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Local Geometry of Circles and Loxodromes

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Abstract. A two-valued Möbius invariant is discussed in terms of differential invariants of smooth curves in the Riemann sphere with respect to Möbius transformations

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The purpose of this note is to present some observations on the local spaces of circular and loxodromic arcs, and to draw attention to a specific Möbius invariant expression (3). In particular, we are presenting in an elementary way some material which was expressed more abstractly in [1]. Our belief is that there may be many interesting things to discover in the local geometry underlying this invariant.

In the action of the group Aut $\widehat{\mathbb{C}}$ of Möbius transformations $z \mapsto (az+b)/(cz+d)$, $a, b, c, d \in \mathbb{C}$, $ad-bc = 1$, on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the Schwarzian derivative

$$\mathcal{S}_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

has special relevance. In particular, we point out two of the many well-known elementary invariance properties for $T \in \text{Aut}\, \widehat{\mathbb{C}}$.

$$T(x) - T(y) = (x-y)T'(x)^{1/2}\, T'(y)^{1/2}, \tag{1}$$

$$\mathcal{S}_{T\circ f} = \mathcal{S}_f. \tag{2}$$

Of course in (1) one needs to define the square root of T' consistently, which is done by choosing a, b, c, d in the formula for T and then noting that $T(z)^{-1/2} = cz + d$.

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1. Two-Valued Möbius Invariant

The Möbius invariant which interests us here is given as follows. Let $z : (-\varepsilon,\varepsilon) \to \widehat{\mathbb{C}}$ be a smooth (C^3) curve. Consider the following third-order differential operator.

$$p, q = z - \frac{2z'^2}{z'' \pm z'\sqrt{-2S_z}}. \tag{3}$$

Proposition A. *The pair $\{p, q\}$ is a (2-valued) Möbius invariant: let $w(t) = T(z(t))$ for any fixed $T \in \operatorname{Aut}\widehat{\mathbb{C}}$. Then*

$$T\left(z - \frac{2z'^2}{z'' \pm z'\sqrt{-2S_z}}\right) = w - \frac{2w'^2}{w'' \pm w'\sqrt{-2S_w}}. \tag{4}$$

Proof: Abbreviate $r = \pm\sqrt{\frac{1}{2}\mathcal{S}_z} = \pm\sqrt{\frac{1}{2}\mathcal{S}_w}$. Write $x = z$, $y = z - \dfrac{2z'^2}{z'' - 2rz'}$.

Then (1) says

$$\begin{aligned} T(x) - T(y) &= \frac{2z'^2}{z'' - 2rz'}\, T'(z)^{1/2}\, T'(z - \frac{2z'^2}{z'' - 2rz'})^{1/2} \\ &= \frac{2T'(z)^{1/2}}{T''(z)^2 z'^2 + T'(z)z'' - 2rT'(z)z'}. \end{aligned} \tag{5}$$

In detail, this is because

$$\left(c\left(z - \frac{2z'^2}{z'' - 2rz'}\right) + d\right)(z'' - 2rz')$$
$$= (cz+d)^3\left(\frac{-2c}{(cz+d)^3}z'^2 + \frac{z''}{(cz+d)^2} + \frac{2rz'}{(cz+d)^2}\right)$$

by simple algebra, and

$$T'\left(z - \frac{2z'^2}{z'' - 2rz'}\right)^{-1/2}(z'' - 2rz')$$
$$= T'\left(z - \frac{2z'^2}{z'' - 2rz'}\right)^{-3/2}(T''(z)^2 z'^2 + T'(z)z'' - 2rT'(z)z')$$

which can be seen by identifying certain appearances of $1/(cz+d)$ with $T'(z)^{1/2}$ and others with $T''(z)/(-2c)$. Finally, combining (5) with (1), (2) and the Chain

