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Hyperbolic geometric characterizations of convex regions

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Abstract. There are a number of known characterizations of convex regions in terms of the density of the hyperbolic metric. For instance, Ω is convex if and only if $1/\lambda_\Omega$ is a concave function, where λ_Ω denotes the density of the hyperbolic metric on Ω. Several new characterizations of convex regions in terms of hyperbolic geometry are given. For example, $\log \lambda_\Omega$ is hyperbolically convex in the sense that it is convex along each hyperbolic geodesic parameterized by hyperbolic arclength if and only if Ω is convex. There are also characterizations in terms of hyperbolic curvature. For instance, Ω is convex if and only each Euclidean segment in Ω parameterized by hyperbolic arclength has absolute hyperbolic curvature at most 1.

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1. Introduction

A region Ω in the complex plane $\mathbb{C}$ is *hyperbolic* if $\mathbb{C} \setminus \Omega$ contains at least two points. Throughout this paper, we assume that Ω is hyperbolic unless stated otherwise. The hyperbolic metric on a hyperbolic region Ω is denoted by $\lambda_\Omega(w)|dw|$ and is normalized to have curvature

$$K(w) = -\frac{\Delta \log \lambda_\Omega(w)}{\lambda_\Omega^2(w)} = -1,$$

where $w = u + iv$ and

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} = 4\frac{\partial^2}{\partial w \partial \bar{w}}$$

denotes the usual Laplacian. The hyperbolic metric on the unit disk $\mathbb{D}$ is

$$\lambda_{\mathbb{D}}(z)|dz| = 2|dz|/(1 - |z|^2).$$

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If $f:\mathbb{D}\to\Omega$ is any holomorphic universal covering projection, then the density λ_Ω of the hyperbolic metric is determined from

$$\lambda_\Omega(f(z))|f'(z)| = 2/(1-|z|^2). \tag{1.1}$$

This paper is concerned with characterizations of convex regions in terms of convexity properties of the density of the hyperbolic metric. Various characterizations of this type are known. For instance, Kim and Minda [3] showed that $1/\lambda_\Omega$ is concave along every Euclidean segment parameterized by Euclidean arclength if and only if Ω is convex. A simply connected region Ω on the Riemann sphere $\mathbb{C}_\infty = \mathbb{C}\cup\{\infty\}$ is called a *Nehari region* if any conformal mapping $f:\mathbb{D}\to\Omega$ satisfies $(1-|z|^2)^2|S_f(z)|\le 2$ for z in $\mathbb{D}$, where

$$S_f = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

is the Schwarzian derivative of f. Chuaqui, Osgood and Pommerenke [1] proved that a simply connected region Ω in $\mathbb{C}_\infty$ is Nehari if and only if $\lambda_\Omega^{1/2}$ is hyperbolically convex on Ω and on every Möbius image of Ω. To say that a real-valued function F defined on Ω is *hyperbolically convex* means that if $\gamma : w = w(s)$ is any hyperbolic geodesic in Ω parameterized by hyperbolic arclength, then the function $F(w(s))$ is convex; that is, has nonnegative second derivative. Because convex regions are a special type of Nehari region, there should be an analogous characterization of convex regions by a stronger property. We provide such a characterization: a hyperbolic region Ω in $\mathbb{C}$ is convex if and only if $\log\lambda_\Omega$ is hyperbolically convex. Moreover, unless the convex region Ω is a strip or a sector with angular opening $\beta\pi$, where $0<\beta\le 1$, then $\log\lambda_\Omega$ is strictly hyperbolically convex in the sense that the second derivative is positive along hyperbolic geodesics. This result is sharp in the following sense. Set

$$v_\alpha = \begin{cases} \frac{\lambda_\Omega^\alpha - 1}{\alpha} & \text{if } \alpha\ne 0,\\ \log\lambda_\Omega & \text{if } \alpha = 0.\end{cases}$$

It is elementary that if v_α is hyperbolically convex, then so is v_β for all $\alpha\le\beta$. Hence, if Ω is a convex region, then the hyperbolic convexity of v_0 implies the hyperbolic convexity of v_α for all $\alpha>0$. On the other hand, there are no regions with v_α hyperbolically convex for any $\alpha<0$; equivalently, there are no regions with $1/\lambda_\Omega^\alpha$ hyperbolically concave for any $\alpha>0$.

The organization of this paper is as follows. Invariant differential operators for holomorphic functions and differential expressions related to the hyperbolic metric are introduced in Section 2. In Section 3 various characterizations of convex regions in terms of conformal maps from the unit disk onto the region are given. That the hyperbolic convexity of $\log\lambda_\Omega$ characterizes convex regions is established in Section 4. Characterizations of convex regions in terms of properties of Euclidean geodesics are presented in Section 5. For instance, a hyperbolic region is convex if and only if every Euclidean segment in Ω has absolute hyperbolic curvature at most 1. Uniformly perfect regions are characterized in terms of the uniform boundedness of the rate of change of hyperbolic curvature along Euclidean geodesics. A hyperbolic region Ω is *uniformly perfect* if there exists $c = c(\Omega) > 0$ such that $c \leq \lambda_\Omega(z)\delta_\Omega(z)$ for all $z \in \Omega$, where $\delta_\Omega(z)$ is the Euclidean distance from z to $\partial\Omega$. The hyperbolic convexity of λ_Ω^α for any $\alpha > 0$ is considered in Section 6. The rate of change of Euclidean curvature along hyperbolic geodesics parameterized by hyperbolic arclength is covered in Section 7. The final section deals with hyperbolic geodesics parameterized by Euclidean arclength.

2. Preliminaries

It is convenient to introduce certain invariant differential operators defined for holomorphic functions on the unit disk. For f is holomorphic on $\mathbb{D}$, set

$$D_1 f(z) = (1-|z|^2)f'(z),$$
$$D_2 f(z) = (1-|z|^2)^2 f''(z) - 2\bar{z}(1-|z|^2)f'(z),$$
$$D_3 f(z) = (1-|z|^2)^3 f'''(z) - 6\bar{z}(1-|z|^2)^2 f''(z) + 6\bar{z}^2(1-|z|^2)f'(z).$$

These differential operators satisfy an important invariance property: if $S(z) = e^{i\phi}(z-b)$, a Euclidean motion of $\mathbb{C}$, and $T(z) = e^{i\theta}(z-a)/(1-\bar{a}z)$, a conformal automorphism of $\mathbb{D}$, then $|D_j(S\circ f\circ T)| = |D_j f|\circ T$, $j = 1, 2, 3$. For more information on these operators, see [5]. Certain combinations of these operators occur frequently. The first is

$$\frac{D_2 f(z)}{D_1 f(z)} = (1-|z|^2)\frac{f''(z)}{f'(z)} - 2\bar{z}.$$

The other combination is

$$\frac{D_3 f(z)}{D_1 f(z)} - \frac{3}{2}\left(\frac{D_2 f(z)}{D_1 f(z)}\right)^2 = (1-|z|^2)^2 S_f(z).$$

Two useful differential quantities for the hyperbolic metric are defined as follows. The first is the connection

$$\Gamma_\Omega(w) = 2\frac{\partial \log \lambda_\Omega(w)}{\partial w},$$

and the second is the Schwarzian

$$(2.1)\qquad S_\Omega(w) = \frac{\partial \Gamma_\Omega(w)}{\partial w} - \frac{1}{2}\Gamma_\Omega^2(w) = 2\left(\frac{\partial^2 \log \lambda_\Omega(w)}{\partial w^2} - \left(\frac{\partial \log \lambda_\Omega(w)}{\partial w}\right)^2\right).$$

Note that

$$\frac{\partial \Gamma_\Omega(w)}{\partial w} = S_\Omega(w) + \frac{1}{2}\,\Gamma_\Omega^2(w). \tag{2.2}$$

The identity

$$\frac{\partial \Gamma_\Omega(w)}{\partial \bar{w}} = \frac{1}{2}\lambda_\Omega^2(w) \tag{2.3}$$

is a direct consequence of the fact that the hyperbolic metric has curvature -1.

There are important relationships between the differential operators and the two differential metric quantities. Suppose $f : \mathbb{D} \to \Omega$ is a holomorphic universal covering projection. Then (1.1) gives

$$\log \lambda_\Omega(f(z)) + \frac{1}{2}\log f'(z) + \frac{1}{2}\log \overline{f'(z)} = \log 2 - \log(1 - z\bar{z}).$$

By applying the operator $\partial/\partial z$ to this identity, we obtain

$$\frac{\partial \log \lambda_\Omega(f(z))}{\partial w} f'(z) + \frac{1}{2}\frac{f''(z)}{f'(z)} = \frac{\bar{z}}{1 - |z|^2}. \tag{2.4}$$

This yields the first relationship

$$\frac{\Gamma_\Omega(f(z))}{\lambda_\Omega(f(z))} = -\frac{1}{2}\frac{|f'(z)|}{f'(z)}\frac{D_2 f(z)}{D_1 f(z)}. \tag{2.5}$$

Next, we relate the Schwarzian of the hyperbolic metric to the differential operators for holomorphic functions. From (2.4), we have

$$\Gamma_\Omega(f(z)) f'(z) = -\frac{f''(z)}{f'(z)} + \frac{2\bar{z}}{1 - z\bar{z}}. \tag{2.6}$$

If we apply the operator $\partial/\partial z$ to this identity, we obtain

$$\frac{\partial \Gamma_\Omega(f(z))}{\partial w} f'(z)^2 + \Gamma_\Omega(f(z)) f''(z) = -\frac{f'''(z)}{f'(z)} + \left(\frac{f''(z)}{f'(z)}\right)^2 + \frac{2\bar{z}^2}{(1 - |z|^2)^2}.$$

By using (2.6), we find

$$\frac{\partial \Gamma_\Omega(f(z))}{\partial w} f'(z)^2 = -\frac{f'''(z)}{f'(z)} + 2\left(\frac{f''(z)}{f'(z)}\right)^2 - \frac{2\bar{z}}{1 - |z|^2}\frac{f''(z)}{f'(z)} + \frac{2\bar{z}^2}{(1 - |z|^2)^2}.$$

From (2.6) we obtain

$$\frac{1}{2}\Gamma_\Omega^2(f(z)) f'(z)^2 = \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 - \frac{2\bar{z}}{1 - |z|^2}\frac{f''(z)}{f'(z)} + \frac{2\bar{z}^2}{(1 - |z|^2)^2},$$

and so

$$S_\Omega(f(z)) f'(z)^2 = \frac{\partial \Gamma_\Omega(f(z))}{\partial w} f'(z)^2 - \frac{1}{2}\Gamma_\Omega^2(f(z)) f'(z)^2 = -S_f(z).$$

Consequently,

$$\frac{S_\Omega(f(z))}{\lambda^2_\Omega(f(z))} = -\frac{1}{4}\left(\frac{|f'(z)|}{f'(z)}\right)^2 (1-|z|^2)^2 S_f(z). \tag{2.7}$$

