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Starlikeness of Nonlinear Integral Transforms

S. Ponnusamy and P. Vasundhra

Abstract. For $n \geq 1$, let $\mathcal{A}_n$ denote the family of all normalized analytic functions f in the unit disk Δ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k.$$

For $\lambda > 0$ and $\mu > 0$, we consider the family $\mathcal{U}_n(\lambda, \mu)$ consisting of functions $f \in \mathcal{A}_n$, $f(z)/z \neq 0$ in Δ, and satisfying the condition

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} - 1 \right| < \lambda, \quad z \in \Delta.$$

For a real-valued, nonnegative integrable function $\varphi(t)$ satisfying the normalized condition $\int_0^1 \varphi(t)\, dt = 1$, and $\mu > 0$, we define

$$[V_\varphi^\mu(f)](z) = z \left[\int_0^1 \varphi(t) \left(\frac{tz}{f(tz)} \right)^\mu dt \right]^{1/\mu}, \quad f \in \mathcal{U}_n(\lambda, \mu).$$

In this paper, we obtain conditions on the parameters λ and μ, and on the function $\varphi(t)$, such that the transform $V_\varphi^\mu(f)$ is univalent or starlike. Our investigation leads to interesting special integral transforms that arise naturally in the setting of the Hadamard product of $z^2/f(z)$ with special families of functions such as the hypergeometric functions, polylogarithms and Hurwitz functions.

Keywords. Univalent, close-to-convex, starlike and convex functions, integral transforms, Hadamard product.

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1. Introduction and Preliminaries

Denote by $\mathcal{A}$ the family of functions f, normalized by $f(0) = f'(0) - 1 = 0$, that are analytic in the unit disk $\Delta = \{z : |z| < 1\}$ and by $\mathcal{S}$ the subfamily of $\mathcal{A}$ consisting of univalent functions in Δ. For $\alpha < 1$, let $\mathcal{S}^*(\alpha)$ (resp. $\mathcal{K}(\alpha)$)

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represent the subfamily of functions in $\mathcal{A}$ that are starlike (resp. convex) of order α. Analytically, they are defined as follows:

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} :\ \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \Delta \right\}$$

and

$$\mathcal{K}(\alpha) = \{ f \in \mathcal{A} :\ zf'(z) \in \mathcal{S}^*(\alpha) \}.$$

It is well known that $\mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*(0) \equiv \mathcal{S}^*$ for $0 \le \alpha < 1$ and $\mathcal{K}(\alpha) \subsetneq \mathcal{K}(0) \equiv \mathcal{K}$ for $0 \le \alpha < 1$ and $\mathcal{K} \subseteq \mathcal{K}(-1/2) \subsetneq \mathcal{S}$, see [20] and [19, p. 71, Theorem 2.24;p.73]. Members of $\mathcal{S}^*$ (resp. $\mathcal{K}$) are called normalized starlike (resp. convex) functions. For $\lambda > 0$ and $\mu \in \mathbb{R}$, we let

$$\mathcal{U}(\lambda, \mu) = \left\{ f \in \mathcal{A} : \left| f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} - 1 \right| < \lambda,\ \frac{f(z)}{z} \neq 0,\ z \in \Delta \right\},$$

and $\mathcal{U}(\lambda, 1) =: \mathcal{U}(\lambda)$. Also, for $n \ge 1$, we let $\mathcal{U}_n(\lambda) = \mathcal{U}(\lambda) \cap \mathcal{A}_n$ and $\mathcal{U}_n(\lambda, \mu) = \mathcal{U}(\lambda, \mu) \cap \mathcal{A}_n$, where

$$\mathcal{A}_n = \left\{ f \in \mathcal{A} :\ f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \right\}.$$

