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The $p(x)$-Laplacian and applications

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Abstract. This is a survey of the application of function spaces in the study of differential equations. It includes a general introduction to the function spaces approach to differential equations, some motivating examples for studying variable exponent spaces, and an overview of recent results on the $p(x)$-Laplace equation.

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1. Overview

We start in the next section with some basic PDE material which in this context serves to motivate the study of function spaces. The section is concluded by some classical results about the solution of the Laplace equation $\triangle\phi = 0$ and some newer results on the non-linear case. In Section 3 we start by giving two applications which lead to variable exponent function spaces. Then we give the precise definitions of these spaces, and review some of the major differences between variable and constant exponent spaces. In Section 4 we present some results, mainly from [16, 18, 19, 20], on the $p(x)$-Laplacian.

If you want to learn more about variable exponent spaces then a good place to start is the web-page

`http://mathstat.helsinki.fi/analysis/varsobgroup/`

Under "Publications" you will find two other overview articles, one by Harjulehto and Hästö [15] and the other by Diening, Hästö and Nekvinda [9]. Mingione has recently published an article [23] on the Calculus of Variations approach to these problems which is also well-worth looking into. Furthermore, the web-page contains links to the home-pages of several other researchers working in the area.

This article is the outgrowth of a talk I gave at the *National Conference on PDE and Applications* in Coimbatore, in March 2005. It is aimed at graduate students and researchers without an extensive background in the field. Sections 3 and 4 contain more recent material which might also be of interest to experts.

2. PDE's, function spaces and applications

Suppose that we are interested in studying the differential equation

$$Du = f, \tag{2.1}$$

where D is a differential operator and f is a given function or mapping. We of course need to specify on what set the differential operator acts. But actually there are also good reasons for specifying more structure on this set, e.g. a topology. For instance we might want to "solve" the equation numerically – if u_ε is an approximate solution such that

$$\sup_x |Du_\varepsilon(x) - f(x)| < \varepsilon,$$

then what does this tell us about how near u_ε is to the true solution of (2.1), e.g. about $|u_\varepsilon(x) - u(x)|$?

A second motivation for the function spaces that we are going to consider is that they allow us to consider differential equations in a weak form. If we are considering a k-th order differential operator, then it would be very nice if we could work in the space C^k of k times continuously differentiable functions. However, we know that this space is not closed. Thus in the example of the previous paragraph u_ε in some sense gets closer and closer to a solution as ε tends to 0, but it may happen that the function actually gets worse and worse (e.g. its norm might increase to infinity) so that the limit $\lim_{\varepsilon\to 0} u_\varepsilon$ does not exist in the function space. Obviously this severely limits the information we can derive about the solution from this approximating sequence.

Weak solutions 1. Recall the definition of the distributional derivative: we say that $v\colon \mathbb{R}^n \to \mathbb{R}^n$ is the distributional derivative of $w\colon \mathbb{R}^n \to \mathbb{R}$ if

$$\int_{\mathbb{R}^n} v\,\varphi\,dx = -\int_{\mathbb{R}^n} w\nabla\varphi\,dx,$$

for all smooth and compactly supported test functions φ. Here $\nabla = (\partial_1, \dots, \partial_n)$ denotes the gradient operator. Of course we use the notation ∇w for v also in the case of distributional derivatives.

We can define a weak solution of a PDE in the same spirit. Suppose we can decompose D as $\partial_1 \circ \tilde{D}$, where ∂_1 is short for $\frac{\partial}{\partial x_1}$. Then a weak solution of (2.1) is a function $\tilde{u}$ which satisfies

$$\int_{\mathbb{R}^n} \tilde{D}\tilde{u}\,\partial_1\varphi\,dx = -\int_{\mathbb{R}^n} f\varphi\,dx$$

for the same class of test functions. Thus we have lowered the differentiability requirement on u by one, and instead we have to impose an integrability requirement on $\tilde{D}\tilde{u}$.