We introduce notation for standard conformal maps of the unit disk onto strips and sectors. For $\beta \in [0,1]$ define $F_\beta : \mathbb{D} \to \mathbb{C}$ by

$$F_\beta(z) = \begin{cases} \frac{1}{2\beta}\left(\left(\frac{1+z}{1-z}\right)^\beta - 1\right) & \text{if } 0 < \beta \le 1, \\ \frac{1}{2}\log\frac{1+z}{1-z} & \text{if } \beta = 0. \end{cases}$$

Then F_β is a normalized ($F_\beta(0) = 0$ and $F'_\beta(0) = 1$) convex univalent function. For $0 < \beta \le 1$, the image $F_\beta(\mathbb{D})$ is a sector with angular opening $\beta\pi$ that is symmetric about the real axis. When $\beta = 1$, $F_1(\mathbb{D})$ is a half-plane. For $0 < \beta < 1$, the real axis is the centerline of symmetry for the sector. In case $\beta = 1$, any horizontal line is considered a centerline of symmetry. For $\beta = 0$, $F_0(\mathbb{D})$ is a horizontal strip of width $\pi/2$ which is symmetric about the real axis. The centerline of the strip is a line of symmetry; we ignore the vertical line segments about which the strip is symmetric. For any sector with angular opening $\beta\pi$, there exist constants A and B so that $AF_\beta + B$ maps $\mathbb{D}$ conformally onto the sector. The general conformal map of $\mathbb{D}$ onto the sector is $A(F_\beta \circ T) + B$, where T is an arbitrary conformal automorphism of the unit disk. Similarly, the general conformal map of $\mathbb{D}$ onto a strip has the form $A(F_0 \circ T) + B$. Any strip or convex sector, except a half-plane, has a unique centerline of symmetry. For a half-plane, any line perpendicular to the edge of the half-plane is considered a centerline of symmetry.

3. Characterizations of convex regions

In this section, we derive several characterizations of convex regions and convex univalent functions that are used in later sections.

Theorem 3.1. *Suppose f is holomorphic and locally univalent on $\mathbb{D}$. Then each of the following is equivalent to f being convex univalent.*
(a) *The invariant form of Trimble's inequality*

$$(1-|z|^2)^2|S_f(z)| + \frac{1}{2}\left|\frac{D_2 f(z)}{D_1 f(z)}\right|^2 \le 2 \tag{3.1}$$

holds for all z in $\mathbb{D}$.
(b) *For all z in $\mathbb{D}$,*

$$\left|(1-|z|^2)^2 S_f(z) + \frac{1}{2}\left(\frac{D_2 f(z)}{D_1 f(z)}\right)^2\right| \le 2. \tag{3.2}$$

(c) *For all z in $\mathbb{D}$ and any $\alpha \geq 0$,*

$$(3.3)\qquad (\alpha+1)\left|\frac{D_2 f(z)}{D_1 f(z)}\right|^2 + \left|2(1-|z|^2)^2 S_f(z) + \alpha\left(\frac{D_2 f(z)}{D_1 f(z)}\right)^2\right| \leq 4(2\alpha+1).$$

(d) *For all z in $\mathbb{D}$ and any $\alpha \geq 0$,*

$$(3.4)\qquad (\alpha+1)\left|\frac{D_2 f(z)}{D_1 f(z)}\right|^2 + \left|2(1-|z|^2)^2 S_f(z) - \alpha\left(\frac{D_2 f(z)}{D_1 f(z)}\right)^2\right| \leq 4(2\alpha+1).$$

Moreover, for a convex univalent function strict inequalities hold unless f maps onto a sector or strip. For a conformal mapping onto a half-plane these inequalities become identities, while for a conformal mapping onto a strip or a sector with angular opening $\beta\pi$, where $0 < \beta < 1$, equality holds if and only if the point z lies on the hyperbolic geodesic in $\mathbb{D}$ that maps onto the centerline.

Proof. The equivalence of f being convex univalent and (3.1) is well known (see [2], [9]). We will prove that (3.1) implies (3.2), (3.3), and (3.4). Then we will show that f is convex univalent if one of (3.2), (3.3) and (3.4) holds. Finally, we will deal with the sharpness of the inequalities.

If (3.1) holds, then f is a convex univalent function and

$$(3.5)\qquad \left|\frac{D_2 f(z)}{D_1 f(z)}\right| \leq 2.$$

It is well known that (3.5) characterizes convexity of f [7]. By using the triangle inequality, (3.2), (3.3) and (3.4) follow from (3.1) and (3.5).

Next, we show that each of (3.2), (3.3) and (3.4) implies (3.5). Because all inequalities are invariant under Koebe transformations, it suffices to show

$$(3.6)\qquad \left|\frac{f''(0)}{f'(0)}\right| \leq 2$$

in order to conclude that f is convex univalent. In all cases, we will study the behavior of the function

$$(3.7)\qquad H_\theta(s) = e^{i\theta}\frac{D_2 f(z(s))}{D_1 f(z(s))} = e^{i\theta}\left((1-z(s)\bar{z}(s))\frac{f''(z(s))}{f'(z(s))} - 2\bar{z}(s)\right),$$

where θ is a fixed real number and $z(s) = \tanh(s/2)e^{i\theta}$, $s \geq 0$, is a hyperbolic arclength parametrization of the hyperbolic geodesic ray $[0, e^{i\theta})$; this means

$z'(s) = \frac{1}{2}(1-|z(s)|^2)e^{i\theta}$. Now,

$$\begin{aligned}\frac{d}{ds}\frac{D_2 f(z(s))}{D_1 f(z(s))} &= (1 - z(s)\bar{z}(s))\left(\frac{f'''(z(s))}{f'(z(s))} - \left(\frac{f''(z(s))}{f'(z(s))}\right)^2\right) z'(s) \\ &\quad - (z'(s)\bar{z}(s) + z(s)\bar{z}'(s))\,\frac{f''(z(s))}{f'(z(s))} - 2\bar{z}'(s) \\ &= \frac{e^{i\theta}}{2}(1-|z(s)|^2)^2\left(S_f(z(s)) + \frac{1}{2}\left(\frac{f''(z(s))}{f'(z(s))}\right)^2\right) \\ &\quad - |z(s)|(1-|z(s)|^2)\frac{f''(z(s))}{f'(z(s))} - (1-|z(s)|^2)e^{-i\theta}.\end{aligned}$$

As

$$\left(\frac{D_2 f(z)}{D_1 f(z)}\right)^2 = (1-|z|^2)^2\left(\frac{f''(z)}{f'(z)}\right)^2 - 4\bar{z}(1-|z|^2)\frac{f''(z)}{f'(z)} + 4\bar{z}^2,$$

we obtain

$$\frac{d}{ds}\left(e^{i\theta}\frac{D_2 f(z(s))}{D_1 f(z(s))}\right) = \frac{e^{2i\theta}}{2}\left((1-|z(s)|^2)^2 S_f(z(s)) + \frac{1}{2}\left(\frac{D_2 f(z(s))}{D_1 f(z(s))}\right)^2\right) - 1.$$

Thus,

$$H'_\theta(s) = \frac{1}{2}e^{i2\theta}(1-|z(s)|^2)^2 S_f(z(s)) + \frac{1}{4}H_\theta^2(s) - 1. \tag{3.8}$$

Now, we show that (3.2) implies (3.6). Suppose (3.6) were not valid. Then by considering a rotation of f if necessary, we may assume

$$A = \frac{f''(0)}{f'(0)} < -2.$$

By using (3.8), condition (3.2) can be rewritten as $|1 + H'_\theta(s)| \le 1$. This implies

$$\frac{d}{ds}\mathrm{Re}\,\{H_\theta(s)\} \le 0.$$

Consequently, for all real θ the function $\mathrm{Re}\,\{H_\theta(s)\}$ is weakly decreasing for $s \ge 0$ and $\lim_{s\to+\infty} \mathrm{Re}\,\{H_\theta(s)\}$ exists, either as a real number or as $-\infty$, for all real θ. Also,

$$\mathrm{Re}\,\{H_\theta(s)\} \le \mathrm{Re}\,\{H_\theta(0)\} = \mathrm{Re}\,(e^{i\theta}A) = A\cos\theta.$$

Because $A < -2$, there is θ_0 in $(0, \pi/2)$ with $\cos\theta_0 = -2/A$. Then for $|\theta| < \theta_0$, $A\cos\theta < A\cos\theta_0 = -2$ and

$$\mathrm{Re}\left\{(1-r^2)\frac{e^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})} - 2r\right\} \le A\cos\theta < -2,$$

so that

$$\operatorname{Re}\left\{\frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right\} \le \frac{r(A\cos\theta + 2r)}{1-r^2} < \frac{r(-2+2r)}{1-r^2} < 0.$$

The preceding results imply that

$$\lim_{r\to 1^-} \operatorname{Re}\left\{\frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right\} = -\infty$$

and that the function $\operatorname{Re}\{zf''(z)/f'(z)\}$ is bounded above by 0 on the sector

$$z = re^{i\theta},\ 0 \le r < 1,\ |\theta| < \theta_0.$$

Now consider the function $F(z) = \exp(zf''(z)/f'(z))$. This function is holomorphic on $\mathbb{D}$, bounded by 1 on the sector $z = re^{i\theta}$, $0 \le r < 1$, $|\theta| < \theta_0$ and has radial limit 0 for $|\theta| < \theta_0$. Plessner's Theorem asserts that for almost all points ζ on the unit circle either F has a finite angular limit at ζ or else $F(S)$ is dense in $\mathbb{C}$ for every Stolz angle S at ζ [8]. Because F is bounded on the sector, it follows that F has a finite angular limit at almost all points $e^{i\theta}$ with $|\theta| < \theta_0$. Hence, F has angular limit 0 at almost all points $e^{i\theta}$ with $|\theta| < \theta_0$. By Privalov's Uniqueness Theorem [8], F is identically zero, a contradiction.

