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Geometric properties of hyperbolic polar coordinates

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Abstract. In a simply connected hyperbolic region in the complex plane $\mathbb{C}$ hyperbolic polar coordinates possess global Euclidean properties similar to those of hyperbolic polar coordinates about the origin in the unit disk if and only if the region is Euclidean convex. For example, the Euclidean distance between travelers moving at unit hyperbolic speed along distinct hyperbolic geodesic rays emanating from an arbitrary common initial point is increasing if and only if the region is convex. Analogous geometric properties of hyperbolic polar coordinates in convex regions in either the spherical plane $\mathbb{C}_\infty$ or in the hyperbolic plane $\mathbb{D}$ are established by making use of characterizations of spherically or hyperbolically convex univalent functions.

Keywords. hyperbolic metric, hyperbolic geodesics, hyperbolic disks, hyperbolically convex univalent functions, spherical metric, spherically convex univalent functions.

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1. Introduction

In [7] the authors investigated geometric properties of hyperbolic polar coordinates on simply connected hyperbolic regions in the Euclidean plane. In particular, we showed that a number of Euclidean properties of hyperbolic polar coordinates hold if and only the region is Euclidean convex. This paper is devoted to the investigation of geometric properties of hyperbolic polar coordinates in simply connected hyperbolic regions that lie in either the spherical plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ or in the hyperbolic plane $\mathbb{D}$. In these two new contexts we are able to establish analogs for many of the results that were proved for Euclidean convex regions in the Euclidean plane. For example, hyperbolic polar coordinates exhibit certain properties relative to spherical (hyperbolic) geometry if and only if the region is spherically (hyperbolically) convex. The proofs of these theorems depend upon characterizations of spherically (hyperbolically) convex univalent functions. Some of the results in this paper were announced in [6].

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A region Ω in the Riemann sphere $\mathbb{C}_\infty$ is *hyperbolic* if $\mathbb{C}_\infty \setminus \Omega$ contains at least three points. The hyperbolic metric on a hyperbolic region Ω is denoted by $\lambda_\Omega(w)|dw|$ and is normalized to have curvature

$$-\frac{\Delta \log \lambda_\Omega(w)}{\lambda_\Omega^2(w)} = -1,$$

where

$$\triangle = 4\frac{\partial^2}{\partial w \partial \bar{w}}$$

denotes the usual Laplacian. The hyperbolic metric on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ is given by

$$\lambda_{\mathbb{D}}(z)|dz| = \frac{2|dz|}{1 - |z|^2}.$$

The induced hyperbolic distance on $\mathbb{D}$ is

$$d_{\mathbb{D}}(a, b) = 2\tanh^{-1}\left|\frac{a - b}{1 - \bar{b}a}\right|$$

and the hyperbolic geodesics are arcs of circles orthogonal to the unit circle. If $f : \mathbb{D} \to \Omega$ is any meromorphic universal covering projection, then the density λ_Ω of the hyperbolic metric is uniquely determined from $\lambda_\Omega(f(z))|f'(z)| = 2/(1-|z|^2)$.

We briefly recall the definition of hyperbolic polar coordinates; see [6] and [7] for more details. Let Ω be a hyperbolic region in $\mathbb{C}_\infty$. Fix a point a in Ω, for θ in $\mathbb{R}$, let $\rho_\theta(a, \Omega)$ denote the unique hyperbolic ray emanating from a that is tangent to $e^{i\theta}$ at a. When a is fixed in Ω, we often write ρ_θ in place of the more precise $\rho_\theta(a, \Omega)$. Of course, $\rho_{\theta+2n\pi} = \rho_\theta$ for all $n \in \mathbb{Z}$. Let $w = w(s, \theta)$, $0 \le s < +\infty$, be the hyperbolic arc length parametrization of ρ_θ. This means that

$$\frac{\partial w(s,\theta)}{\partial s} = \frac{e^{i\Theta(s,\theta)}}{\lambda_\Omega(w(s,\theta))},$$

where $e^{i\Theta(s,\theta)}$ is a Euclidean unit tangent to ρ_θ at the point $w(s, \theta)$. If $\Omega = \mathbb{D}$ and $a = 0$, then ρ_θ is the segment $[0, e^{i\theta})$ with hyperbolic arc length parametrization $z(s, \theta) = \tanh(s/2)e^{i\theta}$. In fact,

$$\frac{\partial z(s,\theta)}{\partial s} = \frac{1}{2}(1 - |z(s,\theta)|^2)e^{i\theta} = \frac{e^{i\theta}}{\lambda_{\mathbb{D}}(z(s,\theta))} = \frac{(1 - |z(s,\theta)|^2)z(s,\theta)}{2|z(s,\theta)|}.$$

Suppose $f : \mathbb{D} \to \Omega$ is a meromorphic covering with $f(0) = a$ and $f'(0) > 0$. Then $w(s, \theta) = f(z(s, \theta))$ is a hyperbolic arc length parametrization of $\rho_\theta(a, \Omega)$.

For the unit disk $\mathbb{D}$, hyperbolic geodesics exhibit various Euclidean properties. For example, when $a = 0$, it is obvious that the Euclidean distance $|z(s, \theta_1) - z(s, \theta_2)|$ is an increasing function of $s \ge 0$ for $e^{i\theta_2} \ne e^{i\theta_1}$. Also, $|z(s, \theta)|$ is an increasing function of $s \ge 0$ for any fixed θ. In [7], we showed that a Euclidean convex region Ω has many analogous properties. For instance, the Euclidean distance function $|w(s,\theta)-a|$ is an increasing function of s for any fixed θ, and $|w(s,\theta_1)-w(s,\theta_2)|$ is an increasing function of s for $\theta_2 \neq \theta_1 + 2n\pi$. We investigate whether similar properties hold for convex regions in either the spherical plane or the hyperbolic plane.

The spherical metric on $\mathbb{C}_\infty$ is given by

$$\sigma(z)|dz| = \frac{2|dz|}{1+|z|^2};$$

it has curvature

$$-\frac{\Delta \log \sigma(z)}{\sigma^2(z)} = 1.$$

The spherical distance on $\mathbb{C}_\infty$ is

$$d_\sigma(z,w) = 2\tan^{-1}\left|\frac{z-w}{1+\bar{w}z}\right|.$$

For antipodal $z, w \in \mathbb{C}_\infty$, that is, $w = -1/\bar{z}$, any of the infinitely many great circular arcs connecting z and w is a spherical geodesic arc. If $z, w \in \mathbb{C}_\infty$ are not antipodal, then the unique spherical geodesic is the shorter arc between z and w of the unique great circle determined by z and w. The orientation preserving conformal isometries of the spherical plane form a group $\mathcal{I}(\mathbb{C}_\infty)$. All of the following groups are identical.

(1) the group of conformal isometries of the spherical distance d_σ;
(2) the group of conformal isometries of the spherical metric $\sigma(z)|dz| = \frac{2|dz|}{1+|z|^2}$;
(3) the group of Möbius maps of the form

$$z \mapsto \frac{az-\bar{c}}{cz+\bar{a}}, \quad |a|^2+|c|^2 = 1;$$

(4) the group of Möbius maps of the form

$$z \mapsto e^{i\theta}\frac{z-c}{1+\bar{c}z},$$

where $c \in \mathbb{C}_\infty$ and $\theta \in \mathbb{R}$.