1.1. The Class $\mathcal{U}(\lambda)$.

Typical members of $\mathcal{U}(1)$ and $\mathcal{U}_2(1)$ are $z/(1-z)^2$ and $z/(1-z^2)$, respectively. These two functions are also in $\mathcal{S}^*$. In [9] (see also [1]), Ozaki and Nunokawa showed that $\mathcal{U}(1) =: \mathcal{U} \subset \mathcal{S}$. Although we have the strict inclusion $\mathcal{U} \subsetneq \mathcal{S}$, functions in $\mathcal{U}$ and $\mathcal{U} \cap \mathcal{A}_2$ are not necessarily starlike in Δ (see for instance, [8]). In view of this observation radii problems have also been discussed recently in Obradović and Ponnusamy [7, 14]. However, because $\mathcal{U} \subset \mathcal{S}$, we have

$$\mathcal{U}(\lambda) \subsetneq \mathcal{S} \quad \text{for } 0 < \lambda \le 1$$

and for $\lambda > 1$, a function in $\mathcal{U}(\lambda)$ is not necessarily univalent in Δ. We include here a simple example to illustrate that the inclusion is strict. Consider $g(z) = z + (\lambda/2)z^2$, where $\lambda > 0$. Then $g'(z) = 1 + \lambda z$ so that g is not univalent whenever $\lambda > 1$. On the other hand, a computation gives

$$\left| g'(z) \left(\frac{z}{g(z)} \right)^2 - 1 \right| = \left| \frac{(\lambda^2/4)z^2}{(1 + (\lambda/2)z)^2} \right| \to \frac{\lambda^2/4}{(1-\lambda/2)^2} \quad \text{as} \quad z \to -1^-.$$

Note that for $\lambda > 1$, $\frac{\lambda^2/4}{(1-\lambda/2)^2} > 1$, from which we conclude that functions in $\mathcal{U}(\lambda)$ need not be univalent when $\lambda > 1$. More recently, the present authors

[17, 18] discussed a general problem from which one obtains conditions on λ such that

$$\mathcal{U}(\lambda) \subsetneq \mathcal{S}^*(\alpha), \quad \mathcal{U}(\lambda) \subsetneq \mathcal{R}(\alpha),$$

where $\mathcal{R}(\alpha) = \{f \in \mathcal{A} : \ {\rm Re} f'(z) > \alpha \text{ for } z \in \Delta\}$.

1.2. The Class $\mathcal{U}(\lambda,\mu)$. In [6, 10, 11, 15], the problem of finding conditions on λ and μ so that each function in $\mathcal{U}(\lambda,\mu)$ is in $\mathcal{S}^*$ or in some subfamilies of $\mathcal{S}$ is considered. For example, Ponnusamy and Singh [11] have shown that

$$\mathcal{U}(\lambda,\mu) \subseteq \mathcal{S}^* \quad \text{if } \mu < 0 \text{ and } 0 < \lambda \leq \frac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}} := \lambda^*(\mu)$$

and in [6], Obradović proved that the above inclusion continues to hold for $0 < \mu < 1$ and with the same bound for λ. Recently, Fournier and Ponnusamy [4] settled sharpness questions. For a ready reference, we recall these results here.

Theorem A. [4] *Let $\mu \in \mathbb{C}$ with ${\rm Re}\,(\mu) < 1$. Then, we have*

1. $\mathcal{U}(\lambda,\mu) \subset \mathcal{S}^*$ *iff* $0 \leq \lambda \leq \dfrac{|1-\mu|}{\sqrt{|1-\mu|^2+|\mu|^2}}$.
2. $\mathcal{U}(\lambda,\mu) \subset \mathcal{S}_p$ *iff* $0 \leq \lambda \leq \min\left(1, \dfrac{|1-\mu|}{|\mu|}\right)$, *where $\mathcal{S}_p$ denotes the class of spirallike functions in $\mathcal{A}$.*

Clearly $\mathcal{U}(1,\mu) \subset \mathcal{S}^*$ iff $\mu = 0$ and $\mathcal{U}(1,\mu) \subset \mathcal{S}_p$ iff ${\rm Re}(\mu) \leq \frac{1}{2}$.