Note that if u is a C^k solution of the equation $Du = f$, then certainly u is a weak solution in this sense; this is shown by integration-by-parts. Let us consider a more concrete example of this, the one dimensional Laplace operator. This operator is defined as the formal dot product of the vector ∇ with itself, i.e. $\triangle = \nabla \cdot \nabla = \sum_i \frac{\partial^2}{\partial x_i^2}$; in one dimension this is just $\partial^2/\partial x^2$. So our original equation would be

$$\tfrac{\partial^2 u}{\partial x^2}(x) = f(x),$$

which would require u to be twice differentiable. We multiply by a test function $\varphi \in C_0^\infty$ and integrate:

$$\int_{\mathbb{R}} \tfrac{\partial^2 u}{\partial x^2}(x)\varphi(x)\,dx = \int_{\mathbb{R}} f(x)\varphi(x)\,dx.$$

Next we integrate the left-hand-side by parts, and use that φ has compact support (i.e. "$\varphi(\pm\infty) = 0$"):

$$-\int_{\mathbb{R}} \tfrac{\partial u}{\partial x}(x)\tfrac{\partial \varphi}{\partial x}(x)\,dx = \int_{\mathbb{R}} f(x)\varphi(x)\,dx,$$

which is just our equation for the weak solution.

Weak solutions 2. For the Laplace problem it turns out that a slightly different notion of weak solution is more convenient. We next recall the Laplace problem and then explain this notion.

In the classical Laplacian Dirichlet boundary value problem we are given a domain Ω in $\mathbb{R}^n$ and a continuous function $w : \partial\Omega \to \mathbb{R}$. The problem is to find a continuous function $u : \overline{\Omega} \to \mathbb{R}$ which satisfies

$$\begin{cases} -\Delta u(x) = 0, & \text{for } x \in \Omega, \\ u(x) = w(x), & \text{for } x \in \partial\Omega. \end{cases}$$

By Weyl's lemma, such a u is always C^2 in Ω, and hence the problem may be considered in the classical sense.

The p-Dirichlet boundary value problem for fixed $p \in (1, \infty)$ is to find a continuous function u on $\overline{\Omega}$ so that the p-Laplace equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \tag{2.2}$$

is satisfied on Ω and $u = w$ on $\partial\Omega$ (see [21] for a comprehensive treatment of this and related problems). When $p \neq 2$, equation (2.2) is nonlinear and it must be understood in a weak sense. The idea here is the following: if g is a C^1 function, then we know that $\nabla g(x) = 0$ at the critical points of g. Even if g is not C^1, we call the critical points of g weak solutions of the equation $\nabla g(x) = 0$. In the p-Laplace equation (2.2) the "independent variable" is the unknown function u, so we need to use variational calculus. With these tools we see that (2.2) is the Euler–Lagrange equation of the variational integral

$$\int_\Omega |\nabla u(x)|^p \, dx, \tag{2.3}$$

which means that smooth minimizers of this integral satisfy (2.2). We call (2.3) the p-Dirichlet energy integral on Ω.

In the non-linear case also the boundary values must be understood in a weak sense, i.e. we look for a minimizer $u \in W^{1,p}(\Omega)$ of (2.3) with the property that $u - w \in W^{1,p}_0(\Omega)$, where $w \in W^{1,p}(\Omega)$ gives the boundary values. Here $W^{1,p}(\Omega)$ denotes the Sobolev space, i.e. the space of functions u such that u and $|\nabla u|$ are both in the Lebesgue space L^p. The zero-boundary-value space $W^{1,p}_0(\Omega)$ is defined as the closure of $C^\infty_0(\Omega)$ (=infinitely differentiable and compactly supported functions in Ω) in $W^{1,p}(\Omega)$.

3. The variable exponent case

We have previously argued for using function spaces and weak formulations of differential equations. We start this section with two examples motivating the use of variable exponent function spaces in particular, and then give the formal definitions of these spaces.

Example 1. Chen, Levine and Rao [5] proposed a framework for image restoration based on a variable exponent Laplacian. To understand their ideas, it is useful first to look at some more classical ideas on the same problem.