Next, we verify that (3.3) implies (3.6) by contradiction. Note that when $\alpha = 0$, (3.3) is equivalent to (3.1), so we only need to investigate the case when $\alpha > 0$. We consider $H_\theta(s)$ and assume $A < -2$. From (3.8), the inequality (3.3) becomes

$$(\alpha+1)\left|H_\theta(s)\right|^2 + \left|4 + 4H'_\theta(s) + (\alpha-1)H^2_\theta(s)\right| \le 4(2\alpha+1).$$

This implies that

$$(\alpha+1)\left|H_\theta(s)\right|^2 + 4 + 4\frac{d}{ds}\operatorname{Re}\left\{H_\theta(s)\right\} + (\alpha-1)\operatorname{Re}\left\{H^2_\theta(s)\right\} \le 4(2\alpha+1).$$

Therefore,

$$\frac{d}{ds}\operatorname{Re}\left\{H_\theta(s)\right\} \le \frac{\alpha}{2}\left(4 - \operatorname{Re}^2\left\{H_\theta(s)\right\}\right) - \frac{1}{2}\operatorname{Im}^2\left\{H_\theta(s)\right\} \le \frac{\alpha}{2}\left(4 - \operatorname{Re}^2\left\{H_\theta(s)\right\}\right).$$

Choose θ_0 the same as above. Then for $|\theta| < \theta_0$,

$$\operatorname{Re}\left\{H_\theta(0)\right\} = \operatorname{Re}\left\{Ae^{i\theta}\right\} = A\cos\theta < A\cos\theta_0 = -2.$$

This together with the inequality above imply that $\operatorname{Re}\{H_\theta(s)\}$ is a decreasing function of s when $|\theta| < \theta_0$. Thus we get $\operatorname{Re}\{H_\theta(s)\} < \operatorname{Re}\{H_\theta(0)\} < -2$ and

$$\frac{\frac{d}{ds}\operatorname{Re}\left\{H_\theta(s)\right\}}{\operatorname{Re}^2\left\{H_\theta(s)\right\} - 4} \le -\frac{\alpha}{2}.$$

By integrating this inequality with respect to s, we obtain

$$\frac{1}{4}\log\frac{\text{Re }\{H_\theta(s)\}-2}{\text{Re }\{H_\theta(s)\}+2}-\frac{1}{4}\log\frac{\text{Re }\{H_\theta(0)\}-2}{\text{Re }\{H_\theta(0)\}+2}\le-\frac{\alpha s}{2},$$

or equivalently,

$$\frac{\text{Re }\{H_\theta(s)\}-2}{\text{Re }\{H_\theta(s)\}+2}\le\frac{\text{Re }\{H_\theta(0)\}-2}{\text{Re }\{H_\theta(0)\}+2}\,e^{-2\alpha s}.$$

Note that $\text{Re }\{H_\theta(s)\}<-2$ implies

$$\frac{\text{Re }\{H_\theta(s)\}-2}{\text{Re }\{H_\theta(s)\}+2}>1.$$

By letting $s\to+\infty$, we see that the right-hand side tends to 0, a contraction.

We now prove that (3.4) yields (3.6). If not, then by performing a rotation if necessary, we may assume $A>2$. By using (3.8), the inequality (3.4) becomes

$$(\alpha+1)\,|H_\theta(s)|^2+\left|4+4H_\theta'(s)-(\alpha+1)H_\theta^2(s)\right|\le4(2\alpha+1).$$

This implies that

$$(\alpha+1)\,|H_\theta(s)|^2-4-4\frac{d}{ds}\text{Re }\{H_\theta(s)\}+(\alpha+1)\text{Re }\left(H_\theta^2(s)\right)\le4(2\alpha+1).$$

Therefore,

$$\frac{d}{ds}\text{Re }\{H_\theta(s)\}\ge\frac{\alpha+1}{2}\left(\text{Re}^2\ \{H_\theta(s)\}-4\right).$$

As $A>2$, this time we choose θ_0 in $(0,\pi/2)$ with $\cos\theta_0=2/A$. Then for $|\theta|<\theta_0$,

$$\text{Re }\{H_\theta(0)\}=\text{Re }\left\{Ae^{i\theta}\right\}=A\cos\theta>A\cos\theta_0=2.$$

This together with the inequality above imply that $\text{Re }\{H_\theta(s)\}$ is an increasing function of s when $|\theta|<\theta_0$. Thus we get $\text{Re }\{H_\theta(s)\}>\text{Re }\{H_\theta(0)\}>2$ and

$$\frac{\frac{d}{ds}\text{Re }\{H_\theta(s)\}}{\text{Re}^2\ \{H_\theta(s)\}-4}\ge\frac{\alpha+1}{2}.$$

By integrating this inequality with respect to s, we find

$$\frac{1}{4}\log\frac{\text{Re }\{H_\theta(s)\}-2}{\text{Re }\{H_\theta(s)\}+2}-\frac{1}{4}\log\frac{\text{Re }\{H_\theta(0)\}-2}{\text{Re }\{H_\theta(0)\}+2}\ge\frac{(\alpha+1)s}{2},$$

or equivalently,

$$\frac{\text{Re }\{H_\theta(s)\}-2}{\text{Re }\{H_\theta(s)\}+2}\ge\frac{\text{Re }\{H_\theta(0)\}-2}{\text{Re }\{H_\theta(0)\}+2}e^{2(\alpha+1)s}.$$

The left-hand side is less than or equal to 1 while the right-hand side tends to infinity as $s\to+\infty$, a contraction.

Finally, we discuss the sharpness of these inequalities. From the example at the end of Section 2, it is straightforward to verify that each of the inequalities becomes an equality where as stated if f is a conformal mapping onto a sector or a strip. So we need to show that f maps on to a sector or a strip if equality holds in one of (3.1), (3.2), (3.3) and (3.4).

Clearly, equality must hold in (3.1) at the same point if one of the inequalities becomes an equality. Assume equality holds at $z_0 \in \mathbb{D}$ in (3.1). By performing a Koebe transformation and a rotation if necessary, we may assume that $z_0 = 0$, $f(0) = f'(0) - 1 = 0$ and $f''(0) \geq 0$. Under our assumption, $f(z) = z + az^2 + a_3 z^3 + \cdots$ with $0 \leq a \leq 1$ and

$$6\left|a_3 - a^2\right| + 2a^2 = 2.$$

If $a = 1$, then we know that $f(z) = z/(1-z)$ with $f(\mathbb{D})$ a half-plane.

Now we consider the case when $0 \leq a < 1$. Define a holomorphic function $\phi(z)$ in $\mathbb{D}$ with $\phi(0) = 0$ and $|\phi(z)| < 1$ by

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1+\phi(z)}{1-\phi(z)}.$$

Then $\phi(z) = b_1 z + b_2 z^2 + \cdots$, $a = b_1 \geq 0$ and $6(a_3 - a^2) = 2b_2$. Moreover, the above equality is equivalent to $|b_2| = 1 - b_1^2$. If we apply the Schwarz Lemma to the function

$$\frac{\phi(z)/z - b_1}{1 - b_1\phi(z)/z} = \frac{b_2}{1-b_1^2} z + \cdots,$$

we see that

$$\frac{\phi(z)/z - b_1}{1 - b_1\phi(z)/z} = e^{i\theta} z$$

for some $\theta \in \mathbb{R}$. Therefore,

$$\phi(z) = z\frac{e^{i\theta}z + b_1}{1 + b_1 e^{i\theta} z}$$

and

$$\frac{f''(z)}{f'(z)} = \frac{2a + 2e^{i\theta}z}{1 + a(e^{i\theta} - 1)z - e^{i\theta}z^2}.$$

Set $B = \sqrt{1 - a^2\sin^2(\theta/2)} + ia\sin(\theta/2)$. Note that $|B| = 1$. Calculations show that

$$\frac{f''(z)}{f'(z)} = \frac{aB - e^{i\theta/2}}{\sqrt{1 - a^2\sin^2(\theta/2)}}\frac{1}{1 + Be^{i\theta/2}z} + \frac{a\bar{B} + e^{i\theta/2}}{\sqrt{1 - a^2\sin^2(\theta/2)}}\frac{1}{1 - \bar{B}e^{i\theta/2}z}.$$

Integration yields

$$\begin{aligned}\log f'(z) &= \frac{aB - e^{i\theta/2}}{Be^{i\theta/2}\sqrt{1-a^2\sin^2(\theta/2)}} \log\left(1 + Be^{i\theta/2}z\right) \\ &\quad - \frac{a\bar{B} + e^{i\theta/2}}{\bar{B}e^{i\theta/2}\sqrt{1-a^2\sin^2(\theta/2)}} \log\left(1 - \bar{B}e^{i\theta/2}z\right) \\ &= \left(\frac{a\cos(\theta/2)}{\sqrt{1-a^2\sin^2(\theta/2)}} - 1\right) \log\left(1 + Be^{i\theta/2}z\right) \\ &\quad - \left(\frac{a\cos(\theta/2)}{\sqrt{1-a^2\sin^2(\theta/2)}} + 1\right) \log\left(1 - \bar{B}e^{i\theta/2}z\right).\end{aligned}$$

Hence,

$$f'(z) = \frac{\left(1 + Be^{i\theta/2}z\right)^{a\cos(\theta/2)/\sqrt{1-a^2\sin^2(\theta/2)}-1}}{\left(1 - \bar{B}e^{i\theta/2}z\right)^{a\cos(\theta/2)/\sqrt{1-a^2\sin^2(\theta/2)}+1}}.$$

If $a\cos(\theta/2) \neq 0$, then direct integration gives us

$$f(z) = \frac{e^{-i\theta/2}}{2a\cos(\theta/2)} \left(\left(\frac{1 + Be^{i\theta/2}z}{1 - \bar{B}e^{i\theta/2}z}\right)^{a\cos(\theta/2)/\sqrt{1-a^2\sin^2(\theta/2)}} - 1\right),$$

so $f(\mathbb{D})$ is a sector. If $a\cos(\theta/2) = 0$, then

$$f'(z) = \frac{B}{2\sqrt{1-a^2\sin^2(\theta/2)}} \frac{1}{1 + Be^{i\theta/2}z} + \frac{\bar{B}}{2\sqrt{1-a^2\sin^2(\theta/2)}} \frac{1}{1 - \bar{B}e^{i\theta/2}z}$$

and

$$f(z) = \frac{e^{-i\theta/2}}{2\sqrt{1-a^2\sin^2(\theta/2)}} \log \frac{1 + Be^{i\theta/2}z}{1 - \bar{B}e^{i\theta/2}z},$$

so $f(\mathbb{D})$ is a strip. ■

By using (2.5) and (2.7), we can restate Theorem 3.1 as characterizations of convex regions.