1.3. Transformations and the Main Problem . One of the classical problems in geometric function theory is to consider transformations that take functions in a subset $\mathcal{F}$ of $\mathcal{A}$ into functions which are univalent or starlike or convex. The subset $\mathcal{F}$ may include some non-univalent functions and some univalent functions in Δ. Before we proceed further, it is important to recall that,

$$F \in \mathcal{K} \Longleftrightarrow zF' \in \mathcal{S}^*,$$

or equivalently,

$$f \in \mathcal{S}^* \Longleftrightarrow \Lambda(f) \in \mathcal{K}, \quad \Lambda(f)(z) = \int_0^1 \frac{f(tz)}{t}\,dt = z + \sum_{n=2}^{\infty} \frac{1}{n} z^n.$$

The operator $\Lambda(f)$ is referred to as the Alexander transform of f. Although $\Lambda(\mathcal{S}) \not\subset \mathcal{S}$ (see [3]), it was shown that there exist sets of functions, having univalent as well as non-univalent functions, whose Alexander transforms are starlike (see for example [10, 5]). As a consequence, this result can be used to generate functions in $\mathcal{S}^*$ that are not convex. Results of this type in the theory of univalent functions are known in the literature. For example, R. Fournier and St. Ruscheweyh [5] studied the following integral transform which obviously includes the Alexander transform as a special case:

$$F(z) = \int_0^1 \varphi(t) \frac{f(tz)}{t}\, dt, \tag{1.4}$$

where φ is a real valued non-negative weight function defined on the unit interval $[0,1]$ with the normalization $\int_0^1 \varphi(t)\, dt = 1$. The authors [5] determined conditions so that $F \in \mathcal{S}^*$. Our work is mainly motivated by this transformation.

The aim of this paper is to determine certain connections between $\mathcal{U}_n(\lambda, \mu)$ and transforms of the type (1.4). More precisely, we shall be interested in the following problem

Problem 1.5. *Let φ be a real valued non-negative integrable function on $[0,1]$ satisfying the normalized condition $\int_0^1 \varphi(t)\, dt = 1$. For $f \in \mathcal{U}_n(\lambda, \mu)$, define*

$$[V_\varphi^\mu(f)](z) = z \left[\int_0^1 \varphi(t) \left(\frac{tz}{f(tz)} \right)^\mu dt \right]^{1/\mu}, \tag{1.6}$$

where for possible multiple-valued power functions principal branches are considered. Given $\varphi(t)$ and $\mu \le n$, find conditions on $\lambda > 0$ so that $V_\varphi^\mu(f)$ is starlike or convex.

1.7. Basic Results . Before we proceed to solve this problem, it would be useful to list down a few basic properties. Observe that, if $g = \Lambda(f)$ then $zg'(z) = f(z)$. Consequently, if $f \in \mathcal{U}_n(\lambda, \mu)$ for $0 < \mu < n$ then we have the following (see [17]):

(i) $f(z) = zg'(z) \in \mathcal{S}^*$ for $0 < \lambda \le \lambda^* = (n-\mu)/\sqrt{(n-\mu)^2 + \mu^2}$

(ii) $\text{Re} \left\{ \left(\frac{z}{f(z)} \right)^\mu \right\} > 1 - \frac{\lambda \mu}{n - \mu}, \quad z \in \Delta.$

In particular, for $f \in \mathcal{U}_n(\lambda)$ with $\lambda \le (n-1)/2$ and $n \ge 2$, one has,

$$\text{Re} \left(\frac{z}{f(z)} \right) > \frac{1}{2}, \quad \text{i.e.} \quad |g'(z) - 1| < 1, \; z \in \Delta,$$

which in turn implies that Re $(g'(z)) > 0$ in Δ. Similarly from [13] it follows that

$$f \in \mathcal{U}_n(\lambda, n) \Rightarrow \text{Re} \left\{ \left(\frac{z}{f(z)} \right)^n \right\} > 1 - \frac{n|f^{(n+1)}(0)|}{(n+1)!} - n\lambda, \quad z \in \Delta.$$

In particular, for $n = 1$ and $f \in \mathcal{U}(\lambda)$ with $\lambda \leq (1 - |f''(0)|)/2$, one has

$$\mathrm{Re}\left(\frac{z}{f(z)}\right) > \frac{1}{2}, \quad \text{i.e. } |g'(z) - 1| < 1 \text{ in } \Delta,$$

which in turn implies that $\mathrm{Re}\,(g'(z)) > 0$ in Δ.