In the context of image restoration we are given an input I which represents shades of gray, typically on a rectangle Ω in the plane. It is assumed that I is made up of the true image corrupted by noise, furthermore, the noise is assumed additive, i.e. $I = T + \eta$, where T is the true image and η is a random variable with zero mean. Thus the effect of the noise can be eliminated by smoothing the input, since this will cause the effect of the zero-mean random variables at nearby locations to cancel. Smoothing corresponds to minimizing the energy

$$E_2(u) = \int_\Omega |\nabla u(x)|^2 + |u(x) - I(x)|^2 \, dx.$$

Unfortunately, the smoothing will also destroy all small details from the image, so this procedure is not very useful.

A better approach is so-called total variation smoothing. In this approach, the smoothing is always applied along the presumed intensity level-sets of the image. Since an edge in the image gives rise to a very large gradient, the level-sets around the edge are very distinct, so this method does a good job of preserving edges. Total variation smoothing corresponds to minimizing the energy

$$E_1(u) = \int_\Omega |\nabla u(x)| + |u(x) - I(x)|^2 \, dx.$$

Unfortunately, total variation smoothing not only preserves edges, it also creates edges where there were none in the original image (the so-called staircase effect).

As the strengths and weaknesses of these two methods for image restoration are opposite, it is natural to try to combine them, which is what Chen, Levine and Rao [5] did. Looking at E_1 and E_2 suggests that an appropriate energy would be

$$E(u) = \int_\Omega |\nabla u(x)|^{p(x)} + |u(x) - I(x)|^2 \, dx,$$

where p is a function varying between 1 and 2. This function should be close to 1 where there are likely to be edges, and close to 2 where there are likely not to be edges. The approximate location of the edges can be determined by just smoothing the input data and looking for where the gradient is large. Although this model is conceptually simple and intuitively appealing, a theoretical analysis of it turns out to be quite complicated, for reasons that we will discuss below.

Example 2. A second application which uses variable exponent type Laplacians is modeling electrorheological fluids [7, 26]. The constitutive equation for the motion of an electrorheological fluid is given by

$$\tag{3.1} \tfrac{\partial}{\partial t} u + \operatorname{div} S(u) + (u \cdot \nabla) u + \nabla \pi = f,$$

where $u \colon \mathbb{R}^{3+1} \to \mathbb{R}^3$ is the velocity of the fluid at a point in space-time, $\nabla = (\partial_1, \partial_2, \partial_3)$ is the gradient operator, $\pi \colon \mathbb{R}^{3+1} \to \mathbb{R}$ is the pressure, $f \colon \mathbb{R}^{3+1} \to \mathbb{R}^3$ represents external forces, and the stress tensor $S \colon W^{1,1}_{loc} \to \mathbb{R}^{3\times 3}$ is of the form

$$Su(x) = \mu(x)(1 + |Du(x)|^2)^{(p(x)-2)/2})Du(x),$$

where, finally, μ is a weight function and $Du = \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetric part of the gradient of u. Note that if $p \equiv 2$, then this equation reduces to the usual non-dimensionalized Navier-Stokes equation.

Although (3.1) is more complicated than the Laplace equations we looked at previously, we see that the highest order differential term is quite similar, namely

$$\operatorname{div} \big((\lambda + |Du(x)|^2)^{(p(x)-2)/2}) Du(x) \big),$$

where $\lambda = 1$. Then the degenerate case $\lambda = 0$ corresponds to the Laplace operator that we considered previously.

Variable exponent spaces. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $p : \Omega \to [1, \infty)$ be a measurable function, called the *variable exponent* on Ω. We write

$$p^+ = \operatorname{ess\,sup}_{x \in \Omega} p(x) \quad \text{and} \quad p^- = \operatorname{ess\,inf}_{x \in \Omega} p(x).$$

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $u : \Omega \to \mathbb{R}$ such that $\varrho_{p(\cdot)}(\lambda u) = \int_\Omega |\lambda u(x)|^{p(x)}\, dx < \infty$ for some $\lambda > 0$. We define a Luxemburg-type norm on this space by the formula

$$\|u\|_{p(\cdot)} = \inf\big\{\lambda > 0 \colon \varrho_{p(\cdot)}(u/\lambda) \leqslant 1\big\}.$$

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consists of all $u \in L^{p(\cdot)}(\Omega)$ such that the absolute value of the distributional gradient ∇u is in $L^{p(\cdot)}(\Omega)$. The norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$$

makes $W^{1,p(\cdot)}(\Omega)$ a Banach space. For the basic theory of variable exponent spaces see [22].