The isometries of the spherical plane are sometimes called rotations since they correspond to rotations of the unit sphere in $\mathbb{R}^3$ after conjugation by stereographic projection; see [6] for more information.

Sections 2 through 4 deal with spherically convex functions and spherically convex regions. Section 2 contains two two-variable characterizations of spherically convex functions. Spherical properties of hyperbolic coordinates are investigated in Section 3; for example, we prove that $d_\sigma(w(s,\theta),a)$ is an increasing function of $s \geq 0$ for each fixed θ if and only if Ω is spherically convex. Also,

$d_\sigma(w(s,\theta_1), w(s,\theta_2))$ is an increasing function of s whenever $e^{i\theta_2} \neq e^{i\theta_1}$ if and only if Ω is spherically convex. Applications of the two-variable characterizations of spherically convex functions to the general study of spherically convex functions are given in Section 4. For example, we obtain sharp lower bounds on $\operatorname{Re}\{a_2 f(z)\}$ and $\operatorname{Re}\{1+\frac{zf''(z)}{f'(z)}\}$ for spherically convex functions $f(z) = \alpha z + a_2 z^2 + \ldots$. Section 5 concerns hyperbolically convex functions and hyperbolically convex regions in the hyperbolic plane. Because this situation closely parallels that for spherically convex regions, we present fewer details.

2. Spherically convex univalent functions

A simply connected region Ω in the spherical plane $\mathbb{C}_\infty$ is called *spherically convex* provided that for each pair of $z, w \in \Omega$ every spherical geodesic connecting z and w also lies in Ω. If Ω is spherically convex and contains a pair of antipodal points, then $\Omega = \mathbb{C}_\infty$. Otherwise, a spherically convex region Ω is a simply connected hyperbolic region in $\mathbb{C}_\infty$. A meromorphic and univalent function f defined on $\mathbb{D}$ is called *spherically convex* if its image $f(\mathbb{D})$ is a spherically convex subset of $\mathbb{C}_\infty$. Since spherical convexity and the spherical metric are invariant under any rotation of $\mathbb{C}_\infty$, we may assume $\Omega \subset \mathbb{C}$ if this is convenient.

Important to our study of hyperbolic polar coordinates on spherically convex regions are characterizations of spherically convex univalent functions. One such characterization obtained by Mejia and Minda [8] (see also [2]) is

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \frac{2zf'(z)\overline{f(z)}}{1+|f(z)|^2}\right\} \geq 0 \tag{2.1}$$

for all z in $\mathbb{D}$. Unfortunately, it is often difficult to use (2.1) because it contains the nonholomorphic term

$$\frac{2zf'(z)\overline{f(z)}}{(1+|f(z)|^2)}.$$

One way to overcome this difficulty is to establish two-variable characterizations for spherically convex univalent functions, that are holomorphic in one of the two variables. We establish such a two-variable characterization of spherically convex univalent functions.

Theorem 2.1. *Let f be meromorphic and locally univalent in $\mathbb{D}$. Then f is spherically convex univalent function if and only if for all z, ζ in $\mathbb{D}$*

$$\operatorname{Re}\left\{\frac{2zf'(z)}{f(z)-f(\zeta)} - \frac{z+\zeta}{z-\zeta} - \frac{2zf'(z)\overline{f(\zeta)}}{1+\overline{f(\zeta)}f(z)}\right\} > 0. \tag{2.2}$$

Proof. For any $g \in \mathcal{I}(\mathbb{C}_\infty)$, $f(z)$ is spherically convex if and only if $g \circ f(z)$ is, and the inequality (2.2) is invariant if we replace f by $g \circ f$. Therefore, we may assume that f is holomorphic in $\mathbb{D}$.

First, we assume that f is spherically convex univalent function. The spherical geodesic γ from a to $b \in \mathbb{C}$ can be parameterized by

$$w(t) = \frac{t\frac{b-a}{1+\bar{a}b} + a}{1 - t\bar{a}\frac{b-a}{1+\bar{a}b}},$$

$0 \le t \le 1$. At a, the tangent vector to γ is

$$w'(0) = \frac{b-a}{1+\bar{a}b}(1+|a|^2).$$

Fix $\zeta = re^{i\theta} \in \mathbb{D}$ and let $z = re^{i\varphi}$, $\theta < \varphi \le \theta + 2\pi$. We consider the spherical geodesic γ_φ connecting $f(\zeta)$ and $f(z)$. The tangent vector to γ_φ at $f(\zeta)$ is

$$w'_\varphi(0) = \frac{f(re^{i\varphi}) - f(\zeta)}{1 + \overline{f(\zeta)}f(re^{i\varphi})}(1 + |f(\zeta)|^2).$$

Since $f(|z| = r)$ is a spherically convex curve [8],

$$L(\varphi) = \arg\left\{w'_\varphi(0)\right\} = \arg\left\{\frac{f(re^{i\varphi}) - f(\zeta)}{1 + \overline{f(\zeta)}f(re^{i\varphi})}(1 + |f(\zeta)|^2)\right\}$$

is nondecreasing in $(\theta, \theta + 2\pi]$. Therefore,

$$\begin{aligned} L'(\varphi) &= \frac{\partial}{\partial\varphi}\arg\left\{\frac{f(re^{i\varphi}) - f(\zeta)}{1 + \overline{f(\zeta)}f(re^{i\varphi})}\right\} \\ &= \mathrm{Re}\left\{\frac{zf'(z)}{f(z) - f(\zeta)} - \frac{zf'(z)\overline{f(\zeta)}}{1 + \overline{f(\zeta)}f(z)}\right\} \\ &\ge 0. \end{aligned}$$

Let

$$p(z,\zeta) = \frac{2zf'(z)}{f(z) - f(\zeta)} - \frac{z+\zeta}{z-\zeta} - \frac{2zf'(z)\overline{f(\zeta)}}{1 + \overline{f(\zeta)}f(z)}. \tag{2.3}$$

As Re $\{(z+\zeta)/(z-\zeta)\} = 0$ for $|z| = r$, $z \ne \zeta$, we conclude that Re $\{p(z,\zeta)\} \ge 0$ when $|z| = |\zeta| = r$; for $z = \zeta$, this follows from continuity. Because Re $\{p(z,\zeta)\}$ is harmonic in both z and ζ, the Maximum Principle yields that Re $\{p(z,\zeta)\} \ge 0$ for $|z| < r$ and $|\zeta| < r$. If we let $r \to 1$, we see that Re $\{p(z,\zeta)\} \ge 0$ for $z, \zeta \in \mathbb{D}$. The Maximum Principle again implies that Re $\{p(z,\zeta)\} > 0$ for $z, \zeta \in \mathbb{D}$ since $p(0,\zeta) = 1$, which means that Re $\{p(z,\zeta)\}$ cannot be identically 0.

Now, we show that if f satisfies the inequality (2.2), then (2.1) holds for all $z \in \mathbb{D}$. The assumption is that Re $\{p(z,\zeta)\} > 0$ for $z, \zeta \in \mathbb{D}$. Since

$$p(z,z) = 1 + \frac{zf''(z)}{f'(z)} - \frac{2zf'(z)\overline{f(z)}}{1+|f(z)|^2},$$

f is a spherically convex univalent function by (2.1). ■

The preceding proof shows that (2.1) is a special case of (2.2).

