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Harmonic extensions on an infinite network

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Abstract. Some extension problems of harmonic and superharmonic functions on an infinite network $N(X,Y,K,r)$ are considered. It is assumed that the infinite graph (X,Y,K) is connected, locally finite and has no cycles. It is proved that a function h defined on $|x| \leq n$ and harmonic in $|x| < n$ can be extended to a harmonic function H on X such that $H = h$ on $|x| \leq n$. The existence of a superharmonic function g_{x_0} on X with point support x_0, such that $\Delta g_{x_0}(x) = -\delta_{x_0}(x)$ is established.

If h is harmonic outside a finite set in a network with a positive potential, we prove that there exists a unique harmonic function u on X and two potentials p_1 and p_2 on X which are bounded and have finite harmonic support such that $h = u + p_1 - p_2$ outside a finite set in X. A P domain and an S domain are defined analogous to the axiomatic case and it is proved that a positive potential exists in X if and only if there exists atleast one P domain in X.

Keywords. Infinite network, extension problems, superharmonic functions with point support.

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1. Introduction

Definition 1.1. Let X be a countable set of nodes, Y be a countable set of arcs, K be the node-arc incidence function, the range of K be $\{-1, 0, 1\}$ and r be a strictly positive function on Y. The quartet $N = \{X, Y, K, r\}$ is called an infinite network if the graph $\{X, Y, K\}$ is connected, locally finite and has no cycles.

For notations and terminologies we mainly follow [4] and [5].

Let $L(X)$ be the set of all real functions on X and $L^+(X)$ the set of all non-negative functions on X. For $x \in X$, denote by W_x the neighbouring nodes z of x. That is $z \sim x$.

$$W_x = \{z \in X | K(x,y)K(z,y) = -1 \text{ for some } y \in Y(x)\}.$$

For every $u \in L(X)$ the Laplacian $\Delta u \in L(X)$ is defined by

$$\Delta u(x) = -t(x)u(x) + \sum_{z \in W_x} t(x,z)u(z)$$

where

$$
\begin{aligned}
t(x) &= \sum_{y \in Y} r(y)^{-1} |K(x,y)| \\
t(x,z) &= \sum_{y \in Y} r(y)^{-1} |K(x,y)k(z,y)| \text{ for } x \neq z.
\end{aligned}
$$

Note that $t(x,z) = t(z,x)$ and $t(x,z) = 0$ for $z \in X | (W_x \cup \{x\})$. Therefore

$$t(x) = \sum_{z \in W_x} t(x,z).$$

Definition 1.2. A function $u \in L(X)$ is superharmonic on a set $A \subseteq X$ if $\Delta u(x) \leq 0$ for all $x \in A$. u is subharmonic on A if $-u$ is superharmonic on A. If u is both superharmonic and subharmonic on A, then u is said to be harmonic on A.

Lemma 1.3. [3, Lemma 4.1] *N is hyperbolic if and only if, there exists a positive potential in X.*

Theorem 1.4. [3, Theorem 4.3] *For any $a \in X$ in a network N with a positive potential, there exists a unique bounded potential $G_a(x)$ such that $\Delta G_a(x) = -\delta_a(x)$.*

2. Extension of harmonic and superharmonic functions

Definition 2.1. We fix a node, called the root node $e \in X$. We denote by $|x|$, the number of edges in the unique geodesic path from x to e.

The infinite graph is divided into concentric balls with the fixed root 'e' as the centre, the radii as the number of edges joining any point on the balls to the node 'e'. In notations, $B_n = \{x \in X | \ |x| = n\}$ and $X = \bigcup_{n=1}^{\infty} B_n$.

First we consider extension problems of harmonic and superharmonic functions defined on a finite set.