Rule we have

$$\begin{aligned} w - \frac{2w'^2}{w'' \pm w'\sqrt{-2S_w}} &= T(z) - \frac{2T'(z)^2 z'^2}{T''(z)^2 z'^2 + T'(z)z'' - 2rT'(z)z'} \\ &= T(z) - \left(T(z) - T(z - \frac{2z'^2}{z'' \pm z'\sqrt{-2S_z}} \right) \\ &= T\left(z - \frac{2z'^2}{z'' \pm z'\sqrt{-2S_z}} \right). \end{aligned}$$

as desired. □

2. Circle and Loxodrome Germs

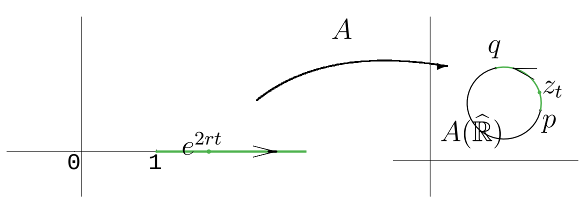
The invariant $\{p, q\}$ deserves some explanation. For a moment let $z_t = e^{2rt}$, $r > 0$, serve as sort of a "model" curve in $\widehat{\mathbb{C}}$. Note that the Schwarzian derivative

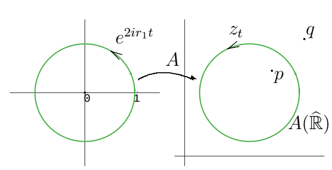
$$\mathcal{S}_z(t) = -2r^2$$

is constant. For this curve the invariants (3) are

$$\begin{aligned} p, q &= e^{2rt} - \frac{2(4r^2)e^{4rt}}{4r^2 e^{2rt} \pm 2(2r)e^{2rt} r} \\ &= e^{2rt} - \begin{cases} e^{2rt} \\ \infty \end{cases} \\ &= \begin{cases} 0 \\ \infty \end{cases}. \end{aligned}$$

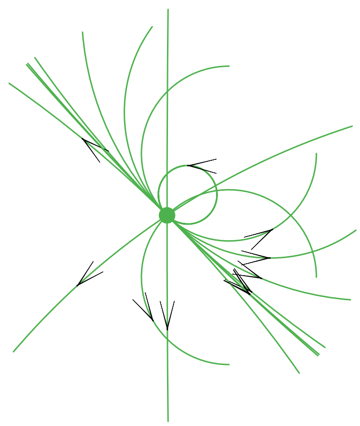
Thus this curve is also special because p, q are constant as well. We now examine a somewhat more general curve, by moving $0, \infty$ to an arbitrary pair of points. Let $A \in \mathrm{Aut}\ \widehat{\mathbb{C}}$ and define $w = A(e^{2rt})$. Here w is a standard "hyperbolic" parametrization of a circular arc. By (2) we have the same Schwarzian derivative, $\mathcal{S}_w(t) = 2r^2$. Similarly, for $w = A(e^{2rt})$, $r = ir_1$ imaginary, we obtain an elliptic parametrization of the circle.





We will use this construction to define "circular germs." A circle $K \subseteq \widehat{\mathbb{C}}$ is an image $A(\widehat{\mathbb{R}})$, $A \in \mathrm{Aut}\,\widehat{\mathbb{C}}$ of the real circle $\widehat{\mathbb{R}} = \mathbb{R} \cup \infty$. In this way circles have natural parameterizations $A(e^{2rt})$. Given a point $z \in \widehat{\mathbb{C}}$, consider all circular paths passing through z parameterized in this way.

Definition. The bundle $\mathcal{B}$ over $\widehat{\mathbb{C}}$ is comprised of all germs α of curves $A(e^{2rt})$ at $t = 0$, where $A \in \mathrm{Aut}\,\widehat{\mathbb{C}}$, $r \in \mathbb{C}$. The projection of α is the point $\pi(\alpha) = A(1) \in \mathbb{C}$.



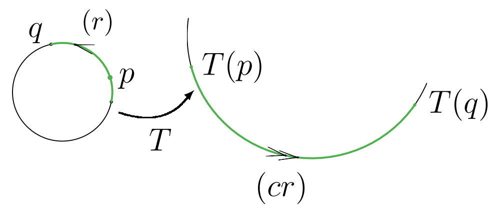
The real-differentiable structure on $\mathcal{B}$ comes from A and r since $\mathrm{Aut}\,\widehat{\mathbb{C}}$ is a manifold in a natural way.