(iii) A simple computation yields

$$1 + \frac{zg''(z)}{g'(z)} = \frac{zf'(z)}{f(z)}$$

and so

$$\left| \frac{1 + \frac{zg''(z)}{g'(z)}}{(g'(z))^{\mu}} - 1 \right| = \left| f'(z) \left(\frac{z}{f(z)}\right)^{\mu+1} - 1 \right|,$$

where for possible multiple-valued power functions principal branches are considered. For instance, this observation shows that if $0 < \mu < n$ and $g \in \mathcal{A}_n$ satisfies the condition

$$\left| 1 + \frac{zg''(z)}{g'(z)} - (g'(z))^{\mu} \right| < b|g'(z)|^{\mu}$$

for $0 < b \leq \lambda^* = \frac{n-\mu}{\sqrt{(n-\mu)^2+\mu^2}}$, then $zg'(z)$ is starlike and hence $g(z)$ is convex. The sharpness part follows as in [4].

2. Main Results

The main aim of this paper is to discuss Problem 1.5. If $f \in \mathcal{U}(\lambda, \mu) \cap \mathcal{A}_n$, then

$$f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} = 1 + (n - \mu)a_{n+1}z^n + \cdots$$

and therefore, the power series representation of functions associated with $\mathcal{U}(\lambda, \mu) \cap \mathcal{A}_n$ helps us to consider two cases, namely, $\mu < n$ and $\mu = n$ independently. In particular, we obtain conditions on $\varphi(t)$, μ and α so that $V_{\varphi}^{\mu}(f) \in \mathcal{S}^*(\alpha)$ whenever $f \in \mathcal{U}(\lambda, \mu)$. Finally, we introduce the family

$$\mathcal{S}_{\alpha}^* = \left\{ f \in \mathcal{S}^*(\alpha) : \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha, \quad z \in \Delta \right\}.$$

Now we are in a position to state our result for the case $\mu < n$.

Theorem 2.1. *For $\mu < n$, let $f \in \mathcal{U}_n(\lambda, \mu)$. Suppose that $\alpha < 1$, λ and $\varphi(t)$ satisfy the condition*

$$0 < \lambda \leq \frac{(1-\alpha)(n-\mu)}{(n+\mu(1-\alpha))\int_0^1 t^n \varphi(t)\, dt}. \tag{2.2}$$

If $V_\varphi^\mu(f)$ is defined by (1.6), then $V_\varphi^\mu(f) \in \mathcal{S}_\alpha^*$.

If $\mu = 1$, then, by Theorem 2.1, we have the following:

Corollary 2.3. *Let $f \in \mathcal{U}_n(\lambda)$ with $n \geq 2$. Suppose that $V_\varphi(f)$ is defined by*

$$V_\varphi^1(f)(z) := V_\varphi(f)(z) = z \int_0^1 \varphi(t) \frac{tz}{f(tz)}\, dt, \tag{2.4}$$

λ and $\varphi(t)$ satisfy the condition

$$0 < \lambda \leq \frac{(1-\alpha)(n-1)}{(n+1-\alpha)\int_0^1 t^n \varphi(t)\, dt}, \quad \alpha < 1.$$

Then $V_\varphi(f) \in \mathcal{S}_\alpha^*$.

