and R is compact, there exists a uniform upper bound on the area of the Riemann surfaces R_n. Hence, there exists a uniform bound on the total number of circles in each packing P_n. There are thus only a finite number of abstract triangulations possible among the circle packings $\{P_n\}_{n=1}^{\infty}$. By the pigeonhole principle, there exists a subsequence $\{R_{n_k}\}_{k=1}^{\infty} \subseteq \{R_n\}_{n=1}^{\infty}$ such that for each $k \in \mathbb{N}$ the circle packing realized in P_{n_k} corresponds to a single abstract triangulation $\mathcal{K}$, (i.e., there exists a subsequence of packings each having the same combinatorial structure). Since the abstract triangulation uniquely determines a surface for which that triangulation is realized as a packing, we find that $\{R_{n_k}\}_{k=1}^{\infty}$ is a constant sequence of Riemann surfaces. Now, $R_n \to R$ as $n \to \infty$, so every subsequence of $\{R_n\}_{n=1}^{\infty}$ must likewise converge to R; that is $R_{n_k} \to R$ as $k \to \infty$. Thus R is an element of the sequence $\{R_n\}_{n=1}^{\infty}$; in fact, R is equivalent to infinitely many elements of this sequence. Therefore, R is a compact, packable Riemann surface. Thus, by our contrapositive argument, if a sequence of packable surfaces, R_n, with degree uniformly bounded through the sequence converge to a Riemann surface that is not, itself, packable, then the mesh of the packings on the surfaces goes to zero. ∎

We now recall another well-known class of compact Riemann surfaces, called **equilateral surfaces**, which will be useful in constructing sequences of packable Riemann surfaces while maintaining control over the combinatorics of their underlying complexes.

Definition 3.3. Suppose S denotes a compact, orientable topological surface and let K denote a triangulation of the surface S. If we paste together equilateral triangles (triangles conformally equivalent to equilateral triangles) in the pattern of K to impose a piecewise affine structure on S, this affine structure defines a conformal structure on S, guaranteeing that S is a Riemann surface. Riemann surfaces thus constructed are called **equilateral surfaces**.

Just as with packable surfaces, compact equilateral surfaces are dense in Teichmüler space, as shown by Belyĭ [3]. We state this result as Theorem 3.4.

Theorem 3.4 (Belyĭ). *If S is a compact Riemann surface of genus $g > 0$, the set of equilateral surfaces of genus g is countable and dense in the Teichmüller space of S.*

This result guarantees the existence of sequences of packable Riemann surfaces with underlying combinatorial structures over which we may exercise a significant degree of control. This ability to control the underlying combinatorics allows us to manipulate the geometry of our packings. In particular, we may force the circles in a sequence of packable (and packed) surfaces to decrease in size.

Proposition 3.5. *Let S be a Riemann surface and let R be an arbitrary point in the Teichmüller space of S. There exists a sequence of packable points $\{R_n\}$ in the Teichmüller space of S such that $R_n \to R$ in the Teichmüller metric as $n \to \infty$, R_n is packable for every n, and the radii of the circles in P_n go to zero as $n \to \infty$, where P_n is the unique packing on the surface R_n for every n.*

Proof. First note that since the set of equilateral surfaces on S is dense in the Teichmüller space of S, there exists a sequence of equilateral surfaces $\{E_n^1\}$ in the Teichmüller space of S such that $E_n^1 \to R$ as $n \to \infty$. Corresponding to the equilateral surface E_1^1 we have a triangulation K_1^1. This triangulation corresponds to a unique packable surface R_1^1, in general, distinct from E_1^1.

Now, refine the triangulation K_1^1 using hex refinement to create a new triangulation K_1^2. Note that refining this triangulation has no effect on the structure of the surface E_1^1; that is, the equilateral surface E_1^2 corresponding to K_1^2 is the same as E_1^1, since hex refinement on an equilateral triangle divides that face into 4 new equilateral triangles. But K_1^2 corresponds to a unique packable surface R_1^2, typically distinct from E_1^2 (and E_1^1). Continuing in this manner, we construct a sequence of packable surfaces $\{R_1^m\}_{m=1}^{\infty}$.