The value r will be called the "invariant velocity" of the curve or germ. This is hyperbolic for r real, elliptic for r imaginary, and parabolic for $r = 0$. We will now look at the natural "circular flow" $\mathcal{B} \times \mathbb{R} \to \mathcal{B}$ which is described in more detail in [1]. Each trajectory of this flow has a constant invariant velocity.

Consider $\alpha \in \mathcal{B}$ lying over $\pi(\alpha) = z_0 \in \widehat{\mathbb{C}}$, with invariant velocity r, and trajectory on the circle $K \ni z_0$. We can create a new germ $\beta \in \mathcal{B}$ in two rather obvious ways: multiplying r by a constant c, and moving to a new circle $T(K)$. Thus we have

$$\beta = (T, c) \cdot \alpha \in \mathcal{B} \tag{6}$$

which is the germ of $TA(e^{crt})$ at $T(z_0)$. We will call the pair $(T, c) \in \mathrm{Aut}\,\widehat{\mathbb{C}} \times \mathbb{R}$ a *multiplier*. All this does is move the circle (and base point) by applying T, while multiplying the invariant velocity by c.



It is a pleasant fact that for germs of the same type and based at the same point, we can *divide* one by the other. Thus we recover the multiplier

$$\frac{\beta}{\alpha} = (T, c) \tag{7}$$

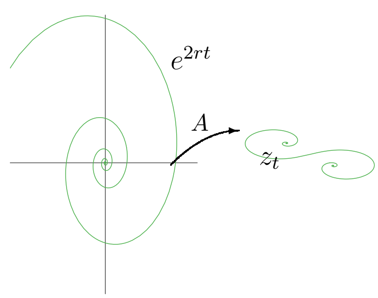
in terms of α and β. Further, given a parameterized family of multipliers we can take *derivatives*:

$$\left.\frac{d}{dt}\right|_{t=0} (T(t), c(t)) = (\dot{T}(t), \dot{c}(t)) \tag{8}$$

Lastly, we mention the operation of *exponentiation:*

$$\exp(\dot{T}, \dot{c}) = (\exp(\dot{T}), e^{\dot{c}}). \tag{9}$$

The not-so-pleasant reality is that not all germs can be divided, because not all curves are of the same type. Observe also that to deal with general smooth curves z_t as in Proposition A, the Schwarzian derivative $\mathcal{S}_z$ will not necessarily be real or purely imaginary. So we are led to consider r in $\mathbb{C}$ instead of $\mathbb{R}$. This gives the notion of *loxodrome*, $z_t = A(e^{2rt})$, $r \in \mathbb{C} \setminus \{0\}$, $A \in \operatorname{Aut} \widehat{\mathbb{C}}$.



The bundle $\mathcal{B}^*$ of loxodromic germs is defined analogously to $\mathcal{B}$. It is also a real manifold. In $\mathcal{B}^*$ we can divide any loxodromic germ by any nonsingular germ (that is, one for which $r \neq 0$).

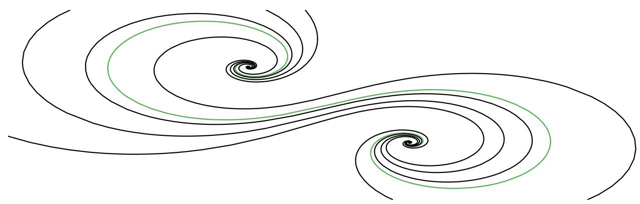
A natural 1-parameter family of Möbius transformations such as $z \mapsto e^{2rt}z$ can in fact be thought of as describing a movement of every $z \in \widehat{\mathbb{C}}$. Thus there is a natural flow on $\widehat{\mathbb{C}}$, and in fact as we will describe below, on $\mathcal{B}^*$. Since ∞ is more natural than 1 as a base point in $\widehat{\mathbb{C}}$ we will henceforth use $\coth rt$ in place of e^{2rt} to describe the natural germs. Thus $A(z)$ is replaced by $A((z+1)/(z-1))$ in the description of α. With this change in mind we define trajectory as follows.