Corollary 3.2. *A hyperbolic region Ω is convex if and only if one of the following holds.*
(a) *For all $w \in \Omega$,*

$$|S_\Omega(w)| + \frac{1}{2}|\Gamma_\Omega(w)|^2 \le \frac{1}{2}\lambda_\Omega^2(w).$$

(b) *For all $w \in \Omega$,*

$$\left|S_\Omega(w) - \frac{1}{2}\Gamma_\Omega^2(w)\right| \le \frac{1}{2}\lambda_\Omega^2(w). \tag{3.9}$$

(c) *For all* $w \in \Omega$ *and any* $\alpha \geq 0$,

$$\frac{\alpha+1}{2}|\Gamma_\Omega(w)|^2 + \left|S_\Omega(w) - \frac{\alpha}{2}\Gamma_\Omega^2(w)\right| \leq \left(\alpha + \frac{1}{2}\right)\lambda_\Omega^2(w).$$

(d) *For all* $w \in \Omega$ *and any* $\alpha \geq 0$,

$$\frac{\alpha+1}{2}|\Gamma_\Omega(w)|^2 + \left|S_\Omega(w) + \frac{\alpha}{2}\Gamma_\Omega^2(w)\right| \leq \left(\alpha + \frac{1}{2}\right)\lambda_\Omega^2(w).$$

Moreover, for a convex region strict inequalities hold unless Ω *is a sector or strip. For a half-plane the inequalities become identities, while for a strip or a sector with angular opening* $\beta\pi$, *where* $0 < \beta < 1$, *equality holds if and only if the point* w *lies on the centerline.*

4. Hyperbolic convexity of $\log \lambda_\Omega$

In order to investigate the hyperbolic convexity of $\log \lambda_\Omega$, we calculate its second derivative along a hyperbolic geodesic parameterized by hyperbolic arclength. Suppose γ is a hyperbolic geodesic in Ω and $\gamma : w = w(s)$ is a hyperbolic arclength parametrization of γ, so $w'(s) = e^{i\theta(s)}/\lambda_\Omega(w(s))$, where $e^{i\theta(s)}$ is a Euclidean unit tangent to γ at $w(s)$. Set $v(s) = \log \lambda_\Omega(w(s))$. We calculate the first and second derivatives of v.

$$\begin{aligned} v'(s) &= \frac{\partial \log \lambda_\Omega(w(s))}{\partial w} w'(s) + \frac{\partial \log \lambda_\Omega(w(s))}{\partial \bar{w}} \overline{w'(s)} \\ &= \operatorname{Re}\left\{\frac{\Gamma_\Omega(w(s))e^{i\theta(s)}}{\lambda_\Omega(w(s))}\right\}. \end{aligned}$$

Next, we find the second derivative.

$$\begin{aligned} v''(s) = {} & \operatorname{Re}\left\{\frac{e^{i\theta(s)}}{\lambda_\Omega(w(s))}\left(\frac{\partial \Gamma_\Omega(w(s))}{\partial w}w'(s) + \frac{\partial \Gamma_\Omega(w(s))}{\partial \bar{w}}\overline{w'(s)}\right)\right\} \\ & - \operatorname{Re}\left\{\frac{e^{i\theta(s)}\Gamma_\Omega(w(s))}{\lambda_\Omega^2(w(s))}\left(\frac{\partial \lambda_\Omega(w(s))}{\partial w}w'(s) + \frac{\partial \lambda_\Omega(w(s))}{\partial \bar{w}}\overline{w'(s)}\right)\right\} \\ & + \operatorname{Re}\left\{\frac{\Gamma_\Omega(w(s))ie^{i\theta(s)}\theta'(s)}{\lambda_\Omega(w(s))}\right\} \\ = {} & \frac{1}{\lambda_\Omega^2(w(s))}\operatorname{Re}\left\{e^{2i\theta(s)}\frac{\partial \Gamma_\Omega(w(s))}{\partial w} + \frac{\partial \Gamma_\Omega(w(s))}{\partial \bar{w}}\right\} \\ & - \frac{1}{\lambda_\Omega^2(w(s))}\operatorname{Re}\left\{\frac{1}{2}e^{2i\theta(s)}\Gamma_\Omega^2(w(s)) + \frac{1}{2}|\Gamma_\Omega(w(s))|^2\right\} \\ & - \frac{\theta'(s)}{\lambda_\Omega(w(s))}\operatorname{Im}\left\{\Gamma_\Omega(w(s))e^{i\theta(s)}\right\}. \end{aligned}$$

Because γ is a hyperbolic geodesic arc, its hyperbolic curvature vanishes; that is, $\kappa_\Omega(w(s),\gamma)=0$, where

$$\kappa_\Omega(w(s),\gamma)=\frac{\kappa_e(w(s),\gamma)+\mathrm{Im}\left\{\Gamma_\Omega(w(s))e^{i\theta(s)}\right\}}{\lambda_\Omega(w(s))} \tag{4.1}$$

and

$$\kappa_e(w(s),\gamma)=\frac{1}{|w'(s)|}\theta'(s)=\lambda_\Omega(w(s))\theta'(s)$$

denotes the Euclidean curvature. Therefore,

$$\theta'(s)=\frac{\kappa_e(w(s)),\gamma)}{\lambda_\Omega(w(s))}=-\frac{\mathrm{Im}\left\{\Gamma_\Omega(w(s))e^{i\theta(s)}\right\}}{\lambda_\Omega(w(s))} \tag{4.2}$$

and so

$$\begin{aligned}v''(s)&=\frac{1}{\lambda_\Omega^2(w(s))}\mathrm{Re}\left\{e^{2i\theta(s)}\left(\frac{\partial\Gamma_\Omega(w(s))}{\partial w}-\frac{1}{2}\Gamma_\Omega^2(w(s))\right)\right\}\\&\quad+\frac{1}{\lambda_\Omega^2(w(s))}\mathrm{Re}\left\{\frac{\partial\Gamma_\Omega(w(s))}{\partial\bar{w}}-\frac{1}{2}|\Gamma_\Omega(w(s))|^2\right\}\\&\quad+\frac{1}{\lambda_\Omega^2(w(s))}\mathrm{Im}^2\left\{\Gamma_\Omega(w(s))e^{i\theta(s)}\right\}.\end{aligned}$$

From (2.2) and (2.3), we obtain

$$\begin{aligned}&v''(s)\\&=\frac{1}{\lambda_\Omega^2(w(s))}\left(\mathrm{Re}\left\{e^{2i\theta(s)}S_\Omega(w(s))\right\}-\frac{1}{2}|\Gamma_\Omega(w(s))|^2+\mathrm{Im}^2\left\{\Gamma_\Omega(w(s))e^{i\theta(s)}\right\}\right)+\frac{1}{2}.\end{aligned}$$

By using $-(1/2)|z|^2+\mathrm{Im}^2\{z\}=-(1/2)\mathrm{Re}\{z^2\}$, we find

$$-\frac{1}{2}|\Gamma_\Omega(w(s))|^2+\mathrm{Im}^2\left\{\Gamma_\Omega(w(s))e^{i\theta(s)}\right\}=-\frac{1}{2}\mathrm{Re}\left\{e^{2i\theta(s)}\Gamma_\Omega^2(w(s))\right\}.$$

Hence,

$$v''(s)=\frac{1}{\lambda_\Omega^2(w(s))}\mathrm{Re}\left\{e^{2i\theta(s)}\left(S_\Omega(w(s))-\frac{1}{2}\Gamma_\Omega^2(w(s))\right)\right\}+\frac{1}{2}. \tag{4.3}$$

Theorem 4.1. *A hyperbolic region Ω is convex if and only if $\log\lambda_\Omega(w)$ is hyperbolically convex. Moreover, $\log\lambda_\Omega$ is strictly hyperbolically convex on any convex region other than a strip or sector. For the exceptional cases it is strictly convex along all hyperbolic geodesics except along centerlines of symmetry.*

Proof. (3.9) provides a characterization of convex regions. From (4.3), we see that $v''(s)\geq 0$ if and only if

$$-\mathrm{Re}\left\{e^{2i\theta(s)}\left(S_\Omega(w(s))-\frac{1}{2}\Gamma_\Omega^2(w(s))\right)\right\}\leq\frac{1}{2}\lambda_\Omega^2(w(s)).$$

This holds for all unit vectors $e^{i\theta(s)}$ at $w(s)$ if and only if (3.9) holds. Thus, Ω is convex if and only if $\log \lambda_\Omega(w)$ is hyperbolically convex. The strict convexity result follows from the case of strict inequality in (3.9) that was asserted in Corollary 3.2. ■

Example 4.2. The density of the hyperbolic metric on the upper half-plane $\mathbb{H}$ is $\lambda_{\mathbb{H}}(w) = 1/\text{Im}\{w\}$. The function $w(s) = u + ie^{-s}$, $s \in \mathbb{R}$, is a hyperbolic arclength parametrization of the intersection of the vertical line $\text{Re}\{w\} = u$ with $\mathbb{H}$; this is a hyperbolic geodesic in $\mathbb{H}$. Then $v(s) = \log \lambda_{\mathbb{H}}(w(s)) = \log e^s = s$ and $v''(s) = 0$.

It is elementary that if $\log \lambda_\Omega$ is hyperbolically convex, then so is λ_Ω^α for all $\alpha > 0$. Likewise, if λ_Ω^α is hyperbolically concave for some $\alpha < 0$, then it is straightforward to verify that $\log(1/\lambda_\Omega)$ is hyperbolically concave, or equivalently, $\log \lambda_\Omega$ is hyperbolically convex. Thus, to show Theorem 4.1 is best possible we show that λ_Ω^α is not hyperbolically concave on any hyperbolic region for any $\alpha < 0$. We employ formula (6.1) that is valid for $\alpha < 0$. In this case, $v_\alpha''(s) \le 0$ along all hyperbolic geodesics in Ω if and only if

$$\frac{1}{\lambda_\Omega^2(w(s))}\left|S_\Omega(w(s)) + \frac{\alpha-1}{2}\Gamma_\Omega^2(w(s))\right| \le \frac{\alpha}{2}\frac{|\Gamma_\Omega(w(s))|^2}{\lambda_\Omega^2(w(s))} + \frac{1}{2}.$$

This implies the inequality in Corollary 3.2(b). Hence, Ω must be convex. A conformal map $f : \mathbb{D} \to \Omega$ is a convex univalent function. By using (2.5) and (2.7), the inequality above is equivalent to

$$\left|(1-|z|^2)^2 S_f(z) + \frac{1-\alpha}{2}\left(\frac{D_2 f(z)}{D_1 f(z)}\right)^2\right| \le \frac{\alpha}{2}\left|\frac{D_2 f(z)}{D_1 f(z)}\right|^2 + 2.$$

Now we consider $H_\theta(s)$ defined in (3.7). From (3.8), we have

$$\left|2 + 2H_\theta'(s) - \frac{\alpha}{2}H_\theta^2(s)\right| \le \frac{\alpha}{2}\,|H_\theta(s)|^2 + 2.$$

This yields

$$\frac{d}{ds}\text{Re}\,\{H_\theta(s)\} \le \frac{\alpha}{2}\text{Re}^2\,\{H_\theta(s)\} \le 0.$$

As

$$\sup\left\{\left|\frac{D_2 f(z)}{D_1 f(z)}\right| : z \in \mathbb{D}\right\} = 2$$

for a convex function f [7], we may assume $f(0) = 0$, $f'(0) = 1$ and $f''(0) < 0$ by using a suitable Koebe transformation and a rotation if necessary. Then for $|\theta| < \frac{\pi}{2}$, Re $\{H_\theta(s)\} \le$ Re $\{H_\theta(0)\} = f''(0)\cos\theta/f'(0) < 0$. Moreover, by integrating

$$\frac{\frac{d}{ds}\mathrm{Re}\,\{H_\theta(s)\}}{\mathrm{Re}^2\,\{H_\theta(s)\}} \le \frac{\alpha}{2}$$

with respect to s, we obtain

$$\frac{1}{\mathrm{Re}\,\{H_\theta(0)\}} - \frac{1}{\mathrm{Re}\,\{H_\theta(s)\}} \le \frac{\alpha s}{2}.$$

Since Re $\{H_\theta(s)\} < 0$ when $|\theta| < \frac{\pi}{2}$, we have $1/\mathrm{Re}\,\{H_\theta(0)\} < \alpha s/2$. By letting $s \to +\infty$, we get a contradiction as $\alpha < 0$.