Corollary 2.2. *Suppose f is meromorphic and locally univalent in $\mathbb{D}$. Then f is a spherically convex univalent function if and only if*

$$\text{Re}\left\{\frac{zf'(z)-\zeta f'(\zeta)}{f(z)-f(\zeta)} - \frac{zf'(z)\overline{f(\zeta)} + \overline{\zeta f'(\zeta)}f(z)}{1+\overline{f(\zeta)}f(z)}\right\} > 0 \tag{2.4}$$

for all z, ζ in $\mathbb{D}$.

Proof. Suppose f is a spherically convex univalent function. Then the inequality (2.2) holds. By interchanging the roles of z and ζ, we obtain

$$\text{Re}\left\{\frac{-2\zeta f'(\zeta)}{f(z)-f(\zeta)} + \frac{z+\zeta}{z-\zeta} - \frac{2\overline{\zeta f'(\zeta)}f(z)}{1+\overline{f(\zeta)}f(z)}\right\} > 0$$

for all z, ζ in $\mathbb{D}$. By adding this inequality to (2.2), we get Re $\{2q(z,\zeta)\} > 0$, where

$$q(z,\zeta) = \frac{zf'(z)-\zeta f'(\zeta)}{f(z)-f(\zeta)} - \frac{zf'(z)\overline{f(\zeta)} + \overline{\zeta f'(\zeta)}f(z)}{1+\overline{f(\zeta)}f(z)}. \tag{2.5}$$

Thus, (2.4) holds.

Conversely, suppose (2.4) holds for all z, ζ in $\mathbb{D}$. Then Re $\{q(z,z)\} > 0$ for all z in $\mathbb{D}$. Since

$$q(z,z) = 1 + \frac{zf''(z)}{f'(z)} - \frac{2zf'(z)\overline{f(z)}}{1+|f(z)|^2},$$

f is spherically convex because (2.1) holds. ■

3. Spherically convex regions

Now, we establish spherical properties of hyperbolic polar coordinates in spherically convex regions. It is convenient to use the density of the hyperbolic metric relative to the spherical metric; that is,

$$\mu_\Omega(w) = \frac{\lambda_\Omega(w)|dw|}{\sigma(w)|dw|} = \frac{1}{2}(1+|w|^2)\lambda_\Omega(w).$$

Then $\lambda_\Omega(w)|dw| = \mu_\Omega(z)\sigma(w)|dw|$ and the function μ_Ω is invariant under the group $\mathcal{I}(\mathbb{C}_\infty)$ of rotations of the sphere.

Theorem 3.1. *Suppose Ω is a simply connected hyperbolic region in $\mathbb{C}_\infty$.*
(a) *If Ω is spherically convex and $a \in \Omega$, then $d_\sigma(w(s,\theta),a)$ is an increasing function of $s \geq 0$ for all θ in $\mathbb{R}$. Moreover, we have the following sharp bounds:*

$$\frac{\tanh(s/2)}{\mu_\Omega(a) + \tanh(s/2)\sqrt{\mu_\Omega^2(a)-1}} \leq \tan\frac{1}{2}d_\sigma(w(s,\theta),a) \leq \frac{\tanh(s/2)}{\mu_\Omega(a) - \tanh(s/2)\sqrt{\mu_\Omega^2(a)-1}}.$$

(b) *If $d_\sigma(w(s,\theta),a)$ is an increasing function of $s \geq 0$ for each a in Ω and all θ in $\mathbb{R}$, then Ω is spherically convex.*

Proof. Since both spherical convexity and spherical distance are invariant under rotations of $\mathbb{C}_\infty$, we may assume $\Omega \subset \mathbb{C}$. Let $f : \mathbb{D} \to \Omega$ be a conformal mapping with $f(0) = a$ and $f'(0) > 0$. Then $w(s,\theta) = f(z(s,\theta))$, and

$$d_\sigma(w(s,\theta),a) = 2\tan^{-1}\left|\frac{w(s,\theta)-a}{1+\bar{a}w(s,\theta)}\right| = 2\tan^{-1}\left|\frac{f(z(s,\theta))-a}{1+\bar{a}f(z(s,\theta))}\right|.$$

This is an increasing function of $s \geq$ if and only if

$$A(s) = \log\left|\frac{f(z(s,\theta))-a}{1+\bar{a}f(z(s,\theta))}\right|$$

is increasing on $[0,+\infty)$. Note that

$$\begin{aligned} A'(s) &= \operatorname{Re}\left\{\frac{f'(z(s,\theta))\frac{\partial z(s,\theta)}{\partial s}}{f(z(s,\theta))-a} - \frac{\bar{a}f'(z(s,\theta))\frac{\partial z(s,\theta)}{\partial s}}{1+\bar{a}f(z(s,\theta))}\right\} \\ &= \frac{1-|z(s,\theta)|^2}{2|z(s,\theta)|}\operatorname{Re}\left\{\frac{z(s,\theta)f'(z(s,\theta))}{f(z(s,\theta))-a} - \frac{\bar{a}z(s,\theta)f'(z(s,\theta))}{1+\bar{a}f(z(s,\theta))}\right\}. \end{aligned} \tag{3.1}$$

(a) Assume Ω is spherically convex. Then f is a spherically convex univalent function. By using Theorem 2.1, we obtain

$$A'(s) > \frac{1-|z(s,\theta)|^2}{4|z(s,\theta)|} > 0.$$

Thus, $A(s)$ is an increasing function of $s \geq 0$.

As $g(z) = (f(z)-a)/(1+\bar{a}f(z))$ is spherically convex with $g(0) = 0$ and $g'(0) > 0$, we have [2]

$$\frac{g'(0)|z|}{1+\sqrt{1-g'(0)^2}\,|z|} \leq |g(z)| \leq \frac{g'(0)|z|}{1-\sqrt{1-g'(0)^2}\,|z|}.$$

Since

$$g'(0) = \frac{f'(0)}{1+|a|^2} = \frac{2}{\lambda_\Omega(a)(1+|a|^2)}$$

and $|z(s,\theta)| = \tanh(s/2)$, we get

$$\begin{aligned}\frac{\tanh(s/2)}{\mu_\Omega(a) + \tanh(s/2)\sqrt{\mu_\Omega^2(a)-1}} &\leq \left|\frac{f(z(s,\theta))-a}{1+\bar{a}f(z(s,\theta))}\right| \\ &\leq \frac{\tanh(s/2)}{\lambda_\mu(a) - \tanh(s/2)\sqrt{\mu_\Omega^2(a)-1}}.\end{aligned}$$

From

$$\tan\frac{1}{2}d_\sigma(w(s,\theta),a) = \left|\frac{w(s,\theta)-a}{1+\bar{a}w(s,\theta)}\right| = \left|\frac{f(z(s,\theta))-a}{1+\bar{a}f(z(s,\theta))}\right|,$$

we obtain the lower and upper bounds on $d_\sigma(w(s,\theta),a)$ in Theorem 3.1(a).