Definition. The *trajectory* of $\alpha \in \mathcal{B}^*$ through $z \in \widehat{\mathbb{C}}$ is

$$\Phi_{\alpha,t}(z) = A\left(\frac{(\cosh rt)z + \sinh rt}{(\sinh rt)z + \cosh rt}\right). \tag{10}$$

The *principal trajectory* of α is the trajectory through $A(\infty)$,

$$\Phi_{\alpha,t}(\infty) = A(\coth rt).$$



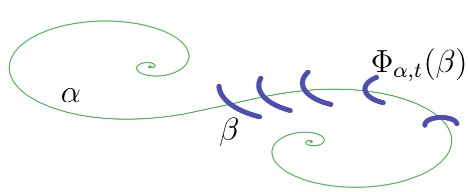
3. Local Loxodrome Geometry

We can imitate in $\mathcal{B}^*$ many constructions which are done with the differential geometry of geodesics on Riemannian manifolds.

Primitive Parallel Transport. Let $\alpha \in \mathcal{B}^*$ and consider any $\beta \in \mathcal{B}^*$ based at the same point $z_0 \in \widehat{\mathbb{C}}$. The trajectory $\Phi_{\beta,u}(\infty)$ of β is moved by α in a natural way as follows. As in (10) write the trajectory of α as $z_t = \Phi_{\alpha,t}(\infty)$, so $z_0 = A(\infty)$. Let β have invariant velocity s, based also at $B(\infty) = z_0$. Then the germ $\beta_t \in \mathcal{B}^*$ based at z_t is characterized by its trajectories,

$$\Phi_{\beta_t,u}(z) = \Phi_{\alpha,t}(\Phi_{\beta_t,u}(\Phi_{\alpha,t}^{-1}(z))).$$

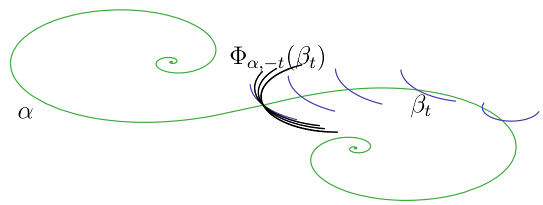
The principal trajectory of β_t is $\Phi_{\alpha,t}(\Phi_{\beta,u}(\infty))$.



Definition. The family β_t described above is the *primitive parallel transport* $\Phi_{\alpha,t}(\beta) = \beta_t \in \mathcal{B}^*$ of β along α.

Now suppose more generally, that $\beta_t \in \mathcal{B}^*$ is based at $z_t = \Phi_{\alpha,t}(\infty)$ on the principal trajectory of α, but β_t is not necessarily equal to $\Phi_{\alpha,t}(\beta)$. How can we measure how much β_t differs from $\Phi_{\alpha,t}(\beta)$? To make the comparison, we first transport β_t along α back to $z(0)$; that is, we look at $\Phi_{\alpha,-t}(\beta_t)$. Then divide by β_0:

$$\frac{\Phi_{\alpha,-t}(\beta_t)}{\beta_0} = (T_t, c_t).$$



Then take the derivative, thus defining

$$\begin{aligned}\frac{d\beta}{d\alpha} &= \left.\frac{d}{dt}\right|_0 \frac{\Phi_{\alpha,-t}(\beta_t)}{\beta_0} \\ &= \left(\left.\frac{d}{dt}\right|_0 T_t, \left.\frac{d}{dt}\right|_0 c_t\right) \\ &= (\dot{T}, \dot{c}) \in \mathrm{sl}_2\,\mathbb{C} \times \mathbb{C}.\end{aligned}$$

Definition. The *covariant derivative* of β_t along α is

$$D_\alpha \beta = \exp\left(\frac{d\beta}{d\alpha}\right) \cdot \beta_0.$$

Here $\alpha \in \mathcal{B}^*$ is fixed, $t \mapsto \beta_t \in \mathcal{B}^*$ is any smooth parametrization, and $D_\alpha \beta \in \mathcal{B}^*$.

From these definitions we have immediately the invariance of the covariant derivative:

$$D_{T\alpha}(T\beta) = T(D_\alpha \beta). \tag{11}$$

Further, the covariant derivative measures how β_t differs from the parallel transport of β_0: if β_t were a primitive parallel transport $\beta_t = \Phi_{\alpha,t}(\beta_0)$, then we would have

$$\frac{\Phi_{\alpha,-t}(\beta_t)}{\beta_0} = (I, 1)$$

and then

$$\frac{d\beta}{d\alpha} = (0, 0).$$

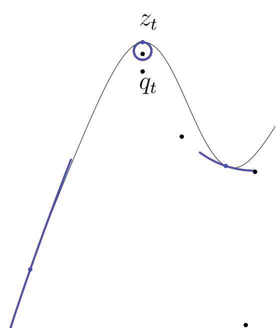
Expressed a different way, we have $D_\alpha\beta = \exp(0,0)\cdot\beta_0 = (1,1)\cdot\beta_0 = \beta_0$.