It follows from the work of Bowers and Stephenson [6] that as $m \to \infty$, the triangles in the carrier of the packing P_1^m on R_1^m converge to equilateral triangles, and the sequence $\{R_1^m\}_{m=1}^{\infty}$ converges to a surface conformally equivalent to E_1^1. Thus, $R_1^m \to E_1^1$ as $m \to \infty$ in the Teicmüller metric. Note, however, that the degree of each packing in the sequence $\{P_1^m\}_{m=1}^{\infty}$ is bounded by the same constant, since the hex packing does not change the degree of any vertex in the initial packing, and every new vertex has degree 6. Now, a standard application of Lemma 2.10, the Length-Area Lemma, guarantees that the radii of the circles in the sequence $\{P_1^m\}_{m=1}^{\infty}$ go to zero as $m \to \infty$. We repeat this process for each equilateral surface E_n^1, generating for each n a sequence of packable surfaces $\{R_n^m\}_{m=1}^{\infty}$ such that $R_n^m \to E_n^1$ in the Teichmüller metric, and the radii of the circles in the sequence of packings $\{P_n^m\}_{m=1}^{\infty}$ corresponding the packable surfaces go to zero as $m \to \infty$.

$$\begin{array}{cccc} R_1^1 & R_2^1 & \cdots & \cdots \\ R_1^2 & R_2^2 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ E_1^1 & E_2^1 & \cdots & R \end{array}$$

Now, for each $n \in \mathbb{N}$, choose a surface R_n from the sequence $\{R_n^m\}_{m=1}^{\infty}$ such that the Teichmüller distance from R_n to E_n^1 is less than 2^{-n} and the maximum radius of any circle in the packing on R_n (in the intrinsic metric on the surface) is similarly less than 2^{-n}. We now have a sequence $\{R_n\}_{n=1}^{\infty}$ such that $R_n \to R$ in the Teichmüller metric as $n \to \infty$, R_n is packable for every n, and the radii of the circles in P_n go to zero as $n \to \infty$, where P_n is the unique packing on the surface R_n for every n. ■

References

1. R. W. Barnard and G. Brock Williams, *Combinatorial Excursiona in Moduli Space*, Pacific J. Math. **205**(2002), no. 1, 3–30.
2. A. F. Beardon and K. Stephenson, *The Uniformization Theorem for Circle Packings*, Indiana Univ. Math. J. **39**(1990), 1383–1425.
3. G. V. Belyĭ, *On Galois Extensions of a Maximal Cyclotomic Field*, Math. USSR Izv. **14**(2)(1980), 247–256.
4. P. L. Bowers and K. Stephenson, *Circle Packings in Surfaces of Finite Type: An In Situ Approach with Application to Modui Space*, Topology **32**(1)(1993), 157–183.
5. ———, *The Set of Circle Packing Points in the Teichmüller Space of a Surface of Finite Conformal Type is Dense*, Math. Proc. Camb. Phil. Soc. **111**(3)(1992), 487–513.
6. ———, *Uniformizing Dessins and Belyĭ Maps Via Circle Packing (Memoirs of the Amer. Math. Soc.)*, Amer. Math. Soc. 2004.
7. R. Brooks, *Circle Packings and Co-compact Extensions of Kleinian Groups*, Invent. Math. **86**(1986), 461–469.
8. P. Buser, *Geometry and Spectra of Compact Riemann Surfaces*, Birkhäuser, Boston, 1992.
9. C. Carathéodory, *Conformal Representation*, Dover, New York, 1932.
10. C. R. Collins and K. Stephenson, *A Circle Packing Algorithm*, Comput. Geom. **25**(2003), 233–256.