For $0 < \alpha \leq 1$, the spherical half-plane (hemisphere)

$$\Omega_\alpha = \left\{w : \left|w - \sqrt{1-\alpha^2}/\alpha\right| < \frac{1}{\alpha}\right\}$$

is spherically convex and

$$k_\alpha(z) = \frac{\alpha z}{1-\sqrt{1-\alpha^2}z}$$

maps $\mathbb{D}$ conformally onto Ω_α. We consider $a = 0$. Then $\mu_{\Omega_\alpha}(0) = 1/\alpha$,

$$w(s,0) = \frac{\alpha\tanh(s/2)}{1-\sqrt{1-\alpha^2}\tanh(s/2)}$$

is the hyperbolic arc length parametrization of $\left[0, \frac{1+\sqrt{1-\alpha^2}}{\alpha}\right)$, and the upper bound is equal to

$$\frac{\alpha\tanh(s/2)}{1-\sqrt{1-\alpha^2}\tanh(s/2)} = \tan\frac{1}{2}d_\sigma(w(s,0),0).$$

This shows that the upper bound is sharp. Similarly,

$$w(s,\pi) = \frac{-\alpha\tanh(s/2)}{1+\sqrt{1-\alpha^2}\tanh(s/2)}$$

is the hyperbolic arc length parametrization of $\left(\frac{-1+\sqrt{1-\alpha^2}}{\alpha}, 0\right]$, and the lower bound is equal to

$$\frac{\alpha\tanh(s/2)}{1+\sqrt{1-\alpha^2}\tanh(s/2)} = \tan\frac{1}{2}d_\sigma(w(s,\pi),0).$$

Hence, the lower bound is also sharp.

(b) Suppose for each θ in $\mathbb{R}$, $d_\sigma(w(s,\theta),a)$ is an increasing function of s. Then $A'(s)\ge 0$. From (3.1), we obtain

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)-a}-\frac{\bar{a}zf'(z)}{1+\bar{a}f(z)}\right\}\ge 0$$

for all $z\in\mathbb{D}$ and $a\in\Omega$. Let $|\zeta|=|z|$ and $a=f(\zeta)$. Then similar to the proof of Theorem 2.1, we can deduce that (2.2) holds. Thus, Ω is spherically convex. ■

Theorem 3.2. *Suppose Ω is a simply connected hyperbolic region in $\mathbb{C}_\infty$.*

(a) *If Ω is spherically convex and $a\in\Omega$, then $d_\sigma(w(s,\theta_1),w(s,\theta_2))$ is an increasing function of $s\ge 0$ for all $e^{i\theta_2}\ne e^{i\theta_1}$.*
(b) *If there exists $a\in\Omega$ such that $d_\sigma(w(s,\theta_1),w(s,\theta_2))$ is an increasing function of $s\ge 0$ whenever $e^{i\theta_2}\ne e^{i\theta_1}$, then Ω is spherically convex.*

Proof. Again, by performing a rotation of $\mathbb{C}_\infty$ if necessary, we may assume $\Omega\subset\mathbb{C}$. Let $f:\mathbb{D}\to\Omega$ be a holomorphic covering with $f(0)=a$ and $f'(0)>0$. Then

$$\begin{aligned}d_\sigma(w(s,\theta_1),w(s,\theta_2))&=d_\sigma(f(z(s,\theta_1)),f(z(s,\theta_2)))\\&=2\tan^{-1}\left|\frac{f(z(s,\theta_1))-f(z(s,\theta_2))}{1+\overline{f(z(s,\theta_2))}f(z(s,\theta_1))}\right|,\end{aligned}$$

and this is an increasing function of s if and only if

$$B(s)=\log\left|\frac{f(z(s,\theta_1))-f(z(s,\theta_2))}{1+\overline{f(z(s,\theta_2))}f(z(s,\theta_1))}\right|$$

is. From

$$\begin{aligned}B'(s)=\operatorname{Re}&\left\{\frac{f'(z(s,\theta_1))\frac{\partial z}{\partial s}(s,\theta_1)-f'(z(s,\theta_2))\frac{\partial z}{\partial s}(s,\theta_2)}{f(z(s,\theta_1))-f(z(s,\theta_2))}\right\}\\=-\operatorname{Re}&\left\{\frac{f'(z(s,\theta_1))\overline{f(z(s,\theta_2))}\,\frac{\partial z}{\partial s}(s,\theta_1)+f(z(s,\theta_1))\overline{f'(z(s,\theta_2))}\,\overline{\frac{\partial z}{\partial s}(s,\theta_2)}}{1+\overline{f(z(s,\theta_2))}f(z(s,\theta_1))}\right\},\end{aligned}$$

we get

$$\begin{aligned}B'(s)=\frac{1-\tanh^2(s/2)}{2\tanh(s/2)}\operatorname{Re}&\left\{\frac{z(s,\theta_1)f'(z(s,\theta_1))-z(s,\theta_2)f'(z(s,\theta_2))}{f(z(s,\theta_1))-f(z(s,\theta_2))}\right.\\&\left.-\frac{z(s,\theta_1)f'(z(s,\theta_1))\overline{f(z(s,\theta_2))}+\overline{z(s,\theta_2)f'(z(s,\theta_2))}f(z(s,\theta_1))}{1+\overline{f(z(s,\theta_2))}f(z(s,\theta_1))}\right\}.\end{aligned}\tag{3.2}$$

(a) If the region Ω is spherically convex, then f is a spherically convex univalent function. Corollary 2.2 implies that $B'(s) > 0$. Therefore, $B(s)$ is an increasing function of $s \geq 0$, and so $d_\sigma(w(s,\theta_1), w(s,\theta_2))$ is increasing on $[0, +\infty)$.

(b) Suppose that there exists a in Ω such that $d_\sigma(w(s,\theta_1), w(s,\theta_2))$ is an increasing function of $s \geq 0$ for any $\theta_2 \neq \theta_1 + 2n\pi$. Then $B'(s) \geq 0$. Thus, from (2.5) and (3.2), we have

$$\operatorname{Re}\{q(z,\zeta)\} \geq 0 \quad \text{for } |z| = |\zeta| < 1.$$

Again, by using the same argument as we did in the proof of Theorem 2.1, we get $\operatorname{Re}\{q(z,\zeta)\} > 0$ for all z, ζ in $\mathbb{D}$. By Corollary 2.2, f is a spherically convex univalent function, so Ω is spherically convex.

■

Geometrically, Theorem 3.2(a) indicates that in a spherically convex region Ω, two hyperbolic geodesics starting off in different directions from a point a in Ω will spread farther apart relative to the spherical distance.

Theorem 3.3. *Suppose Ω is a simply connected hyperbolic region in $\mathbb{C}_\infty$.*
(a) *If Ω is spherically convex and $a \in \Omega$, then $\left|\frac{\partial w(s,\theta)}{\partial\theta}\right| / (1 + |w(s,\theta)|^2)$ is an increasing function of $s \geq 0$ for all θ in $\mathbb{R}$. Furthermore, the sharp lower bound*

$$\frac{|\partial w(s,\theta)/\partial\theta|}{1 + |w(s,\theta)|^2} \geq \frac{\tanh(s/2)}{\mu_\Omega(a)(1 + \tanh^2(s/2)) + 2\tanh(s/2)\sqrt{\mu_\Omega^2(a) - 1}}$$

holds.
(b) *If there exists $a \in \Omega$ such that $\left|\frac{\partial w(s,\theta)}{\partial\theta}\right| / (1 + |w(s,\theta)|^2)$ is an increasing function of $s \geq 0$ for all θ in $\mathbb{R}$, then Ω is spherically convex.*

Proof. Without losing of generality, we may assume $\Omega \subset \mathbb{C}$. Let $f : \mathbb{D} \to \Omega$ be a conformal mapping with $f(0) = a$ and $f'(0) > 0$. Then $w(s,\theta) = f(z(s,\theta))$, and

$$\left|\frac{\partial w(s,\theta)}{\partial\theta}\right| = \left|f'(z(s,\theta))\frac{\partial z(s,\theta)}{\partial\theta}\right| = |z(s,\theta)f'(z(s,\theta))|.$$