Example 2.5. Corollary 2.3 for $\alpha = 0$ implies the following:

$$f \in \mathcal{U}_n(\lambda)\ (n \geq 2) \implies V_\varphi(f) \in \mathcal{S}_1^* \ \text{ if } \ 0 < \lambda \leq \frac{n-1}{(n+1)\int_0^1 t^n \varphi(t)\, dt}.$$

In particular, we have

(i) when the second coefficient of f is zero, we deduce that

$$f \in \mathcal{U}_2(\lambda) \text{ implies that } V_\varphi(f) \in \mathcal{S}_1^* \text{ whenever } 0 < \lambda \leq \frac{1}{3\int_0^1 t^2 \varphi(t)\, dt}.$$

(ii) If the second and the third coefficients of f are zero, then

$$f \in \mathcal{U}_3(\lambda) \text{ implies that } V_\varphi(f) \in \mathcal{S}_1^* \text{ whenever } 0 < \lambda \leq \frac{1}{2\int_0^1 t^3 \varphi(t)\, dt}.$$

Our next intention is to deal with the case $\mu = n$ and formulate the next result.

Theorem 2.6. *Let $n \geq 1$, and $f \in \mathcal{U}_n(\lambda, n)$ with $a_n = f^{(n)}(0)/n!$. Suppose that λ and $\varphi(t)$ satisfy the condition*

$$0 < \lambda \leq \frac{1-\alpha-n|a_{n+1}|(2-\alpha)\int_0^1 t^n \varphi(t)\, dt}{(2n+1-n\alpha)\int_0^1 t^{n+1}\varphi(t)\, dt} \tag{2.7}$$

for some $\alpha < 1$. If $V_\varphi^n(f)$ is defined by (1.6), then $V_\varphi^n(f) \in \mathcal{S}_\alpha^*$.

If we apply Theorem 2.6 with $n = 1$, we immediately obtain

Corollary 2.8. *Let* $f \in \mathcal{U}(\lambda)$ *and* $V_\varphi(f)$ *be given by* (2.4). *If* λ *and* $\varphi(t)$ *satisfy the condition*

$$0 < \lambda \le \frac{1-\alpha-(2-\alpha)|a_2|\int_0^1 t\varphi(t)\,dt}{(3-\alpha)\int_0^1 t^2\varphi(t)\,dt},$$

for $\alpha < 1$. *Then* $V_\varphi(f) \in \mathcal{S}^*_\alpha$.

The case $\alpha = 0$ of Corollary 2.8 gives the following:

$$f \in \mathcal{U}(\lambda) \quad \Longrightarrow \quad V_\varphi(f) \in \mathcal{S}^*_1 \quad \text{if } 0 < \lambda \le \frac{1-2|a_2|\int_0^1 t\varphi(t)\,dt}{3\int_0^1 t^2\varphi(t)\,dt}.$$

Note that if $a_2 = 0$, then the last implication leads to Example 2.5(i) but not necessarily the converse.

3. Proofs of Main Theorems

3.1. Proof of Theorem 2.1. Let $f \in \mathcal{U}_n(\lambda,\mu)$ where $\mu < n$. Then, we can write

$$f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} = 1 + \lambda w(z) = 1 + (n-\mu)a_{n+1}z^n + \cdots,$$

where $w \in \mathcal{B}_n$, i.e. w is analytic in Δ, $|w(z)| < 1$ with $w(0) = w'(0) = \cdots = w^{n-1}(0) = 0$. In view of the Schwarz lemma, we then have $|w(z)| \le |z|^n$. Further, it is easy to obtain that (see [17])

$$\left(\frac{z}{f(z)}\right)^{\mu} = 1 - \lambda\int_0^1 \frac{w(s^{1/\mu}z)}{s^2}\,ds. \tag{3.2}$$

Now, we define

$$P(z) = z\left[\int_0^1 \varphi(t)\left(\frac{tz}{f(tz)}\right)^{\mu} dt\right].$$

Then we have

$$P(z) = z\left(\frac{[V_\varphi^\mu(f)](z)}{z}\right)^{\mu} \tag{3.3}$$

where $V_\varphi^\mu(f)$ is given by (1.6). In view of the representation (3.2), we may write

$$\begin{aligned} P(z) &= z\int_0^1 \varphi(t)\left[1 - \lambda\int_0^1 \frac{w(s^{1/\mu}tz)}{s^2}\,ds\right]dt \\ &= z - z\lambda\int_0^1\int_0^1 \varphi(t)\frac{w(s^{1/\mu}tz)}{s^2}\,ds\,dt. \qquad (3.4) \end{aligned}$$