Theorem 2.2. *Let $u(x)$ be a finite valued function defined on $|x| = n-1$ and $|x| = n$. Then there exists a function $v(x)$ on $|x| \geq n-1$ such that $v(x) = u(x)$ on $|x| = n-1$ and $|x| = n$ and $v(x)$ is harmonic on $|x| > n-1$.*

Proof. Let $x_0 \in X$ be an arbitrary point on $|x| = n$. Let z be the unique neighbour of x_0 with $|z| = n-1$, and z_j $(j = 1, 2, \ldots, m)$ be the neighbours of x_0 with $|z_i| = n+1$. Define

$$v(x) = \begin{cases} u(x) & \text{on} \quad |x| = n-1 \text{ and } |x| = n \\ \alpha & \text{at} \quad z_j, \ j = 1, \ldots, m \end{cases}$$

where α is the constant given by

$$\alpha = \frac{t(x_0)u(x_0) - t(x_0, z)u(z)}{\sum_{j=1}^{m} t(x_0, z_j)}.$$

Then v is harmonic at x_0. This can be done with each vertex in B_n. Then v is defined on $B_{n-1} \cup B_n \cup B_{n+1}$ and harmonic on B_n.

Now we consider v defined on $B_n \cup B_{n+1}$ and proceed as before to get an extension of v to B_{n+2} such that it is harmonic on B_{n+1}. By induction, we get a function v which is harmonic on $|x| \geq n$ such that $v(x) = u(x)$ on $B_{n-1} \cup B_n$. ∎

Corollary 2.3. *Let $h(x)$ be a function defined on $|x| \leq n$, harmonic on $|x| < n$. Then there exists a harmonic function H on X such that $H = h$ on $|x| \leq n$.*

Proof. By the theorem, there exists a function u defined on $|x| \geq n-1$ such that $u = h$ on $|x| = n-1$ and $|x| = n$ and u is harmonic on $|x| \geq n$. Define

$$H = \begin{cases} u & \text{on} \quad |x| \geq n+1 \\ h & \text{on} \quad |x| \leq n. \end{cases}$$

Then H satisfies the required conditions. ∎

Corollary 2.4. *Let $u(x)$ be defined on $|x| \leq n$, superharmonic in $|x| \leq n-1$, harmonic in $|x| \leq n-2$. Then there exists a superharmonic function $v(x)$ on X such that $v(x) = u(x)$ on $|x| \leq n, v(x)$ is harmonic on $|x| \leq n-2$ and $|x| \geq n$.*

Proof. Let s be a function defined on $|x| \geq n-1$ such that $s = u$ on $|x| = n-1$ and $|x| = n$ and s is harmonic on $|x| \geq n$. Define

$$v(x) = \begin{cases} u(x) & \text{on} \quad |x| \leq n \\ s(x) & \text{on} \quad |x| \geq n+1. \end{cases}$$

Then v satisfies the required properties. ∎

Corollary 2.5. *There exists a superharmonic function g_{x_0} on X with point support $x_0 \in X$ such that*

$$\Delta g_{x_0}(x) = -\delta_{x_0}(x)$$

Proof. Let u be a function such that $u(x_0) = 1$ and $u(x) = 0$ for any $x \sim x_0$. By Theorem 2.2 we can extend u to X such that u is harmonic in $X - \{x_0\}$ and at x_0, u is superharmonic as $\Delta u(x_0) < 0$.

Now by defining

$$g_{x_0}(x) = -\frac{u(x)}{\Delta u(x_0)},\ \forall\, x \in X,$$

the required result is obtained. ■

Note: If N is a network without a positive potential, then g_{x_0} will not be lower bounded.

A modification of the above corollary gives the following:

Corollary 2.6. *In any network , there exists a subharmonic function $S \geq 0$ having point support at any given point x_0 which is necessarily unbounded if there is no positive potential in N.*

Proof. Consider the function u defined by $u(x_0) = 0, u(x) = 1$ for $x \sim x_0$. We can extend u to a function S on X such that S is harmonic on $X \setminus \{x_0\}$. Clearly S is subharmonic at x_0 and $S \geq 0$ on X. Since S is nonconstant, S is unbounded if there is no positive potential in N. ■

Now we consider extension of functions which are defined outside a finite set. An example was given in [2] to show that a harmonic function defined outside a finite set in X, does not extend to a harmonic function on the whole space X. However, we prove the following theorems.