Set

$$C(s) = \log\frac{|\partial w(s,\theta)/\partial\theta|}{1 + |w(s,\theta)|^2} = \log\frac{|z(s,\theta)f'(z(s,\theta))|}{1 + |f(z(s,\theta))|^2}.$$

Then

$$C'(s) = \frac{1 - |z(s,\theta)|^2}{2|z(s,\theta)|}\operatorname{Re}\left\{1 + \frac{z(s,\theta)f''(z(s,\theta))}{f'(z(s,\theta))} - \frac{2z(s,\theta)f'(z(s,\theta))\overline{f(z(s,\theta))}}{1 + |f(z(s,\theta))|^2}\right\}.$$

(a) If Ω is spherically convex, then f is a spherically convex univalent function. Inequality (2.1) implies that $C'(s) > 0$, which shows that $\left|\frac{\partial w(s,\theta)}{\partial\theta}\right|/(1+|w(s,\theta)|^2)$ is an increasing function of s.

Now we derive the sharp lower bound on $\left|\frac{\partial w(s,\theta)}{\partial\theta}\right|/(1+|w(s,\theta)|^2)$. The function $g(z) = (f(z)-a)/(1+\bar{a}f(z))$ is spherically convex with $g(0)=0$ and $g'(0) = \frac{f'(0)}{1+|a|^2} > 0$, so [3]

$$\frac{1+|g(z)|^2+\sqrt{(1+|g(z)|^2)^2-(1-|z|^2)^2|g'(z)|^2}}{(1-|z|^2)|g'(z)|} \le \frac{1+\sqrt{1-g'(0)^2}}{g'(0)}\frac{1+|z|}{1-|z|}.$$

Or equivalently,

$$\sqrt{1-\frac{(1-|z|^2)^2|g'(z)|^2}{(1+|g(z)|^2)^2}} \le \frac{1+\sqrt{1-g'(0)^2}}{g'(0)}\frac{1+|z|}{1-|z|}\frac{(1-|z|^2)|g'(z)|}{1+|g(z)|^2}-1.$$

By squaring both sides and solving the resulting inequality for $|g'(z)|/(1+|g(z)|^2)$, we find

$$\frac{|g'(z)|}{1+|g(z)|^2} \ge \frac{g'(0)}{1+|z|^2+2|z|\sqrt{1-g'(0)^2}}.$$

As

$$\frac{|f'(z)|}{1+|f(z)|^2} = \frac{|g'(z)|}{1+|g(z)|^2}, \quad g'(0) = \frac{2}{\lambda_\Omega(a)(1+|a|^2)} \quad \text{and} \quad |z(s,\theta)| = \tanh(s/2),$$

we obtain

$$\begin{aligned}\frac{|\partial w(s,\theta)/\partial\theta|}{1+|w(s,\theta)|^2} &= \frac{|z(s,\theta)f'(z(s,\theta))|}{1+|f(z(s,\theta))|^2} \\ &\ge \frac{\tanh(s/2)}{\mu_\Omega(a)(1+\tanh^2(s/2))+2\tanh(s/2)\sqrt{\mu_\Omega^2(a)-1}}.\end{aligned}$$

In order to show sharpness, we consider the function k_α, $0<\alpha\le 1$, which is a conformal map of $\mathbb{D}$ onto the hemisphere Ω_α and is a spherically convex univalent function. If $a=0$, then $w(s,\theta) = k_\alpha(\tanh(s/2)e^{i\theta})$ and

$$\begin{aligned}\frac{|\partial w(s,\theta)/\partial\theta|}{1+|w(s,\theta)|^2} &= \frac{\left|\tanh(s/2)k'_\alpha(\tanh(s/2)e^{i\theta})\right|}{1+|k_\alpha(\tanh(s/2)e^{i\theta})|^2} \\ &= \frac{\alpha\tanh(s/2)}{\left|1-\sqrt{1-\alpha^2}\tanh(s/2)e^{i\theta}\right|^2+\alpha^2\tanh^2(s/2)}.\end{aligned}$$

When $\theta=\pi$,

$$\frac{|\partial w(s,\theta)/\partial\theta|}{1+|w(s,\theta)|^2} = \frac{\alpha\tanh(s/2)}{1+\tanh^2(s/2)+2\sqrt{1-\alpha^2}\tanh(s/2)},$$

which is equal to the lower bound as $a = 0$ and $\lambda_{\Omega_\alpha}(a) = 2/\alpha$.

(b) Now, suppose for some $a \in \Omega$,

$$\frac{\left|\frac{\partial w(s,\theta)}{\partial\theta}\right|}{1+|w(s,\theta)|^2}$$

is an increasing function of $s \geq 0$ for every θ in $\mathbb{R}$. Then $C'(s) \geq 0$, that is, (2.1) holds. Thus, f is a spherically convex univalent function and so Ω is spherically convex. ■

Theorem 3.3(a) is geometrically plausible. Note that $2\left|\frac{\partial w(s,\theta)}{\partial\theta}\right|/(1+|w(s,\theta)|^2)$ is the speed relative to the spherical metric with which $w(s,\theta)$ is moving around the hyperbolic circle $\{w : d_\Omega(w,a) = s\}$. If $s_1 < s_2$, then the point $w(s_2,\theta)$ must move faster than $w(s_1,\theta)$ in order to traverse the larger hyperbolic circle in the same time period 2π. Theorem 3.3(a) indicates that it travels not only faster in an average sense but also pointwise faster

$$\frac{2\,|\partial w(s_1,\theta)/\partial\theta|}{1+|w(s_1,\theta)|^2} < \frac{2\,|\partial w(s_2,\theta)/\partial\theta|}{1+|w(s_2,\theta)|^2}$$

when using the natural hyperbolic arc length parametrization $w(s,\theta)$.

4. Applications of the two-variable characterizations

As we pointed out earlier, we cannot easily deduce properties of spherically convex univalent functions from (2.1) since it contains a nonholomorphic term. Theorem 2.1 overcomes this difficulty in some cases. For example, if f is spherically convex, then $p(z,\zeta)$ is holomorphic as a function of $z \in \mathbb{D}$, has positive real part, and satisfies

$$\left|p(z,\zeta) - \frac{1+|z|^2}{1-|z|^2}\right| \leq \frac{2|z|}{1-|z|^2}. \tag{4.1}$$

As

$$p(z,z) = 1 + \frac{zf''(z)}{f'(z)} - \frac{2zf'(z)\overline{f(z)}}{1+|f(z)|^2},$$

the nonholomorphic function $p(z,z)$ still satisfies the inequality (4.1), which is well known for the class consisting of holomorphic functions $p(z)$ in $\mathbb{D}$ with $p(0) = 1$ and $\operatorname{Re}\{p(z)\} > 0$. Note that

$$\left|p(z,z) - \frac{1+|z|^2}{1-|z|^2}\right| \leq \frac{2|z|}{1-|z|^2}$$

also characterizes spherically convex functions and implies the inequality (2.1), see [3].