Most of the problems in the development of the theory of $L^{p(\cdot)}$ spaces arise from the fact that these spaces are virtually never translation invariant. This is easily seen by considering translating critical singularities to locations with higher value of the exponent. The use of convolution is also limited: it was shown in [8] that Young's inequality $\|f * g\|_{p(\cdot)} \le c\,\|f\|_{p(\cdot)}\|g\|_1$ holds if and only if p is constant. These two problems significantly restrict the techniques available and slowed down the development of the theory.

If the Hardy–Littlewood maximal operator is bounded, then these problems are alleviated to some extent, and for instance convolution becomes possible. If one assumes that p is log-Hölder continuous, i.e. that the inequality

$$|p(x) - p(y)| \le \frac{c}{\log(1/|x-y|)}$$

holds for all points with $|x-y| < \frac{1}{2}$, then the maximal operator is locally bounded on $L^{p(\cdot)}$ [8]. This condition is in fact optimal in the sense of modulus of continuity [25] – if it is weakened, then there exists an exponent p which satisfies the weaker conditions for which M is not bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Global boundedness results were derived in [6, 24].

4. The $p(x)$-Laplacian

We have now seen that the p-Laplacian problem for $p \neq 2$ has to be understood in a weak sense as the minimization of a certain energy integral, and we have seen examples of problems which lead to variable exponent Laplacians. The generalization of (2.3) to the variable exponent case is immediate: we want to minimize

$$\int_\Omega |\nabla u(x)|^{p(x)}\, dx, \tag{4.1}$$

subject to the appropriate boundary conditions. The boundary values are again understood in a weak sense, using the space $W_0^{1,p(\cdot)}$ studied in [17].

Early work on this problem was done by Zhikov [27, 28] and Alkhutov [4]. This work was carried on by Acerbi & Mingione and their collaborators e.g. [1, 2, 3, 10], and, independently, by Fan and collaborators e.g. [11, 12, 13, 14]. Much of this work was directed at proving regularity results or generalizing other classical results in the case when the variable exponent satisfies certain quite strong regularity assumption (e.g. log-Hölder or α-Hölder continuity). Harjulehto, Hästö, Koskenoja and Varonen [18] showed that it is possible to derive existence and uniqueness results under much weaker conditions on the exponent.

If $p^+ < \infty$ and if there exists $\delta > 0$ such that for every $x \in \Omega$ either

$$p^-_{B(x,\delta)} \geqslant n \qquad \text{or} \qquad p^+_{B(x,\delta)} \leqslant \frac{n\, p^-_{B(x,\delta)}}{n - p^-_{B(x,\delta)}}$$

holds, then the variable exponent p is said to satisfy the *jump condition* in Ω. Here $p^-_{B(x,\delta)}$ denotes the infimum of p in the ball $B(x,\delta)$, and $p^+_{B(x,\delta)}$ the supremum. Roughly, the jump condition guarantees that p does not jump too much locally in Ω. Note that if Ω is bounded and if p is uniformly continuous, then p satisfies the jump condition in Ω.

Let us define an "energy integral" operator on the Sobolev space by

$$I_{p(\cdot)}(u) = \varrho_{p(\cdot)}(|\nabla u|).$$

The following is the main theorem from [18]:

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $w \in W^{1,p(\cdot)}(\Omega)$. Assume that p satisfies the jump condition in Ω and that $1 < p^- \leqslant p^+ < \infty$. Then $I_{p(\cdot)}$ has a unique minimizer in the set*

$$\{u\colon u - w \in W_0^{1,p(\cdot)}(\Omega)\}.$$

To shed further light on what assumptions on p are relevant, Harjulehto, Hästö and Koskenoja [16] studied the Dirichlet problem in one dimension. In this case the energy to minimize is

$$\int_a^b |u'(x)|^{p(x)}\, dx,$$

where (a,b) is an interval of the real line and the values of $u(a)$ and $u(b)$ are given. Assuming that $u(a) < u(b)$ we directly see that the minimizer should be chosen to be increasing. With this in mind we see that the Euler-Lagrange equation is just $p(x)u'(x)^{p(x)-1} = c$, so we have the simple formula

$$u'(x) = (c/p(x))^{1/(p(x)-1)}$$

for the derivative. Notice that in the fixed exponent case this implies that the solution is linear.