Theorem 2.7. *If $h(x)$ is a harmonic function outside a finite set in a network with a positive potential, then there exists a unique harmonic function u on X and two potentials p_1 and p_2 on X which are bounded and have finite harmonic support such that $h = u + p_1 - p_2$ outside a finite set in X.*

Proof. By hypothesis for large n, h is defined in $|x| \geq n$. Let

$$v(x) = \begin{cases} h(x) & \text{on} \quad |x| \geq n \\ 0 & \text{on} \quad |x| \leq n-1. \end{cases}$$

Then clearly $v(x)$ is harmonic in $|x| \geq n+1$ and $|x| \leq n-2$. Let $y_1, y_2, \ldots, y_k$ be the vertices on $|x| = n-1$ and $|x| = n$. Let

$$u(x) = v(x) + \sum_{j=1}^{k} \Delta v(y_j) G_{y_j}(x).$$

Then

$$\Delta u(x) = \Delta v(x) + \sum_{j=1}^{k} \Delta v(y_j) \Delta G_{y_j}(x).$$

If $x \neq y_j$ then $\Delta v(x) = 0$, $\Delta G_{y_j}(x) = 0$ and hence $\Delta u(x) = 0$. If $x = y_j$ for some j then

$$\Delta u(x) = \Delta v(y_j) + \Delta v(y_j)(-1) = 0.$$

Therefore u is harmonic on X. Hence

$$\begin{aligned} v(x) &= u(x) - \sum_{j=1}^{k} \Delta v(y_j) G_{y_j}(x) \\ &= u(x) - \sum_{j=1}^{k} \left[(\Delta v(y_j))^+ - (\Delta v(y_j))^- \right] G_{y_j}(x) \\ &= u(x) - \sum_{j=1}^{k} (\Delta v(y_j))^+ G_{y_j}(x) + \sum_{j=1}^{k} (\Delta v(y_j))^- G_{y_j}(x) \\ &= u(x) + p_1(x) - p_2(x), \end{aligned}$$

where

$$\begin{aligned} p_1(x) &= \sum_{j=1}^{k} (\Delta v(y_j))^- G_{y_j}(x) \\ &= \sum_{j=1}^{k} \sup(0, -\Delta v(y_j)) G_{y_j}(x) \end{aligned}$$

and

$$\begin{aligned} p_2(x) &= \sum_{j=1}^{k} (\Delta v(y_j))^+ G_{y_j}(x) \\ &= \sum_{j=1}^{k} \sup(0, \Delta v(y_j)) G_{y_j}(x). \end{aligned}$$

Then p_1 and p_2 are potentials with support on $|x| = n-1$ and $|x| = n$. Since $G_j's$ are bounded, p_1, p_2 are also bounded. Since $v(x) = h(x)$ in $|x| \geq n$, we get $h(x) = u(x) + p_1(x) - p_2(x)$ in $|x| \geq n$.

To prove the uniqueness of u, suppose $h(x) = u_1(x) + q_1(x) - q_2(x)$ is another such representation for h. Let $s(x) = |u(x) - u_1(x)|$. Then s is subharmonic on X and $s(x) = |q_1(x) - q_2(x) - p_1(x) + p_2(x)|$ outside a finite set. Hence

$$s(x) \leq p_1(x) + p_2(x) + q_1(x) + q_2(x)$$

outside a finite set. $s - (p_1 + p_2 + q_1 + q_2)$ is subharmonic on a finite set and ≤ 0 outside the finite set.

Therefore by maximum principle, $s \leq p_1 + p_2 + q_1 + q_2$ on X. By [3, Theorem 2.4] we get $s \leq 0$. But by definition, $s \geq 0$. Hence $s = 0$. This proves the theorem. ■

Theorem 2.8. [1, Theorem 1.2] *Let h be a harmonic function outside a finite set in X. Then there exists a harmonic function u on X such that $|u - h|$ is bounded outside a finite set if and only if there exists a positive potential in N.*

Proof. The if part is proved in the above theorem.