We use this idea to derive a number of results for spherically convex functions. Recall that a holomorphic and univalent function f in $\mathbb{D}$ with $f(0) = f'(0) - 1 = 0$ is called starlike of order $\beta \geq 0$ if $\operatorname{Re}\{zf'(z)/f(z)\} > \beta$ in $\mathbb{D}$. By using Theorem 2.1, we show that spherically convex functions are closely related to starlike functions.

Theorem 4.1. *If f is a spherically convex univalent function and $f(0) = 0$, then for every $\zeta \in \mathbb{D}$,*

$$F_\zeta(z) = \frac{z\zeta}{f(\zeta)} \frac{f(z) - f(\zeta)}{(z - \zeta)\left(1 + \overline{f(\zeta)}f(z)\right)}$$

is starlike of order $1/2$.

Proof. Direct calculations yield

$$\frac{2zF'_\zeta(z)}{F_\zeta(z)} - 1 = p(z, \zeta),$$

so Theorem 2.1 implies the result. ■

Since Re $\{F(z)/z\} > 1/2$ and $F(z)^2/z$ is starlike if F is starlike of order $1/2$ (see [11, p. 49]), we get the following results as corollaries of Theorem 4.1.

Corollary 4.2. *If f is a spherically convex univalent function and $f(0) = 0$, then for every $\zeta \in \mathbb{D}$,*

$$\operatorname{Re}\left\{\frac{\zeta}{f(\zeta)} \frac{f(z) - f(\zeta)}{(z - \zeta)\left(1 + \overline{f(\zeta)}f(z)\right)}\right\} > \frac{1}{2}$$

for all z in $\mathbb{D}$.

Corollary 4.3. *If f is a spherically convex univalent function and $f(0) = 0$, then for every $\zeta \in \mathbb{D}$,*

$$\frac{F_\zeta^2(z)}{z} = \frac{z\zeta^2}{f(\zeta)^2} \frac{(f(z) - f(\zeta))^2}{(z - \zeta)^2\left(1 + \overline{f(\zeta)}f(z)\right)^2}$$

is starlike in $\mathbb{D}$.

Mejia and Pommerenke [9] obtained a number of results for spherically convex functions by observing that f is Euclidean convex if f is spherically convex and $f(0) = 0$. We now provide the sharp order of Euclidean convexity for such spherically convex univalent functions.

Corollary 4.4. *Let $f(z) = \alpha z + a_2 z^2 + \dots$, $0 < \alpha < 1$, be a spherically convex univalent function. Then for all z in $\mathbb{D}$*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left(\frac{\alpha}{1+\sqrt{1-\alpha^2}}\right)^2. \tag{4.2}$$

This result is best possible for each α.

Proof. By letting $\zeta \to z$ in Corollary 4.2, we get

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \frac{1+|f(z)|^2}{2}.$$

By using (2.1), we obtain

$$u(z) := \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \frac{2|f(z)|^2}{1+|f(z)|^2}\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq |f(z)|^2. \tag{4.3}$$

If $u(z)$ is identically 1, then inequality (4.2) clearly holds. If $u(z)$ is not a constant, then inequality (4.3) implies

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \inf\left\{|w|^2 : w \notin f(\mathbb{D})\right\}$$

since $u(z)$ is harmonic in $\mathbb{D}$. The result then follows from the fact that $|w| \geq \frac{\alpha}{1+\sqrt{1-\alpha^2}}$ if $w \notin f(\mathbb{D})$ [8].

Sharpness follows by considering the function k_α.

$$\inf\left\{\operatorname{Re}\left\{1 + \frac{zk_\alpha''(z)}{k_\alpha'(z)}\right\} : z \in \mathbb{D}\right\} = \left(\frac{\alpha}{1+\sqrt{1-\alpha^2}}\right)^2.$$

■

Next, we give the sharp lower bound on Re $\{a_2 f(z)\}$ for spherically convex univalent functions $f(z) = \alpha z + a_2 z^2 + \dots$. Similar results hold for Euclidean convex univalent functions [1] and hyperbolically convex univalent functions [5].

Theorem 4.5. *Let $f(z) = \alpha z + a_2 z^2 + \dots$, $0 < \alpha \leq 1$, be a spherically convex univalent function. Then for all z in $\mathbb{D}$*

$$\operatorname{Re}\{a_2 f(z)\} \geq 1 - \alpha^2 - \sqrt{1-\alpha^2}.$$

This result is best possible for all $\alpha \in (0, 1]$.

Proof. For any fixed $\zeta \in \mathbb{D}$, we consider the function $p_\zeta(z) = p(z, \zeta)$ given in (2.3). Then $p_\zeta(0) = 1$ and Re $\{p_\zeta(z)\} > 0$ in $\mathbb{D}$ from Theorem 2.1. Differentiation of $p_\zeta(z)$ with respect to z produces

$$\frac{1}{2}p_\zeta'(0) = -\frac{\alpha}{f(\zeta)} + \frac{1}{\zeta} - \alpha\overline{f\zeta)}$$

and

$$\frac{1}{8}\left(p''_\zeta(0)-p'_\zeta(0)^2\right)=(1+|f(\zeta)|^2)\left(-\frac{a_2}{f(\zeta)}+\frac{\alpha^2}{f(\zeta)^2}\left(\frac{f(\zeta)}{\alpha\zeta}-1\right)\right).$$

By using the inequality $\left|p''_\zeta(0)-p'_\zeta(0)^2\right|+|p'_\zeta(0)|^2\le 4$ [5], we obtain

$$(1+|f(\zeta)|^2)\left|-\frac{a_2}{f(\zeta)}+\frac{\alpha^2}{f(\zeta)^2}\left(\frac{f(\zeta)}{\alpha\zeta}-1\right)\right|\le\frac{1}{2}\left(1-\left|-\frac{\alpha}{f(\zeta)}+\frac{1}{\zeta}-\alpha\overline{f(\zeta)}\right|^2\right),$$

or,

$$\left|a_2f(\zeta)-\alpha^2\left(\frac{f(\zeta)}{\alpha\zeta}-1\right)\right|\le\frac{1}{2(1+|f(\zeta)|^2)}\left(|f(\zeta)|^2-\left|\frac{f(\zeta)}{\zeta}-\alpha(1+|f(\zeta)|^2)\right|^2\right).$$

This implies

$$\begin{aligned}\text{Re }\{a_2f(\zeta)\}&\ge\alpha^2\text{Re}\left\{\frac{f(\zeta)}{\alpha\zeta}-1\right\}\\&\quad-\frac{1}{2(1+|f(\zeta)|^2)}\left(|f(\zeta)|^2-\left|\frac{f(\zeta)}{\zeta}-\alpha(1+|f(\zeta)|^2)\right|^2\right)\\&=\frac{1}{2}\left|\frac{f(\zeta)}{\zeta}\right|^2\frac{1-|\zeta|^2}{1+|f(\zeta)|^2}-\frac{1}{2}\alpha^2+\frac{1}{2}\alpha^2|f(\zeta)|^2\\&>-\frac{1}{2}\alpha^2+\frac{1}{2}\alpha^2|f(\zeta)|^2.\end{aligned}$$

Because Re $\{a_2f(\zeta)\}$ is harmonic for $\zeta\in\mathbb{D}$, we obtain

$$\text{Re }\{a_2f(\zeta)\}\ge-\frac{1}{2}\alpha^2+\frac{1}{2}\alpha^2\inf\{|w|^2:w\notin f(\mathbb{D})\}.$$

The desired inequality follows from the fact that $|w|\ge\frac{\alpha}{1+\sqrt{1-\alpha^2}}$ if $w\notin f(\mathbb{D})$ [8].