5. Euclidean geodesics in hyperbolic regions

We begin by investigating hyperbolic curvature of Euclidean line segments.

Theorem 5.1. *Let Ω be a hyperbolic region. Ω is convex if and only if every Euclidean segment in Ω has absolute hyperbolic curvature at most 1.*

Proof. First, by using (2.5) and (3.5), we see that Ω is convex if and only if $|\Gamma_\Omega(w)| \le \lambda_\Omega(w)$ on Ω. Let γ be a Euclidean segment in Ω parameterized by hyperbolic arclength, say $w = w(s)$, where $w'(s) = e^{i\theta}/\lambda_\Omega(w(s))$. Note that θ is independent of s since γ is a Euclidean segment. As $\kappa_e(w(s),\gamma) = 0$, (4.1) gives

$$\kappa_\Omega(w(s),\gamma) = \mathrm{Im}\left\{\frac{e^{i\theta}\Gamma_\Omega(w(s))}{\lambda_\Omega(w(s))}\right\}. \tag{5.1}$$

Hence,

$$|\kappa_\Omega(w(s),\gamma)| \le \frac{|\Gamma_\Omega(w(s))|}{\lambda_\Omega(w(s))}.$$

If Ω is convex, then $|\Gamma_\Omega(w)| \le \lambda_\Omega(w)$, and so $|\kappa_\Omega(w(s),\gamma)| \le 1$ for any Euclidean segment γ in Ω.

Conversely, assume that any Euclidean segment in Ω has absolute hyperbolic curvature at most 1. Given w_0 in Ω, choose a Euclidean segment γ through w_0, say $w_0 = w(0)$, which is parallel to $e^{i\theta}$ at w_0, where this unit vector is selected so that

$$\kappa_\Omega(w(0),\gamma) = \frac{|\Gamma_\Omega(w(0))|}{\lambda_\Omega(w(0))}.$$

Then $|\kappa_\Omega(w(s),\gamma)| \le 1$ produces $|\Gamma_\Omega(w(0))| \le \lambda_\Omega(w(0))$. Since $w_0 = w(0)$ is arbitrary in Ω, we conclude that Ω is convex. ■

In a uniformly perfect region, a hyperbolic geodesic arc is not too far from being a Euclidean geodesic. The next result shows that Euclidean geodesic arcs in a uniformly perfect region are nearly hyperbolic geodesic arcs in the sense that the absolute hyperbolic curvature is bounded by a fixed constant.

Theorem 5.2. *Let Ω be a hyperbolic region. Ω is uniformly perfect if and only if there is a finite constant $K \geq 1$ such that every Euclidean segment in Ω has absolute hyperbolic curvature at most K.*

Proof. The proof of Theorem 5.1 shows that

$$K = \sup\left\{\frac{|\Gamma_\Omega(w)|}{\lambda_\Omega(w)} : w \in \Omega\right\},$$

which is finite if and only if Ω is uniformly perfect [6]. ■

Not only is the hyperbolic curvature of Euclidean segments uniformly bounded in uniformly perfect regions, the rate of change of the hyperbolic curvature of Euclidean segments is uniformly bounded.

Theorem 5.3. *Let Ω be a hyperbolic region. Ω is uniformly perfect if and only if there is a finite constant $B \geq 0$ such that every Euclidean segment in Ω parameterized by hyperbolic arclength has absolute rate of change of hyperbolic curvature at most B.*

Proof. Consider any Euclidean segment γ contained in Ω. Then (5.1) holds, from which we have

$$\begin{aligned}\frac{d}{ds}\kappa_\Omega(w(s),\gamma) &= \mathrm{Im}\left\{\frac{e^{i\theta}}{\lambda_\Omega(w(s))}\left(\frac{\partial\Gamma_\Omega(w(s))}{\partial w}w'(s) + \frac{\partial\Gamma_\Omega(w(s))}{\partial\bar{w}}\overline{w'(s)}\right)\right\}\\ &\quad - \mathrm{Im}\left\{\frac{e^{i\theta}\Gamma_\Omega(w(s))}{\lambda_\Omega^2(w(s))}\left(\frac{\partial\lambda_\Omega(w(s))}{\partial w}w'(s) + \frac{\partial\lambda_\Omega(w(s))}{\partial\bar{w}}\overline{w'(s)}\right)\right\}\\ &= \mathrm{Im}\left\{\frac{e^{2i\theta}}{\lambda_\Omega^2(w(s))}\frac{\partial\Gamma_\Omega(w(s))}{\partial w} + \frac{1}{\lambda_\Omega^2(w(s))}\frac{\partial\Gamma_\Omega(w(s))}{\partial\bar{w}}\right\}\\ &\quad - \frac{1}{2}\mathrm{Im}\left\{\frac{e^{2i\theta}\Gamma_\Omega^2(w(s))}{\lambda_\Omega^2(w(s))} + \frac{|\Gamma_\Omega(w(s))|^2}{\lambda_\Omega^2(w(s))}\right\}.\end{aligned}$$

As $|\Gamma_\Omega(w(s))|/\lambda_\Omega(w(s))$ is real-valued, we obtain from (2.3)

$$\begin{aligned}\frac{d}{ds}\kappa_\Omega(w(s),\gamma) &= \mathrm{Im}\left\{\frac{e^{2i\theta}}{\lambda_\Omega^2(w(s))}\left(\frac{\partial\Gamma_\Omega(w(s))}{\partial w} - \frac{1}{2}\Gamma_\Omega^2(w(s))\right)\right\}\\ &= \mathrm{Im}\left\{\frac{e^{2i\theta}S_\Omega(w(s))}{\lambda_\Omega^2(w(s))}\right\}.\end{aligned}$$

Thus

$$\left|\frac{d}{ds}\kappa_\Omega(w(s),\gamma)\right| \le \frac{|S_\Omega(w(s))|}{\lambda_\Omega^2(w(s))}.$$

It is known that Ω is uniformly perfect if and only if [2]

$$\beta(\Omega) = \sup\left\{2\frac{|S_\Omega(w)|}{\lambda_\Omega^2(w)} : w \in \Omega\right\} < \infty. \tag{5.2}$$

Therefore,

$$\left|\frac{d}{ds}\kappa_\Omega(w(s),\gamma)\right| \le \frac{\beta(\Omega)}{2}.$$

Conversely, one can show that $\beta(\Omega) \le 2B$, so in fact, $\beta(\Omega) = 2B$. ■

Corollary 5.4. *A hyperbolic region Ω in $\mathbb{C}$ is Nehari if and only if every Euclidean segment γ in Ω parameterized by hyperbolic arclength satisfies*

$$\left|\frac{d}{ds}\kappa_\Omega(w(s),\gamma)\right| \le \frac{1}{2}.$$

Next, we turn to the question of determining when $\log\lambda_\Omega$ is hyperbolically convex along Euclidean geodesics. Suppose γ is a Euclidean line segment in Ω and $\gamma : w = w(s)$ is a hyperbolic arclength parametrization of γ. We calculate the first and second derivatives of $v(s) = \log\lambda_\Omega(w(s))$.

$$\begin{aligned} v'(s) &= \frac{\partial \log\lambda_\Omega(w(s))}{\partial w} w'(s) + \frac{\partial \log\lambda_\Omega(w(s))}{\partial \bar{w}} \overline{w'(s)} \\ &= \operatorname{Re}\left\{e^{i\theta}\frac{\Gamma_\Omega(w(s))}{\lambda_\Omega(w(s))}\right\}. \end{aligned}$$

The second derivative is

$$\begin{aligned} v''(s) &= \operatorname{Re}\left\{\frac{e^{i\theta}}{\lambda_\Omega(w(s))}\left(\frac{\partial\Gamma_\Omega(w(s))}{\partial w}w'(s) + \frac{\partial\Gamma_\Omega(w(s))}{\partial\bar{w}}\overline{w'(s)}\right)\right\} \\ &\quad - \operatorname{Re}\left\{\frac{e^{i\theta}\Gamma_\Omega(w(s))}{\lambda_\Omega^2(w(s))}\left(\frac{\partial\lambda_\Omega(w(s))}{\partial w}w'(s) + \frac{\partial\lambda_\Omega(w(s))}{\partial\bar{w}}\overline{w'(s)}\right)\right\} \\ &= \frac{1}{\lambda_\Omega^2(w(s))}\operatorname{Re}\left\{e^{2i\theta}\frac{\partial\Gamma_\Omega(w(s))}{\partial w} + \frac{\partial\Gamma_\Omega(w(s))}{\partial\bar{w}}\right\} \\ &\quad - \frac{1}{2\lambda_\Omega^2(w(s))}\operatorname{Re}\left\{e^{2i\theta}\Gamma_\Omega^2(w(s)) + |\Gamma_\Omega(w(s))|^2\right\}. \end{aligned}$$

From (2.2) and (2.3), we have

$$v''(s) = \frac{1}{\lambda_\Omega^2(w(s))}\left(\operatorname{Re}\left\{e^{2i\theta}S_\Omega(w(s))\right\} - \frac{1}{2}|\Gamma_\Omega(w(s))|^2\right) + \frac{1}{2}. \tag{5.3}$$

Theorem 5.5. *Let Ω be a hyperbolic region.* $\log \lambda_\Omega(w)$ *is hyperbolically convex on Euclidean segments if and only if*

$$|S_\Omega(w)| + \frac{1}{2}|\Gamma_\Omega(w)|^2 \le \frac{1}{2}\lambda_\Omega^2(w), \tag{5.4}$$

that is, if and only if Ω is convex.

Proof. From (5.3), we see that $v''(s) \ge 0$ if and only if

$$-\mathrm{Re}\left\{e^{2i\theta} S_\Omega(w(s))\right\} + \frac{1}{2}\Gamma_\Omega^2(w(s)) \le \frac{1}{2}\lambda_\Omega^2(w(s)).$$

This holds for all unit vectors $e^{i\theta}$ at $w(s)$ if and only if (5.4) holds, which characterizes convex regions by Corollary 3.2. ■

Example 5.6. From Example 4.2, we see that for the upper half-plane $\mathbb{H}$, $v''(s) = 0$ along vertical half-lines, so Theorem 5.5 is sharp.