By (3.4), we note that

$$\begin{aligned}\left|\frac{P(z)}{z}-1\right| &= \lambda\left|\int_0^1\int_0^1 \varphi(t)\frac{w(s^{1/\mu}tz)}{s^2}\,ds\,dt\right|\\ &\le \lambda\int_0^1 \varphi(t)t^n\left(\int_0^1 s^{(n/\mu)-2}\,ds\right)dt\\ &\le \frac{\lambda\mu}{n-\mu}\int_0^1 \varphi(t)t^n\,dt\\ &< \lambda\le 1, \quad \text{by the condition (2.2) on } \lambda,\end{aligned}$$

which shows that $P(z)/z$ is nonvanishing for $|z|<1$. In particular, $P(z)$ is analytic in Δ and has a simple zero only at the origin and nowhere else. Taking the logarithmic differentiation of (3.3) and taking into account of the proper branch, it is clear that

$$\frac{z[V_\varphi^\mu(f)]'(z)}{[V_\varphi^\mu(f)](z)}-1=\frac{1}{\mu}\left[\frac{zP'(z)}{P(z)}-1\right].$$

Hence, to complete the proof, it suffices to obtain the required conclusion about the function P. Now, differentiating (3.4) with respect to z, we get

$$P'(z)=1-\lambda\mu\int_0^1 \varphi(t)w(tz)\,dt-\lambda(\mu+1)\int_0^1\int_0^1 \frac{w(s^{1/\mu}tz)}{s^2}\,ds\,dt. \tag{3.5}$$

From (3.4) and (3.5), it follows that

$$\frac{zP'(z)}{P(z)}-1=-\frac{\lambda\mu\displaystyle\int_0^1 \varphi(t)w(tz)\,dt+\lambda\mu\int_0^1\int_0^1 \varphi(t)\frac{w(s^{1/\mu}tz)}{s^2}\,ds\,dt}{1-\lambda\displaystyle\int_0^1\int_0^1 \varphi(t)\frac{w(s^{1/\mu}tz)}{s^2}\,ds\,dt}.$$

Again, as $|w(z)|\le |z|^n$, we see that

$$\begin{aligned}\frac{1}{\mu}\left|\frac{zP'(z)}{P(z)}-1\right| &< \frac{n\lambda\displaystyle\int_0^1 t^n\varphi(t)\,dt}{n-\mu-\lambda\mu\displaystyle\int_0^1 t^n\varphi(t)\,dt}\\ &\le 1-\alpha, \quad \text{by (2.2)}.\end{aligned}$$

This gives the required conclusion. □

3.6. Proof of Theorem 2.6. Let $f \in \mathcal{U}_n(\lambda, n)$. Then, it is a simple exercise to see that

$$f'(z)\left(\frac{z}{f(z)}\right)^{n+1} = 1 + a_{n+2}z^{n+1} + \cdots = 1 + \lambda w(z),$$

for some $w \in \mathcal{B}_{n+1}$. By the Schwarz lemma we then have $|w(z)| \leq |z|^{n+1}$. It follows easily that

$$\left(\frac{z}{f(z)}\right)^n = 1 - na_{n+1}z^n - \lambda \int_0^1 \frac{w(s^{1/n}z)}{s^2}\, ds.$$

Indeed, since $f(z) \neq 0$ in $0 < |z| < 1$, we see that

$$\left(\frac{z}{f(z)}\right)^n - \left(\frac{z}{f(z)}\right)^{n-1}\left[-\left(\frac{z}{f(z)}\right)^2 f'(z) + \frac{z}{f(z)}\right] = \left(\frac{z}{f(z)}\right)^{n+1} f'(z).$$