It is easy to see that for the spherically convex univalent function $k_\alpha(z)$ the infimum of Re $\{a_2k_\alpha(z)\}$ over $z\in\mathbb{D}$ is $1-\alpha^2-\sqrt{1-\alpha^2}$, so the lower bound is sharp for each $\alpha\in[0,1)$. ■

5. Hyperbolically convex univalent functions and regions

A simply connected region Ω in the hyperbolic plane $\mathbb{D}$ is called *hyperbolically convex* if for all points $a,b\in\Omega$ the arc of the hyperbolic geodesic in $\mathbb{D}$ connecting a and b also lies in Ω. A holomorphic and univalent function f defined on $\mathbb{D}$ with $f(\mathbb{D})\subset\mathbb{D}$ is called *hyperbolically convex* if its image $f(\mathbb{D})$ is a hyperbolically convex subset of $\mathbb{D}$. In this section, we establish properties of hyperbolic polar coordinates in hyperbolically convex regions in the hyperbolic plane. It is convenient to introduce the notation

$$\nu_\Omega(w) = \frac{\lambda_\Omega(w)|dw|}{\lambda_{\mathbb{D}}(w)|dw|} = \frac{1}{2}(1-|w|^2)\lambda_\Omega(w)$$

for the density of the hyperbolic metric of a region $\Omega \subset \mathbb{D}$ relative to the hyperbolic metric $\lambda_{\mathbb{D}}(w)|dw|$. The quantity ν_Ω is invariant $\mathcal{A}(\mathbb{D})$, the group of all isometries of the hyperbolic plane.

There are several known characterizations of hyperbolically convex functions. For example, a holomorphic and locally univalent function f with $f(\mathbb{D}) \subset \mathbb{D}$ is hyperbolically convex if and only if for all z in $\mathbb{D}$ [4]

$$\text{(5.1)} \qquad \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)} + \frac{2zf'(z)\overline{f(z)}}{1-|f(z)|^2}\right\} \geq 0.$$

Mejia and Pommerenke [10] (also see [5]) proved that a holomorphic and locally univalent function f with $f(\mathbb{D}) \subset \mathbb{D}$ is hyperbolically convex if and only if

$$\text{(5.2)} \qquad \operatorname{Re}\left\{\frac{2zf'(z)}{f(z)-f(\zeta)} - \frac{z+\zeta}{z-\zeta} + \frac{2zf'(z)\overline{f(\zeta)}}{1-\overline{f(\zeta)}f(z)}\right\} > 0$$

for all z, ζ in $\mathbb{D}$. Similar to the proof of Corollary 2.2, we get the following characterization of hyperbolically convex functions from (5.2).

Theorem 5.1. *A holomorphic and locally univalent function f with $f(\mathbb{D}) \subset \mathbb{D}$ is hyperbolically convex if and only if*

$$\text{(5.3)} \qquad \operatorname{Re}\left\{\frac{zf'(z)-\zeta f'(\zeta)}{f(z)-f(\zeta)} + \frac{zf'(z)\overline{f(\zeta)} + \overline{\zeta f'(\zeta)}f(z)}{1-\overline{f(\zeta)}f(z)}\right\} > 0$$

for all z, ζ in $\mathbb{D}$.

Now we use these characterizations to derive properties of hyperbolic polar coordinates for hyperbolically convex regions in $\mathbb{D}$.

Theorem 5.2. *Let $\Omega \subset \mathbb{D}$ be simply connected.*

(a) *If Ω is hyperbolically convex and $a \in \Omega$, then $d_{\mathbb{D}}(w(s,\theta), a)$ is an increasing function of $s \geq 0$ for all θ in $\mathbb{R}$. Moreover, we have the following sharp* *bounds:*

$$\begin{aligned}
&\frac{2\tanh(s/2)}{\nu_\Omega(a)(1+\tanh(s/2))+\sqrt{\nu_\Omega^2(a)(1+\tanh(s/2))^2-4\tanh(s/2)}}\\
&\le \tanh\frac{1}{2}d_{\mathbb{D}}(w(s,\theta),a)\\
&\le \frac{2\tanh(s/2)}{\nu_\Omega(a)(1-\tanh(s/2))+\sqrt{\nu_\Omega^2(a)(1-\tanh(s/2))^2+4\tanh(s/2)}}.
\end{aligned}$$

(b) *If $d_{\mathbb{D}}(w(s,\theta),a)$ is an increasing function of $s \ge 0$ for each a in Ω and all θ in $\mathbb{R}$, then Ω is hyperbolically convex.*

Proof. Since the proof of the equivalence between $d_{\mathbb{D}}(w(s,\theta),a)$ being an increasing function of s and Ω being hyperbolically convex parallels the proof of Theorem 3.1, using (5.2) instead of (2.2), we omit this part of the proof.

Assume that Ω is hyperbolically convex, we derive the sharp bounds on $d_{\mathbb{D}}(w(s,\theta),a)$. Let $f:\mathbb{D}\to\Omega$ be the conformal mapping with $f(0)=a$ and $f'(0)>0$. Then f is hyperbolically convex. This implies that $g(z)=(f(z)-a)/(1-\bar{a}f(z))$ is hyperbolically convex with $g(0)=0$ and

$$g'(0)=\frac{f'(0)}{1-|a|^2}=\frac{2}{\lambda_\Omega(a)(1-|a|^2)}>0.$$

We know that [4]

$$\frac{2g'(0)|z|}{1+|z|+\sqrt{(1+|z|)^2-4g'(0)^2|z|}}\le|g(z)|\le\frac{2g'(0)|z|}{1-|z|+\sqrt{(1-|z|)^2+4g'(0)^2|z|}}.$$

That is,

$$\begin{aligned}
&\frac{2|z|}{\nu_\Omega(a)(1+|z|)+\sqrt{\nu_\Omega^2(a)(1+|z|)^2-4|z|}}\\
&\le\left|\frac{f(z)-a}{1-\bar{a}f(z)}\right|\\
&\le\frac{2|z|}{\nu_\Omega(a)(1-|z|)+\sqrt{\nu_\Omega^2(a)(1-|z|)^2+4|z|}}\,.
\end{aligned}$$

As

$$\tanh\frac{1}{2}d_{\mathbb{D}}(w(s,\theta),a)=\left|\frac{w(s,\theta)-a}{1-\bar{a}w(s,\theta)}\right|=\left|\frac{f(z(s,\theta))-a}{1-\bar{a}f(z(s,\theta))}\right|$$

and $|z(s,\theta)|=\tanh(s/2)$, this gives the bounds in Theorem 5.2(a).