In the variable exponent case, however, we can draw a number of interesting conclusions which are relevant also for higher dimensions. First of all, if $p \to 1$ somewhere, then we might be in trouble, because the exponent tends to ∞. Assuming that $p^- > 1$, we see that $u \in C^1$ if p is continuous. In fact, for every $\alpha \in [0,1]$, $u \in C^{1,\alpha}$ if $p \in C^\alpha$. (Recall the definition of $C^{1,\alpha}$: this space consists of those functions with α-Hölder continuous first derivative.)

Therefore, it is natural to look for existence results under weaker assumptions also in higher dimensions. Recently, the results of [18] were complemented in [20] in two ways: first, it was shown that the Dirichlet energy integral always has a minimizer if the boundary values are in L^∞:

Theorem 4.3. *Let Ω be bounded and $1 < p^- \leqslant p^+ < \infty$. Suppose that $w \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. Then $I_{p(\cdot)}$ has a unique minimizer in the set*

$$\big\{u \colon u - w \in W_0^{1,p(\cdot)}(\Omega)\big\}.$$

Note that this in some sense corresponds to the one-dimensional case, as the boundary values are necessarily bounded in one dimension.

Second, an example was given which shows that if the boundary values are allowed to be unbounded, and the "jump-condition" from [18] is violated, then a minimizer need not exist:

Theorem 4.4. *Let $q_1 \in (1, n/(n-1))$ and $q_2 \in (q_1^*, \infty)$. Then there exist a smooth exponent p with $p^- = q_1$ and $p^+ = q_2$, a bounded domain Ω and a boundary value function $w \in W^{1,p(\cdot)}(\Omega)$ such that $I_{p(\cdot)}$ does not have a minimizer in the set $\{u \colon u - w \in W_0^{1,p(\cdot)}(\Omega)\}$.*

Thus we have a fairly clear picture of the existence of minimizers in the case when $p^- > 1$. However, for applications to the image restoration problem mentioned above it is of critical importance that p approaches and equals 1 in some parts of the domain.

It is well-known that the space $W^{1,1}$ is not reflexive. Similarly, it was shown in [22] that the space $W^{1,p(\cdot)}$ is reflexive if and only if $1 < p^- \leq p^+ < \infty$. This limits our ability to work with Laplacian Dirichlet problems in which the exponent approaches 1. Moreover, the usual concept of a weak solution is not sufficient for the case $p = 1$, rather so-called viscosity solutions are needed. This concept has not yet been successfully transferred to the variable exponent setting.

Another open problem is the minimal assumptions on the exponent needed to give some regularity (e.g. continuity) of the solutions. By an example in [19], we know that minimizers are not always continuous, even for uniformly continuous exponents. On the other hand the results by Mingione *et al.* and Fan *et al.* cited previously are derived under much stronger continuity assumptions on the exponent, so there remains a gap to fill here.