Next, we consider the behavior of $\lambda_\Omega^{-\alpha}$ along Euclidean segments. Let $v_\alpha(s) = \lambda_\Omega^{-\alpha}(w(s))$, where γ is a Euclidean line segment in Ω and $\gamma : w = w(s)$ is a hyperbolic arclength parametrization of γ. We calculate the first and second derivatives of v_α.

$$\begin{aligned} v_\alpha'(s) &= -\alpha\lambda_\Omega^{-\alpha-1}(w(s))\left(\frac{\partial\lambda_\Omega(w(s))}{\partial w}w'(s) + \frac{\partial\lambda_\Omega(w(s))}{\partial \bar{w}}\overline{w'(s)}\right) \\ &= -\alpha v_\alpha(s)\mathrm{Re}\left\{\frac{e^{i\theta}\Gamma_\Omega(w(s))}{\lambda_\Omega(w(s))}\right\}. \end{aligned}$$

Now, we calculate the second derivative.

$$\begin{aligned} v_\alpha''(s) &= -\alpha v_\alpha'(s)\mathrm{Re}\left\{\frac{e^{i\theta}\Gamma_\Omega(w(s))}{\lambda_\Omega(w(s))}\right\} \\ &\quad - \alpha v_\alpha(s)\mathrm{Re}\left\{\frac{e^{i\theta}}{\lambda_\Omega(w(s))}\left(\frac{\partial\Gamma_\Omega(w(s))}{\partial w}w'(s) + \frac{\partial\Gamma_\Omega(w(s))}{\partial\bar{w}}\overline{w'(s)}\right)\right\} \\ &\quad + \alpha v_\alpha(s)\mathrm{Re}\left\{\frac{e^{i\theta}\Gamma_\Omega(w(s))}{\lambda_\Omega^2(w(s))}\left(\frac{\partial\lambda_\Omega(w(s))}{\partial w}w'(s) + \frac{\partial\lambda_\Omega(w(s))}{\partial\bar{w}}\overline{w'(s)}\right)\right\} \\ &= \alpha^2 v_\alpha(s)\mathrm{Re}^2\left\{\frac{e^{i\theta}\Gamma_\Omega(w(s))}{\lambda_\Omega(w(s))}\right\} \\ &\quad - \alpha v_\alpha(s)\mathrm{Re}\left\{\frac{e^{2i\theta}}{\lambda_\Omega^2(w(s))}\left(\frac{\partial\Gamma_\Omega(w(s))}{\partial w} - \frac{1}{2}\Gamma_\Omega^2(w(s))\right)\right\} \\ &\quad - \alpha v_\alpha(s)\mathrm{Re}\left\{\frac{1}{\lambda_\Omega^2(w(s))}\left(\frac{\partial\Gamma_\Omega(w(s))}{\partial\bar{w}} - \frac{1}{2}|\Gamma_\Omega(w(s))|^2\right)\right\}. \end{aligned}$$

From (2.2) and (2.3), we have

$$v_\alpha''(s) = \alpha^2 v_\alpha(s)\mathrm{Re}^2\left\{\frac{e^{i\theta}\Gamma_\Omega(w(s))}{\lambda_\Omega(w(s))}\right\} - \alpha v_\alpha(s)\mathrm{Re}\left\{\frac{e^{2i\theta}}{\lambda_\Omega^2(w(s))}S_\Omega(w(s))\right\}$$
$$-\frac{\alpha}{2}v_\alpha(s) + \frac{\alpha}{2}v_\alpha(s)\frac{|\Gamma_\Omega(w(s))|^2}{\lambda_\Omega^2(w(s))}.$$

By using $\mathrm{Re}^2 z := (\mathrm{Re}\, z)^2 = \frac{1}{2}|z|^2 + \frac{1}{2}\mathrm{Re}\{z^2\}$, we get

(5.5)

$$v_\alpha''(s) =$$
$$\alpha v_\alpha(s)\left[\frac{\alpha+1}{2}\frac{|\Gamma_\Omega(w(s))|^2}{\lambda_\Omega^2(w(s))} - \mathrm{Re}\left\{\frac{e^{2i\theta}}{\lambda_\Omega^2(w(s))}\Big(S_\Omega(w(s)) - \frac{\alpha}{2}\Gamma_\Omega^2(w(s))\Big)\right\} - \frac{1}{2}\right].$$

Theorem 5.7. $v_\alpha(s) = \lambda_\Omega^{-\alpha}(w(s))$, $\alpha > 0$, *satisfies* $v_\alpha''(s) \le \alpha^2 v_\alpha(s)$ *along all Euclidean segments in* Ω *if and only if* Ω *is convex.*

Proof. From (5.5), we see that $v_\alpha''(s) \le \alpha^2 v_\alpha(s)$ along all Euclidean segments in Ω if and only if

$$\frac{\alpha+1}{2}|\Gamma_\Omega(w(s))|^2 - \mathrm{Re}\left\{e^{2i\theta}\Big(S_\Omega(w(s)) - \frac{\alpha}{2}\Gamma_\Omega^2(w(s))\Big)\right\} \le \left(\alpha+\frac{1}{2}\right)\lambda_\Omega^2(w(s)),$$

which is equivalent to the inequality in Corollary 3.2(c) because θ is arbitrary. Corollary 3.2 implies the equivalence with the convexity of Ω. ∎

The differential inequality $v'' \le 4v$ (similar to those in Theorem 5.7) was utilized in the characterization of hyperbolically convex regions by Kim and Sugawa [4].

6. Hyperbolic convexity of λ_Ω^α

In order to study the hyperbolic convexity of λ_Ω^α, we perform similar calculations for the function $v_\alpha(s) = \lambda_\Omega^\alpha(w(s))$, where $w(s)$ is a hyperbolic arclength parametrization of a hyperbolic geodesic.

$$v_\alpha'(s) = \alpha\lambda_\Omega^{\alpha-1}(w(s))\left(\frac{\partial\lambda_\Omega(w(s))}{\partial w}w'(s) + \frac{\partial\lambda_\Omega(w(s))}{\partial\bar{w}}\overline{w'(s)}\right)$$
$$= \alpha v_\alpha(s)\mathrm{Re}\left\{\frac{\Gamma_\Omega(w(s))e^{i\theta(s)}}{\lambda_\Omega(w(s))}\right\}.$$

Now, we calculate the second derivative.

$$\begin{aligned}
v''_\alpha(s) &= \alpha v'_\alpha(s)\mathrm{Re}\left\{\frac{\Gamma_\Omega(w(s))e^{i\theta(s)}}{\lambda_\Omega(w(s))}\right\} + \alpha v_\alpha(s)\mathrm{Re}\left\{\frac{\Gamma_\Omega(w(s))ie^{i\theta(s)}\theta'(s)}{\lambda_\Omega(w(s))}\right\}\\
&\quad + \alpha v_\alpha(s)\mathrm{Re}\left\{\frac{e^{i\theta(s)}}{\lambda_\Omega(w(s))}\left(\frac{\partial\Gamma_\Omega(w(s))}{\partial w}w'(s) + \frac{\partial\Gamma_\Omega(w(s))}{\partial\bar{w}}\overline{w'(s)}\right)\right\}\\
&\quad - \alpha v_\alpha(s)\mathrm{Re}\left\{\frac{\Gamma_\Omega(w(s))e^{i\theta(s)}}{\lambda^2_\Omega(w(s))}\left(\frac{\partial\lambda_\Omega(w(s))}{\partial w}w'(s) + \frac{\partial\lambda_\Omega(w(s))}{\partial\bar{w}}\overline{w'(s)}\right)\right\}\\
&= \alpha^2 v_\alpha(s)\mathrm{Re}^2\left\{\frac{\Gamma_\Omega(w(s))e^{i\theta(s)}}{\lambda_\Omega(w(s))}\right\} - \alpha v_\alpha(s)\theta'(s)\mathrm{Im}\left\{\frac{\Gamma_\Omega(w(s))e^{i\theta(s)}}{\lambda_\Omega(w(s))}\right\}\\
&\quad + \alpha v_\alpha(s)\mathrm{Re}\left\{\frac{1}{\lambda^2_\Omega(w(s))}\left(e^{2i\theta(s)}\frac{\partial\Gamma_\Omega(w(s))}{\partial w} + \frac{\partial\Gamma_\Omega(w(s))}{\partial\bar{w}}\right)\right\}\\
&\quad - \frac{\alpha v_\alpha(s)}{2}\mathrm{Re}\left\{\frac{1}{\lambda^2_\Omega(w(s))}\left(e^{2i\theta(s)}\Gamma^2_\Omega(w(s)) + |\Gamma_\Omega(w(s))|^2\right)\right\}.
\end{aligned}$$

By using (4.2), (2.2) and (2.3), we obtain

$$\begin{aligned}
v''_\alpha(s) &= \alpha^2 v_\alpha(s)\mathrm{Re}^2\left\{\frac{\Gamma_\Omega(w(s))e^{i\theta(s)}}{\lambda_\Omega(w(s))}\right\} + \alpha v_\alpha(s))\mathrm{Im}^2\left\{\frac{\Gamma_\Omega(w(s))e^{i\theta(s)}}{\lambda_\Omega(w(s))}\right\}\\
&\quad + \alpha v_\alpha(s)\mathrm{Re}\left\{\frac{e^{2i\theta(s)}}{\lambda^2_\Omega(w(s))}\left(\frac{\partial\Gamma_\Omega(w(s))}{\partial w} - \frac{1}{2}\Gamma^2_\Omega(w(s))\right)\right\}\\
&\quad + \alpha v_\alpha(s)\mathrm{Re}\left\{\frac{1}{\lambda^2_\Omega(w(s))}\left(\frac{\partial\Gamma_\Omega(w(s))}{\partial\bar{w}} - \frac{1}{2}|\Gamma_\Omega(w(s))|^2\right)\right\}\\
&= \alpha^2 v_\alpha(s)\mathrm{Re}^2\left\{\frac{\Gamma_\Omega(w(s))e^{i\theta(s)}}{\lambda_\Omega(w(s))}\right\} + \alpha v_\alpha(s)\mathrm{Re}\left\{\frac{e^{2i\theta(s)}}{\lambda^2_\Omega(w(s))}S_\Omega(w(s))\right\}\\
&\quad + \frac{\alpha}{2}v_\alpha(s) - \frac{\alpha}{2}v_\alpha(s)\frac{|\Gamma_\Omega(w(s))|^2}{\lambda^2_\Omega(w(s))} + \alpha v_\alpha(s)\mathrm{Im}^2\left\{\frac{\Gamma_\Omega(w(s))e^{i\theta(s)}}{\lambda_\Omega(w(s))}\right\},
\end{aligned}$$

so that

$$\begin{aligned}
v''_\alpha(s) &= \alpha v_\alpha(s)\left(\alpha\mathrm{Re}^2\left\{\frac{\Gamma_\Omega(w(s))e^{i\theta(s)}}{\lambda_\Omega(w(s))}\right\} + \mathrm{Im}^2\left\{\frac{\Gamma_\Omega(w(s))e^{i\theta(s)}}{\lambda_\Omega(w(s))}\right\}\right)\\
&\quad + \alpha v_\alpha(s)\mathrm{Re}\left\{\frac{e^{2i\theta(s)}}{\lambda^2_\Omega(w(s))}S_\Omega(w(s))\right\} + \frac{\alpha}{2}v_\alpha(s) - \frac{\alpha}{2}v_\alpha(s)\frac{|\Gamma_\Omega(w(s))|^2}{\lambda^2_\Omega(w(s))}.
\end{aligned}$$