For $0 < \alpha \le 1$, the hyperbolic half-plane

$$H_\alpha = \mathbb{D} \setminus \left\{ w : \left| w + \frac{1}{\alpha} \right| \le \frac{\sqrt{1-\alpha^2}}{\alpha} \right\}$$

is hyperbolically convex and

$$K_\alpha(z) = \frac{2\alpha z}{1 - z + \sqrt{(1-z)^2 + 4\alpha^2 z}}$$

maps $\mathbb{D}$ conformally onto H_α. When $a = 0$, $\nu_{H_\alpha}(0) = 1/\alpha$, $w(s,0) = K_\alpha(\tanh(s/2))$ is the hyperbolic arc length parametrization of $[0,1)$, and the upper bound is equal to

$$\frac{2\alpha \tanh(s/2)}{1 - \tanh(s/2) + \sqrt{(1-\tanh(s/2))^2 + 4\alpha^2 \tanh(s/2)}} = \tanh \frac{1}{2} d_{\mathbb{D}}(w(s,0), 0).$$

This shows that the upper bound is sharp. Similarly, $w(s,\pi) = K_\alpha(-\tanh(s/2))$ is the hyperbolic arc length parametrization of $\left(\frac{-1+\sqrt{1-\alpha^2}}{\alpha}, 0 \right]$, and the lower bound is equal to

$$\frac{2\alpha \tanh(s/2)}{1 + \tanh(s/2) + \sqrt{(1+\tanh(s/2))^2 - 4\alpha^2 \tanh(s/2)}} = \tanh \frac{1}{2} d_{\mathbb{D}}(w(s,\pi), 0).$$

Thus, the lower bound is also sharp. ∎

By using the characterization (5.3) for hyperbolically convex univalent functions, we can prove the following theorem in the same manner as Theorem 3.2 was proved.

Theorem 5.3. *Suppose $\Omega \subset \mathbb{D}$.*

(a) *If Ω is hyperbolically convex and $a \in \Omega$, then $d_{\mathbb{D}}(w(s,\theta_1), w(s,\theta_2))$ is an increasing function of $s \ge 0$ for all $\theta_2 \ne \theta_1 + 2n\pi$.*

(b) *If there exists $a \in \Omega$ such that $d_{\mathbb{D}}(w(s,\theta_1), w(s,\theta_2))$ is an increasing function of $s \ge 0$ whenever $e^{i\theta_2} \ne e^{i\theta_1}$, then Ω is hyperbolically convex.*

Theorem 5.4. *Let $\Omega \subset \mathbb{D}$ be simply connected.*

(a) *If Ω is hyperbolically convex and $a \in \Omega$, then $\left| \frac{\partial w(s,\theta)}{\partial \theta} \right| / (1 - |w(s,\theta)|^2)$ is an increasing function of $s \ge 0$ for all θ in $\mathbb{R}$. Furthermore, we have the* *following sharp estimates:*

$$\frac{\tanh(s/2)}{(1+\tanh(s/2))\sqrt{\nu_\Omega^2(a)(1+\tanh(s/2))^2-4\tanh(s/2)}}$$
$$\le \frac{|\partial w(s,\theta)/\partial\theta|}{1-|w(s,\theta)|^2}$$
$$\le \frac{\tanh(s/2)}{(1-\tanh(s/2))\sqrt{\nu_\Omega^2(a)(1-\tanh(s/2))^2+4\tanh(s/2)}}.$$

(b) *If there exists $a \in \Omega$ such that*

$$\frac{\left|\frac{\partial w(s,\theta)}{\partial\theta}\right|}{1-|w(s,\theta)|^2}$$

is an increasing function of $s \ge 0$ for all θ in $\mathbb{R}$, then Ω is hyperbolically convex.

Proof. We again omit the proof of the equivalence between Ω being hyperbolically convex and

$$\frac{\left|\frac{\partial w(s,\theta)}{\partial\theta}\right|}{1-|w(s,\theta)|^2}$$

being an increasing function of s since it parallels the proof of Theorem 3.3, using (5.1) instead of (2.1). Assume that Ω is hyperbolically convex. We establish the sharp bounds on the quantity

$$\frac{\left|\frac{\partial w(s,\theta)}{\partial\theta}\right|}{1-|w(s,\theta)|^2}.$$

Let $f : \mathbb{D} \to \Omega$ be the conformal mapping with $f(0) = a$ and $f'(0) > 0$. Then f is hyperbolically convex, $w(s,\theta) = f(z(s,\theta))$, and $g(z) = (f(z)-a)/(1-\bar{a}f(z))$ is hyperbolically convex with $g(0) = 0$ and

$$g'(0) = \frac{f'(0)}{1-|a|^2} = \frac{2}{\lambda_\Omega(a)(1-|a|^2)} > 0.$$

Note that

$$\frac{|\partial w(s,\theta)/\partial\theta|}{1-|w(s,\theta)|^2} = \frac{|z(s,\theta)f'(z(s,\theta))|}{1-|f(z(s,\theta))|^2} = \frac{|z(s,\theta)g'(z(s,\theta))|}{1-|g(z(s,\theta))|^2}.$$

In [4], we established

$$\frac{g'(0)}{(1+|z|)\sqrt{(1+|z|)^2-4g'(0)^2|z|}} \le \frac{|g'(z)|}{1-|g(z)|^2} \le \frac{g'(0)}{(1-|z|)\sqrt{(1-|z|)^2+4g'(0)^2|z|}}.$$

Since $|z(s,\theta)| = \tanh(s/2)$, we have

$$\begin{aligned}
&\frac{\tanh(s/2)}{(1+\tanh(s/2))\sqrt{\nu_\Omega^2(a)(1+\tanh(s/2))^2-4\tanh(s/2)}}\\
&\le \frac{|z(s,\theta)g'(z(s,\theta))|}{1-|g(z(s,\theta))|^2}\\
&= \frac{|\partial w(s,\theta)/\partial\theta|}{1-|w(s,\theta)|^2}\\
&\le \frac{\tanh(s/2)}{(1-\tanh(s/2))\sqrt{\nu_\Omega^2(a)(1-\tanh(s/2))^2+4\tanh(s/2)}}.
\end{aligned}$$

Therefore, the bounds in part (a) are valid.

Now, we verify the sharpness of the bounds. For $0 < \alpha \le 1$, the conformal map K_α of $\mathbb{D}$ onto the hyperbolic half-plane H_α is hyperbolically convex. When $a = 0$, $w(s,\theta) = K_\alpha(\tanh(s/2)e^{i\theta})$, and

$$\frac{|\partial w(s,\theta)/\partial\theta|}{1-|w(s,\theta)|^2} = \frac{\tanh(s/2)|K_\alpha'(\tanh(s/2)e^{i\theta})|}{1-|K_\alpha(\tanh(s/2)e^{i\theta})|^2}.$$

When $\theta = 0$,

$$\begin{aligned}
\frac{|\partial w(s,\theta)/\partial\theta|}{1-|w(s,\theta)|^2} &= \frac{\tanh(s/2)|K_\alpha'(\tanh(s/2))|}{1-|K_\alpha(\tanh(s/2))|^2}\\
&= \frac{\alpha\tanh(s/2)}{(1-\tanh(s/2))\sqrt{(1-\tanh(s/2))^2+4\alpha^2\tanh(s/2)}},
\end{aligned}$$

which equals the upper bound in this case since $\nu_{H_\alpha}(0) = 1/\alpha$. For $\theta = \pi$,

$$\begin{aligned}
\frac{|\partial w(s,\theta)/\partial\theta|}{1-|w(s,\theta)|^2} &= \frac{\tanh(s/2)|K_\alpha'(-\tanh(s/2))|}{1-|K_\alpha(-\tanh(s/2))|^2}\\
&= \frac{\alpha\tanh(s/2)}{(1+\tanh(s/2))\sqrt{(1+\tanh(s/2))^2-4\alpha^2\tanh(s/2)}},
\end{aligned}$$

which is equal to the lower bound in this case since $\nu_{H_\alpha}(0) = 1/\alpha$. ■

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