To prove the converse, assume that for every function h which is harmonic outside a finite set in X, there exists a harmonic function u on X such that $|u - h|$ is bounded outside a finite set.

Suppose that N has no positive potential. From Corollary 2.6, we have a function $S \geq 0$ in X which is harmonic outside a finite set and unbounded in X.

By assumption, there exists a harmonic function u on X such that $|S - u|$ is bounded outside a finite set.

Since $S \geq 0$, we get u is lower bounded outside a finite set and hence lower bounded on X. Since N has no positive potential, u is a constant. This means S is bounded, which is a contradiction. This proves the theorem. ■

Using the extension results we define a P domain and an S domain on an infinite network [1].

Take all points $x_i \in X, i = 1, 2, \cdots n$ such that $x_i \sim e$.

For each i, define

$$C(e, x_i) = \{x \in X \mid \text{The geodesic joining } e \text{ and } x \text{ passes through } x_i\}.$$

We assume e and x_i belong to $C(e, x_i)$.

Then

$$X = \bigcup_{i=1}^{n} C(e, x_i).$$

For a fixed i, consider the function u defined by $u(e) = 0, u(x_i) = 1$.

Then u can be extended to a function h on $C(e, x_i)$ as in Theorem 2.2 such that $h \geq 0$ on $C(e, x_i)$ and h is harmonic on $C(e, x_i) \setminus \{e\}$.

Let $H(e, x_i)$ denote the class of all such extension functions.

Definition 2.9. If $C(e, x_i)$ contains an infinite number of vertices then $C(e, x_i)$ is called a P domain if and only if there exists a bounded function in $H(e, x_i)$; otherwise $C(e, x_i)$ is called an S domain.

Theorem 2.10. *A positive potential exists in X if and only if there exists at least one P domain $C(e, x_{i_0})$ in X.*

Proof. Suppose that there exists a P domain $C(e, x_{i_0})$ in X. Then there exists a bounded function h on $C(e, x_{i_0})$ which is harmonic on $C(e, x_{i_0}) \backslash \{e\}$, $h(e) = 0, h(x_{i_0}) = 1$.

Then the function u on X defined by

$$u = \begin{cases} h & \text{in} \quad C(e, x_{i_0}) \\ 0 & \text{otherwise} \end{cases}$$

is subharmonic on X, nonconstant and bounded. Hence there exists a positive potential in X.

Conversely assume that there exists a positive potential p in X. Let $u(x) = R_1^e(x)$(§4 in [3]).

Then u is a positive superharmonic function, harmonic on $X - \{e\}$ and $u(e) = 1, u(x) \leq 1$. Also $u(x) \leq \dfrac{p(x)}{p(e)}$ as $\dfrac{p(x)}{p(e)}$ is a positive superharmonic function and equal to 1 at e.

Hence $u(x)$ is a potential and is non-constant.

Consider all $C(e, x_i)$ with infinite number of vertices. If $u \equiv 1$ in all these, then u is a superharmonic function which is 1 outside a finite set. Therefore by minimum principle, $u \geq 1$ on X.

Hence $u \equiv 1$ on X, a contradiction.

Thus there exists at least one $C(e, x_i)$ say $C(e, x_{i_0})$ with an infinite number of vertices such that for some s in $C(e, x_{i_0}), u(s) < 1$.

Now $u(x_{i_0}) \neq 1$. For, if $u(x_{i_0}) = 1$, $u(e) = 1, u \leq 1$ and u is harmonic outside e will imply $u = 1$ on $C(e, x_{i_0})$ which is a contradiction to $u(s) < 1$.

Define $h(x) = \dfrac{1 - u(x)}{1 - u(x_{i_0})}$ for $x \in C(e, x_{i_0})$.

Then $h(e) = 0, h(x_{i_0}) = 1, h$ is bounded and harmonic in $C(e, x_{i_0})$ except at 'e' which means that $C(e, x_{i_0})$ is a P domain in X. ■

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