From

$$\alpha\mathrm{Re}^2\, z + \mathrm{Im}^2\, z = \frac{1+\alpha}{2}|z|^2 + \frac{\alpha-1}{2}\mathrm{Re}\,\{z^2\},$$

we get

$$v_\alpha''(s) = \alpha v_\alpha(s)\left(\frac{1+\alpha}{2}\frac{|\Gamma_\Omega(w(s))|^2}{\lambda_\Omega^2(w(s))} + \frac{\alpha-1}{2}\mathrm{Re}\left\{\frac{\Gamma_\Omega^2(w(s))e^{2i\theta(s)}}{\lambda_\Omega^2(w(s))}\right\}\right)$$
$$+\alpha v_\alpha(s)\mathrm{Re}\left\{\frac{e^{2i\theta(s)}}{\lambda_\Omega^2(w(s))}S_\Omega(w(s))\right\} + \frac{\alpha}{2}v_\alpha(s) - \frac{\alpha}{2}v_\alpha(s)\frac{|\Gamma_\Omega(w(s))|^2}{\lambda_\Omega^2(w(s))}.$$

Finally,

(6.1)

$$v_\alpha''(s) =$$
$$\alpha v_\alpha(s)\left(\frac{\alpha}{2}\frac{|\Gamma_\Omega(w(s))|^2}{\lambda_\Omega^2(w(s))} + \mathrm{Re}\left\{\frac{e^{2i\theta(s)}}{\lambda_\Omega^2(w(s))}\left(S_\Omega(w(s)) + \frac{\alpha-1}{2}\Gamma_\Omega^2(w(s))\right)\right\} + \frac{1}{2}\right).$$

Theorem 6.1. *Let Ω be a hyperbolic region. For $\alpha > 0$, λ_Ω^α is hyperbolically convex if and only if*

$$\left|S_\Omega(w(s)) + \frac{\alpha-1}{2}\Gamma_\Omega^2(w(s))\right| \le \frac{\alpha}{2}|\Gamma_\Omega(w(s))|^2 + \frac{1}{2}\lambda_\Omega^2(w(s)). \tag{6.2}$$

In particular, for any $\alpha > 0$, λ_Ω^α is hyperbolically convex if Ω is convex.

Proof. From (6.1), we see that $v_\alpha(s)$ is convex along all hyperbolic geodesics in Ω if and only if

$$\frac{\alpha}{2}\frac{|\Gamma_\Omega(w(s))|^2}{\lambda_\Omega^2(w(s))} + \frac{1}{2} - \frac{1}{\lambda_\Omega^2(w(s))}\left|S_\Omega(w(s)) + \frac{\alpha-1}{2}\Gamma_\Omega^2(w(s))\right| \ge 0,$$

or equivalently (6.2).

If Ω is convex, (3.9) holds by Corollary 3.2. (6.2) then follows from (3.9). ■

Observe that (6.2) becomes (3.9) when $\alpha = 0$. Also, if Ω is a Nehari region in $\mathbb{C}$, we see that (6.2) holds for $\alpha = 1/2$ by using (2.7), so $\lambda_\Omega^{1/2}$ is hyperbolically convex as established by Chuaqui, Osgood and Pommerenke [1].

If γ is a hyperbolic geodesic parameterized by hyperbolic arclength $w = w(s)$ and $v_\alpha(s) = \lambda_\Omega^\alpha(w(s))$, then (6.1) holds. Rather than asking when $v_\alpha(s)$ is hyperbolically convex, we ask when $v_\alpha(s)$ is *α-hyperbolically concave* in the sense that

$$v_\alpha''(s) \le \alpha^2 v_\alpha(s)$$

for some $\alpha \ge 0$. By (6.1), this holds if and only if

$$\frac{\alpha}{2}|\Gamma_\Omega(w(s))|^2 + \mathrm{Re}\left\{e^{2i\theta(s)}\left(S_\Omega(w(s)) + \frac{\alpha-1}{2}\Gamma_\Omega^2(w(s))\right)\right\} \le \left(\alpha - \frac{1}{2}\right)\lambda_\Omega^2(w(s)).$$

Because this holds for all $\theta(s)$, we must have

$$\frac{\alpha}{2}|\Gamma_\Omega(w(s))|^2 + \left|S_\Omega(w(s)) + \frac{\alpha-1}{2}\Gamma_\Omega^2(w(s))\right| \le \left(\alpha - \frac{1}{2}\right)\lambda_\Omega^2(w(s)). \tag{6.3}$$

For $\alpha \ge 1$ and Ω a convex region, Corollary 3.2(d) implies that $v_\alpha''(s) \le \alpha^2 v_\alpha(s)$.

Example 6.2. For the upper half-plane $\mathbb{H}$, we have from Example 4.2 $v_\alpha(s) = e^{\alpha s}$ for any vertical half-line and $v_\alpha''(s) = \alpha^2 v_\alpha$, so the differential inequality $v_\alpha''(s) \le \alpha^2 v_\alpha(s)$ for $\alpha \ge 1$ is best possible.

Theorem 6.3. λ_Ω^α, $\alpha \ge 1$, *is α-hyperbolically concave if and only if Ω is convex.*

Proof. Preceding work shows that $v_\alpha''(s) \le \alpha^2 v_\alpha(s)$ for a convex region. That is, λ_Ω^α is α-hyperbolically concave for convex Ω. Now, suppose $v_\alpha''(s) \le \alpha^2 v_\alpha(s)$ holds along all hyperbolic geodesics in a hyperbolic region. This means that (6.3) holds. Corollary 3.2 then implies that Ω is convex. ■

7. Rate of change of Euclidean curvature of hyperbolic geodesics

We investigate the Euclidean curvature of hyperbolic geodesics. In any disk or half-plane, the hyperbolic geodesics are circular arcs and so have constant Euclidean curvature $\kappa_e(w(s), \gamma)$. Then the rate of change of Euclidean curvature vanishes. It is plausible that in convex regions, the rate of change of Euclidean curvature of hyperbolic geodesics should not be too large. We show that the rate of change of the quantity $\kappa_e(w(s), \gamma)/\lambda_\Omega(w(s))$ is bounded in a convex region.

If γ is a hyperbolic geodesic, then $\kappa_\Omega(w(s), \gamma)$ vanishes and (4.1) gives

$$\frac{\kappa_e(w(s), \gamma)}{\lambda_\Omega(w(s)} = -\mathrm{Im}\left\{\frac{e^{i\theta(s)}\Gamma_\Omega(w(s))}{\lambda_\Omega(w(s))}\right\}.$$

We are interested in the rate of change of the quantity $\kappa_e(w(s),\gamma)/\lambda_\Omega(w(s))$, where γ is a hyperbolic geodesic parameterized by hyperbolic arclength. Now,

$$\begin{aligned}\frac{d}{ds}\frac{\kappa_e(w(s),\gamma)}{\lambda_\Omega(w(s))} &= -\mathrm{Im}\left\{\frac{e^{i\theta(s)}}{\lambda_\Omega(w(s))}\left(\frac{\partial\Gamma_\Omega(w(s))}{\partial w}w'(s)+\frac{\partial\Gamma_\Omega(w(s))}{\partial\bar w}\overline{w'(s)}\right)\right\}\\ &\quad+\mathrm{Im}\left\{\frac{e^{i\theta(s)}\Gamma_\Omega(w(s))}{\lambda_\Omega^2(w(s))}\left(\frac{\partial\lambda_\Omega w(s)}{\partial w}w'(s)+\frac{\partial\lambda_\Omega w(s)}{\partial\bar w}\overline{w'(s)}\right)\right\}\\ &\quad-\mathrm{Im}\left\{\frac{ie^{i\theta(s)}\theta'(s)\Gamma_\Omega(w(s))}{\lambda_\Omega(w(s))}\right\}\\ &= -\mathrm{Im}\left\{\frac{e^{2i\theta(s)}}{\lambda_\Omega^2(w(s))}\frac{\partial\Gamma_\Omega(w(s))}{\partial w}+\frac{1}{\lambda_\Omega^2(w(s))}\frac{\partial\Gamma_\Omega(w(s))}{\partial\bar w}\right\}\\ &\quad+\frac{1}{2}\mathrm{Im}\left\{\frac{e^{2i\theta(s)}\Gamma_\Omega^2(w(s))}{\lambda_\Omega^2(w(s))}+\frac{|\Gamma_\Omega(w(s))|^2}{\lambda_\Omega^2(w(s))}\right\}\\ &\quad-\theta'(s)\mathrm{Re}\left\{\frac{e^{i\theta(s)}\Gamma_\Omega(w(s))}{\lambda_\Omega(w(s))}\right\}.\end{aligned}$$

Because $|\Gamma_\Omega(w(s))|/\lambda_\Omega(w(s))$ is real-valued, (2.3) and (4.2) give

$$\begin{aligned}\frac{d}{ds}\frac{\kappa_e(w(s),\gamma)}{\lambda_\Omega(w(s))} &= -\mathrm{Im}\left\{\frac{e^{2i\theta(s)}}{\lambda_\Omega^2(w(s))}\left(\frac{\partial\Gamma_\Omega(w(s))}{\partial w}-\frac{1}{2}\Gamma_\Omega^2(w(s))\right)\right\}\\ &\quad+\mathrm{Im}\left\{\frac{e^{i\theta(s)}\Gamma_\Omega(w(s))}{\lambda_\Omega(w(s))}\right\}\mathrm{Re}\left\{\frac{e^{i\theta(s)}\Gamma_\Omega(w(s))}{\lambda_\Omega(w(s))}\right\}.\end{aligned}$$

From $\mathrm{Re}\{z\}\mathrm{Im}\{z\}=(1/2)\mathrm{Im}\{z^2\}$ and (2.2), we obtain

$$\frac{d}{ds}\frac{\kappa_e(w(s),\gamma)}{\lambda_\Omega(w(s))} = -\mathrm{Im}\left\{\frac{e^{2i\theta(s)}}{\lambda_\Omega^2(w(s))}\left(S_\Omega(w(s))-\frac{1}{2}\Gamma_\Omega^2(w(s))\right)\right\}. \tag{7.1}$$

Theorem 7.1. *Let Ω be a hyperbolic region. Ω is convex if and only if*

$$\left|\frac{d}{ds}\frac{\kappa_e(w(s),\gamma)}{\lambda_\Omega(w(s))}\right| \le \frac{1}{2} \tag{7.2}$$

for every hyperbolic geodesic γ in Ω parameterized by hyperbolic arclength.