Parallel Transport. First we explain the Möbius invariants p, q given in (3). Let $z : (-\varepsilon,\varepsilon) \to \widehat{\mathbb{C}}$ be any smooth curve.

Definition. The *contact germ* $\alpha \in \mathcal{B}^*$ to the curve z_t at z_0 is the germ whose principal trajectory has third-order contact with z_t at $t = 0$, i.e.,

$$\left.\frac{d^k}{dt^k}\right|_{t=0} z_t = \left.\frac{d^k}{dt^k}\right|_{t=0} \Phi_{\alpha,t}(A(\infty))$$

for $k = 0, 1, 2, 3$.



We define the *associated fixed point set* $\{p_t, q_t\}$ and *invariant velocity* r_t at time t to be those of the unique contact germ at z_t (valid at nonsingular points, i.e., $r_t \neq 0$). By construction the contact germ is a Möbius invariant; that is, the contact germ to $w_t = T(z_t)$ is $T\alpha = (T,1)\alpha$ when α is the contact germ to z_t. This is the content of Proposition A.

The notion of contact germ permits us to relax the condition that z_t be the trajectory of α in the definition of parallel transport.

Definition. Given $z : (-\varepsilon,\varepsilon) \to \widehat{\mathbb{C}}$, at each z_t let $\beta_t \in \mathcal{B}^*$ a loxodrome germ, varying smoothly. Define the *change of* β *along* z as the germ derivative

$$\left.\frac{d\beta_t}{dz_t}\right|_{t=0} = \left.\frac{d\beta_t}{d\alpha}\right|_{t=0}$$

where α is the contact loxodrome germ to z_t at z_0. We say β_t is *parallel along* z_t when $d\beta/dz = (0,0)$ along z_t. When this is the case, call β_t the *parallel transport* of β_0 along z_t.

The following statements are analogous to well-known results in differential geometry.

Proposition 1. *Suppose that the invariant velocity* $r_t \in \mathbb{C}$ *of the nonsingular curve* z *in* $\widehat{\mathbb{C}}$ *is constant. Then* z *is a loxodrome with a natural parametrization.*

Proposition 2. *Along a nonsingular curve, there exists a unique parallel transport for any loxodrome germ.*

Proposition 3. *The only self-parallel curves are the loxodromes.*

Proposition 4. *Parallel transport preserves angles.*

4. Questions about Loxodromes

1. There is a natural Möbius-invariant inner product $\alpha_t \cdot \beta_t$ on loxodrome germs based at the same point (see [1] for the precise definition). Strangely, parallel translate does *not* preserve this inner product. Why not?

2. To prove Proposition 2 one uses only the contact of order $k = 1$, and Proposition 4 uses $k = 2$. What Proposition would result from interpreting $k = 3$?

3. Take a smooth curve $z : (-\varepsilon, \varepsilon) \to \widehat{\mathbb{C}}$, and fix $x_0, x_1, x_2 \in \widehat{\mathbb{C}}$. Let $C_t \in \mathrm{Aut}\, \widehat{\mathbb{C}}$ take these three points to z_t, p_t, q_t. Then one sees that $\dot{C}_t C_t^{-1} \in \mathrm{sl}_2\, \mathbb{C}$ and that this product is a Möbius-invariant which does not depend on the choice of x_0, x_1, x_2. One calculates

$$\det(C_t C_t^{-1}) = -r^2 + \frac{r'^2}{4r^2},$$

Explain the term $r'^2/4r^2$, which involves d^4z/dt^4.

4. On Riemannian manifolds, one can always reparameterize a curve to have constant velocity. By Proposition 1 this is not possible for invariant velocity for curves in $\widehat{\mathbb{C}}$. Is there a "best" reparameterization for a smooth curve in the context of Möbius transformations?

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