References

1. E. Acerbi and G. Mingione, *Regularity results for stationary electro-rheological fluids*, Arch. Ration. Mech. Anal. **164**(2002), 213–259.
2. E. Acerbi and G. Mingione, *Regularity results for electrorheological fluids: the stationary case*, C. R. Acad. Sci. Paris **334**(2002), 817–822.
3. E. Acerbi and G. Mingione, *Gradient estimates for the $p(x)$-Laplacean system*, J. Reine Angew. Math. **584**(2005), 117–148.
4. Yu. A. Alkhutov, *The Harnack inequality and the Hölder property of solutions of nonlinear elliptic equations with a nonstandard growth condition*, (Russian) Differential Equations **33**(1997), no. 12, 1653–1663 (1998). [Translation of *Differ. Uravn.* **33**(1997), no. 12, 1651–1660, 1726.]
5. Y. Chen, S. Levine and M. Rao, *Variable Exponent, Linear Growth Functionals in Image Restoration*, SIAM J. Appl. Math. **66**(2006), no. 4, 1383–1406.
6. D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer, *The maximal function on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **28**(2003), 223–238; **29**(2004), 247–249.
7. L. Diening, *Theoretical and Numerical Results for Electrorheological Fluids*, Ph.D. thesis, University of Freiburg, Germany, 2002.
8. ———, *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$*, Math. Inequal. Appl. **7**(2004), no. 2, 245–254.
9. L. Diening, P. Hästö and A. Nekvinda, *Open problems in variable exponent Lebesgue and Sobolev spaces*, FSDONA 2004 Proceedings (Drabek and Rakosnik (eds.); Milovy, Czech Republic, 2004), 38–58.
10. M. Eleuteri, *Hölder continuity results for a class of functionals with non standard growth*, Boll. Unione Mat. Ital. **7-B**(2004), 129–157.
11. X.-L. Fan and X. Han, *Existence and multiplicity of solutions for $p(x)$-Laplacian equations in R^N*, Nonlinear Anal. **59**(2004), no. 1–2, 173–188.
12. X.-L. Fan and Q.-H. Zhang, *Existence of solutions for $p(x)$-Laplacian Dirichlet problem*, Nonlinear Anal. **52**(2003), 1843–1852.
13. X.-L. Fan, Q.-H. Zhang and D. Zhao, *Eigenvalues of $p(x)$-Laplacian Dirichlet problem*, J. Math. Anal. Appl. **302**(2005), 306–317.
14. X.-L. Fan, Y. Z. Zhao and Q. H. Zhang, *A strong maximum principle for $p(x)$-Laplace equations*, Chinese J. Contemp. Math. **24**(2003), no. 3, 277–282. [Translation of *Chinese Ann. Math. Ser. A* **24**(2003), no. 4, 495–500.]
15. P. Harjulehto and P. Hästö, *An overview of variable exponent Lebesgue and Sobolev spaces*, Future Trends in Geometric Function Theory (D. Herron (ed.), RNC Workshop, Jyväskylä, 2003), 85–93.

16. P. Harjulehto, P. Hästö and M. Koskenoja, *The Dirichlet energy integral on intervals in variable exponent Sobolev spaces*, Z. Anal. Anwend. **22**(2003), no. 4, 911–923.
17. P. Harjulehto, P. Hästö, M. Koskenoja and S. Varonen, *Sobolev capacity on the space* $W^{1,p(\cdot)}(\mathbb{R}^n)$, J. Funct. Spaces Appl. **1**(2003), no. 1, 17–33.
18. ______, *The Dirichlet energy integral and variable exponent Sobolev spaces with zero boundary values*, Potential Anal. **25**(2006), no. 3, 205–222.
19. P. Hästö, *Counter-examples of regularity in variable exponent Sobolev spaces*, The *p*-Harmonic Equation and Recent Advances in Analysis (Manhattan, KS, 2004), Contemp. Math. 367, Amer. Math. Soc., Providence, RI, 2005.
20. ______, *On the variable exponent Dirichlet energy integral*, Comm. Pure Appl. Anal. **5**(2006), no. 3, 413–420.
21. J. Heinonen, T. Kilpelnen and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1993.
22. O. Kováčik and J. Rákosník, *On spaces* $L^{p(x)}$ *and* $W^{1,p(x)}$, Czechoslovak Math. J. **41(116)**(1991), 592–618.
23. G. Mingione, *Regularity of minima: an invitation to the dark side of the Calculus of Variations*, Appl. Math. **51**(2006), 355–425.
24. A. Nekvinda, *Hardy-Littlewood maximal operator on* $L^{p(x)}(\mathbb{R}^n)$, Math. Inequal. Appl. **7**(2004), no. 2, 255–266.
25. L. Pick and M. Růžička, *An example of a space* $L^{p(x)}$ *on which the Hardy-Littlewood maximal operator is not bounded*, Expo. Math. **19**(2001), 369–371.
26. M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer-Verlag, Berlin, 2000.
27. V. V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR-Izv. **29**(1987), no. 1, 33–66. [Translation of *Izv. Akad. Nauk SSSR Ser. Mat.* **50**(1986), no. 4, 675–710, 877.]
28. V. V. Zhikov, *On some variational problems*, Russian J. Math. Phys. **5**(1997), no. 1, 105–116 (1998).

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