Proof. From (7.1), we have

$$\left|\frac{d}{ds}\frac{\kappa_e(w(s),\gamma)}{\lambda_\Omega(w(s))}\right| \le \frac{1}{\lambda_\Omega^2(w(s))}\left|S_\Omega(w(s))-\frac{1}{2}\Gamma_\Omega^2(w(s))\right|.$$

If Ω is convex, then (7.2) follows from Corollary 3.2.

Conversely, assume (7.2) holds. Given w_0 in Ω choose a hyperbolic geodesic γ through w_0, say $w(0) = w_0$, in a direction $e^{i\theta(0)}$ so that

$$\frac{d}{ds}\frac{\kappa_e(w(0),\gamma)}{\lambda_\Omega(w(0))} = \frac{1}{\lambda_\Omega^2(w(0))}\left|S_\Omega(w(0)) - \frac{1}{2}\Gamma_\Omega^2(w(0))\right|.$$

Then (7.2) implies that

$$\left|S_\Omega(w(0)) - \frac{1}{2}\Gamma_\Omega^2(w(0))\right| \leq \frac{1}{2}\lambda_\Omega^2(w(0)).$$

Because $w_0 = w(0)$ is arbitrary, Corollary 3.2 implies that Ω is convex. ■

Example 7.2. This result is sharp for $\mathbb{H}$. Note that $\lambda_\mathbb{H}(w) = 1/\mathrm{Im}\,\{w\}$. For any $a \in \mathbb{R}$ and $b > 0$, $\gamma : w(s) = a + be^{i2\arctan e^s}$ is a hyperbolic geodesic in $\mathbb{H}$ with $\kappa_e(w(s),\gamma) = 1/b$. Let $t = 2\arctan e^s$, then $\frac{\kappa_e(w(s),\gamma)}{\lambda_\Omega(w(s))} = \sin t$, $\frac{dt}{ds} = \sin t$, and

$$\frac{d}{ds}\frac{\kappa_e(w(s),\gamma)}{\lambda_\Omega(w(s))} = \cos t \sin t = \frac{1}{2}\sin(2t),$$

which achieves its maximum value 1/2 when $t = \pi/4$.

In Theorem 7.1 we considered the rate of change of Euclidean curvature divided by the density of the hyperbolic metric. Now, we consider the rate of change of the Euclidean curvature.

Theorem 7.3. *Let Ω be a hyperbolic region. Ω is uniformly perfect if and only if there is a finite constant $C \geq 0$ such that every hyperbolic geodesic in Ω parameterized by hyperbolic arclength satisfies*

$$\left|\frac{d}{ds}\kappa_e(w(s),\gamma)\right| \leq C\lambda_\Omega(w(s)).$$

Proof. Consider any hyperbolic geodesic γ in Ω parameterized by hyperbolic arclength by $w = w(s)$. From (4.1),

$$\kappa_e(w(s),\gamma) = -\mathrm{Im}\,\left\{e^{i\theta(s)}\Gamma_\Omega(w(s))\right\}.$$

Then by using (4.2),

$$\begin{aligned}
&\frac{d}{ds}\kappa_e(w(s),\gamma)\\
&= -\mathrm{Im}\,\left\{e^{i\theta(s)}\left(\frac{\partial\Gamma_\Omega(w(s))}{\partial w}w'(s) + \frac{\partial\Gamma_\Omega(w(s))}{\partial\bar{w}}\overline{w'(s)}\right) + ie^{i\theta(s)}\theta'(s)\Gamma_\Omega(w(s))\right\}\\
&= -\mathrm{Im}\,\left\{\frac{e^{2i\theta(s)}}{\lambda_\Omega(w(s))}\frac{\partial\Gamma_\Omega(w(s))}{\partial w}\right\} + \mathrm{Im}\,\left\{\frac{e^{i\theta(s)}\Gamma_\Omega(w(s))}{\lambda_\Omega(w(s))}\right\}\mathrm{Re}\,\left\{e^{i\theta(s)}\Gamma_\Omega(w(s))\right\}.
\end{aligned}$$

As $\mathrm{Re}\{z\}\mathrm{Im}\{z\} = (1/2)\mathrm{Im}\{z^2\}$, we obtain

$$\begin{aligned}\frac{d}{ds}\kappa_e(w(s),\gamma) &= -\mathrm{Im}\left\{\frac{e^{2i\theta(s)}}{\lambda_\Omega(w(s))}\left(\frac{\partial\Gamma_\Omega(w(s))}{\partial w} - \frac{1}{2}\Gamma_\Omega^2(w(s))\right)\right\} \\ &= -\mathrm{Im}\left\{\frac{e^{2i\theta(s)}S_\Omega(w(s))}{\lambda_\Omega(w(s))}\right\}.\end{aligned}$$

Hence,

$$\left|\frac{d}{ds}\kappa_e(w(s),\gamma)\right| \le \frac{|S_\Omega(w(s))|}{\lambda_\Omega(w(s))}.$$

Identity (5.2) implies $C = \beta(\Omega)/2$. ∎

Corollary 7.4. *A hyperbolic region Ω in $\mathbb{C}$ is Nehari if and only if every hyperbolic geodesic γ in Ω parameterized by hyperbolic arclength satisfies*

$$\left|\frac{d}{ds}\kappa_e(w(s),\gamma)\right| \le \frac{1}{2}\lambda_\Omega(w(s)).$$

8. Hyperbolic geodesics with Euclidean parametrization

Finally, we consider the case of a hyperbolic geodesic with Euclidean arclength parametrization. Let γ be a hyperbolic geodesic with Euclidean arclength parametrization $w = w(s)$, so $w'(s) = e^{i\theta(s)}$. Set

$$v_\alpha(s) = \lambda_\Omega^\alpha(w(s)).$$

Then

$$\begin{aligned}v_\alpha'(s) &= \alpha\lambda_\Omega^{\alpha-1}(w(s))\left(\frac{\partial\lambda_\Omega(w(s))}{\partial w}w'(s) + \frac{\partial\lambda_\Omega(w(s))}{\partial\bar{w}}\overline{w'(s)}\right) \\ &= \alpha v_\alpha(s)\mathrm{Re}\left\{e^{i\theta(s)}\Gamma_\Omega(w(s))\right\}.\end{aligned}$$

Next,

$$\begin{aligned}v_\alpha''(s) &= \alpha v_\alpha'(s)\mathrm{Re}\left\{e^{i\theta(s)}\Gamma_\Omega(w(s))\right\} + \alpha v_\alpha(s)\mathrm{Re}\left\{ie^{i\theta(s)}\theta'(s)\Gamma_\Omega(w(s))\right\} \\ &\quad + \alpha v_\alpha(s)\mathrm{Re}\left\{e^{i\theta(s)}\left(\frac{\partial\Gamma_\Omega(w(s))}{\partial w}w'(s) + \frac{\partial\Gamma_\Omega(w(s))}{\partial\bar{w}}\overline{w'(s)}\right)\right\} \\ &= \alpha^2 v_\alpha(s)\mathrm{Re}^2\left\{e^{i\theta(s)}\Gamma_\Omega(w(s))\right\} - \alpha v_\alpha(s)\theta'(s)\mathrm{Im}\left\{e^{i\theta(s)}\Gamma_\Omega(w(s))\right\} \\ &\quad + \alpha v_\alpha(s)\mathrm{Re}\left\{e^{2i\theta(s)}\frac{\partial\Gamma_\Omega(w(s))}{\partial w} + \frac{\partial\Gamma_\Omega(w(s))}{\partial\bar{w}}\right\}.\end{aligned}$$

From (2.3), (4.1), $\kappa_\Omega(w(s),\gamma) = 0$ and $\theta'(s) = \kappa_e(w(s),\gamma)$, we have

$$\theta'(s) = -\mathrm{Im}\left\{e^{i\theta(s)}\Gamma_\Omega(w(s))\right\}$$

and

$$v_\alpha''(s) = \alpha v_\alpha(s)\left(\alpha \mathrm{Re}^2\left\{e^{i\theta(s)}\Gamma_\Omega(w(s))\right\} + \mathrm{Im}^2\left\{e^{i\theta(s)}\Gamma_\Omega(w(s))\right\}\right) + \alpha v_\alpha(s)\left(\mathrm{Re}\left\{e^{2i\theta(s)}\frac{\partial\Gamma_\Omega(w(s))}{\partial w}\right\} + \frac{1}{2}\lambda_\Omega^2(w(s))\right).$$

As

$$\alpha \mathrm{Re}^2\{z\} + \mathrm{Im}^2\{z\} = \frac{\alpha+1}{2}|z|^2 + \frac{\alpha-1}{2}\mathrm{Re}\,\{z^2\},$$

we obtain

$$v_\alpha''(s) = \alpha v_\alpha(s)\left(\frac{\alpha+1}{2}|\Gamma_\Omega(w(s))|^2 + \mathrm{Re}\,\left\{e^{2i\theta(s)}\left(S_\Omega(w(s)) + \frac{\alpha}{2}\Gamma_\Omega^2(w(s))\right)\right\} + \frac{1}{2}\lambda_\Omega^2(w(s))\right).$$

For $\alpha = -1$, this simplifies to

$$v_{-1}''(s) = -v_{-1}(s)\left(\mathrm{Re}\left\{e^{2i\theta(s)}\left(S_\Omega(w(s)) - \frac{1}{2}\Gamma_\Omega^2(w(s))\right)\right\} + \frac{1}{2}\lambda_\Omega^2(w(s))\right).$$

Consequently, $v_{-1}''(s) \le 0$ if and only if

$$\mathrm{Re}\left\{e^{2i\theta(s)}\left(S_\Omega(w(s)) - \frac{1}{2}\Gamma_\Omega^2(w(s))\right)\right\} + \frac{1}{2}\lambda_\Omega^2(w(s)) \ge 0.$$

This holds for all unit vectors at $w(s)$ if and only if the inequality in Corollary 3.2(b) holds, which characterizes convex regions by Corollary 3.2. Thus, we have established the following result.

Theorem 8.1. *Let Ω be a hyperbolic region. $1/\lambda_\Omega$ is Euclidean concave in the sense that $v_{-1}''(s) \le 0$ along all hyperbolic geodesics in Ω with Euclidean arclength parametrization if and only if Ω is convex.*

Example 8.2. This result is sharp for $\mathbb{H}$. Since $\lambda_{\mathbb{H}}(w) = 1/\mathrm{Im}(w)$ and $w(s) = u + is$ is a Euclidean arclength parametrization of a hyperbolic geodesic in $\mathbb{H}$, we have $v_{-1}(s) = s$. Then $v_{-1}''(s) = 0$.

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