11. T. Dubejko and K. Stephenson, *Circle Packing: Experiments in Discrete Analytic Function Theory*, Experiment. Math. **4**(4)(1995), 307–348.
12. F. P. Gardiner and N. Lakic, *Quasiconformal Teichmüller Theory*, Mathematical Surveys and Monographs, **76**, Amer. Math. Soc. 2000.
13. Zheng-Xu He and B. Rodin, *Convergence of Circle Packings of Finite Valence to Riemann Mappings*, Comm. Anal. Geom. **1**(1993), 31–41.
14. Y. Imayoshi and M. Taniguchi, *An Introduction to Teichmüller Spaces*, Springer-Verlag, New York, 1992.
15. G. A. Jones and D. Singerman, *Complex Functions: An Algebraic and Geometric Viewpoint*, Cambridge University Press, Cambridge, 1987.
16. O. Lehto and K. I. Virtanen, *Quasiconformal Mappings of the Plane*, 2ed. Springer-Verlag, Berlin, Heidelberg, New York, 1973.
17. D. Minda and B. Rodin, *Circle Packing and Riemann Surfaces*, J. Analyse Math. **57**(1991), 221–249.
18. B. Mohar, *Circle Packing of Maps in Polynomial Time*, European J. Combin. **18**(1997), 785–805.
19. E. M. Murphy, *Discrete conformal approximation of complex earthquake maps*, Ph.D. thesis, Texas Tech Universty, July 2005.
20. B. Rodin and D. Sullivan, *The Convergence of Circle Packings to the Riemann Mapping*, J. Differential Geom. **26**(1987), 349–360.
21. K. Stephenson, *An Introduction to Circle Packing: The Theory of Discrete Analytic Functions*, Cambridge University Press, 2005.
22. ———, *A Probabalistic Proof of Thurston's Conjectire on Circle Packings*, Rendiconti del Seminario Mate. e Fisico di Milano **66**(1996), 201–291.
23. ———, *Circle Packing: A Mathematical Tale*, Notices Amer. Math. Soc. **50**(11)(2003), 1376–1388.
24. ———, *Circle Packing and Discrete Analytic Function Theory*, Handbook of Complex Analysis, **1**: Geometric Function Theory, R. Kühnau, Editor, Elsevier, 2002, pp. 333–370.
25. J. Stillwell, *Geometry of Surfaces*, Springer-Verlag, New York, 1992.
26. O. Schramm, *Existence and Uniqueness of Packings with Specified Combinatorics*, Israel J. Math. **73**(1991), 321–341.
27. W. P. Thurston, *The Finite Riemann Mapping Theorem*, Invited Talk, An International Symposium at Purdue University on the occasion of the proof of the Bieberbach Conjecture, 1985.
28. ———,*The Geometry and Topology of 3-Manifolds*, Princeton University Notes, preprint.
29. G. Brock Williams, *Earthquakes and Circle Packing*, J. Analyse Math. **85**(2001), 371–396.
30. ———,*Discrete Conformal Welding*, Indiana Univ. Math. J. **53**(2004), 765–804.
31. ———,*Circle Packings, Quasiconformal mappings and Applications*, In: Quasiconformal Mappings and their Applications (ed. by S. Ponnusamy, T. Sugawa and M. Vuorinen), Narosa Publishing House, New Delhi, India (2007), 327–346.

Roger W. Barnard E-mail: barnard@math.ttu.edu
Address:
Department of Mathematics and Statistics,
Texas Tech University,
Lubbock, Texas 79409, U.S.A.

Eric M. Murphy E-mail: eric.murphy@pentagon.af.mil
Address:
Air Force Studies and Analyses Agency,
Washington, DC 20330-1570, U.S.A.

G. Brock Williams E-mail: brock.williams@ttu.edu
Address:
Department of Mathematics and Statistics,
Texas Tech University,
Lubbock, Texas 79409, U.S.A.
URL: http://www.math.ttu.edu/∼williams