By hypothesis, we can write

$$\left(\frac{z}{f(z)}\right)^n - \left(\frac{z}{f(z)}\right)^{n-1}\left[-\left(\frac{z}{f(z)}\right)^2 f'(z) + \frac{z}{f(z)}\right] = 1 + \lambda w(z), \tag{3.7}$$

where $w \in \mathcal{B}_{n+1}$. Suppose that

$$\left(\frac{z}{f(z)}\right)^n = 1 + \sum_{k=n}^{\infty} p_k z^k \quad \text{and} \quad w(z) = \sum_{k=n+1}^{\infty} b_k z^k.$$

Then

$$\left(\frac{z}{f(z)}\right)^n - \left(\frac{z}{f(z)}\right)^{n-1}\left[-\left(\frac{z}{f(z)}\right)^2 f'(z) + \frac{z}{f(z)}\right] = 1 + \sum_{k=n+1}^{\infty}\left(1 - \frac{k}{n}\right) p_k z^k.$$

A comparison of the coefficient of z^k on both sides of (3.7) shows that

$$\left(1 - \frac{k}{n}\right) p_k = \lambda b_k \quad (k \geq n+1)$$

so that

$$\left(\frac{z}{f(z)}\right)^n = 1 - na_{n+1}z^n + \lambda \sum_{k=n}^{\infty} \frac{b_k}{1 - k/n} z^k.$$

We can rewrite the last equality in integral form

$$\left(\frac{z}{f(z)}\right)^n = 1 - na_{n+1}z^n - \lambda \int_0^1 \frac{w(t^{1/n}z)}{t^2}\, dt.$$

Now, as in the proof of Theorem 2.1, define

$$P(z) = z\left[\int_0^1 \varphi(t)\left(\frac{tz}{f(tz)}\right)^n dt\right], \quad f \in \mathcal{U}_n(\lambda, n).$$

Therefore,

$$P(z) = z\int_0^1 \varphi(t)\left[1 - na_{n+1}t^n z^n - \lambda\int_0^1 \frac{w(s^{1/n}tz)}{s^2}\,ds\right]dt$$

$$= z - na_{n+1}z^{n+1}\int_0^1 t^n\varphi(t)\,dt - z\lambda\int_0^1\int_0^1 \varphi(t)\frac{w(s^{1/n}tz)}{s^2}\,ds\,dt. \tag{3.8}$$

By hypothesis, we see that $\mathrm{Re}\,(P(z)/z) > 0$ for all $z \in \Delta$ and hence $P(z)/z$ has no zeros in Δ. Indeed, as $|w(z)| \leq |z|^{n+1}$ for $z \in \Delta$, one has

$$\begin{aligned}\left|\frac{P(z)}{z} - 1\right| &\leq n|a_{n+1}|\int_0^1 t^n\varphi(t)\,dt + \lambda\int_0^1\int_0^1 \varphi(t)s^{1/n-1}t^{n+1}\,ds\,dt\\ &= n|a_{n+1}|\int_0^1 t^n\varphi(t)\,dt + n\lambda\int_0^1 t^{n+1}\varphi(t)\,dt\\ &< \lambda \leq 1, \quad \text{by (2.7).}\end{aligned}$$

Then, from (1.6) with $\mu = n$, we see that

$$P(z) = z\left(\frac{[V_\varphi^n(f)](z)}{z}\right)^n.$$

Taking logarithmic differentiation, it is clear that

$$\frac{z[V_\varphi^n(f)]'(z)}{[V_\varphi^n(f)](z)} - 1 = \frac{1}{n}\left[\frac{zP'(z)}{P(z)} - 1\right]. \tag{3.9}$$

Differentiating both sides of (3.8) with respect to z, we get

$$\begin{aligned}P'(z) &= 1 - n(n+1)a_{n+1}z^n\int_0^1 t^n\varphi(t)\,dt - \lambda n\int_0^1 \varphi(t)w(tz)\,dt\\ &\quad -\lambda(n+1)\int_0^1\int_0^1 \varphi(t)\frac{w(s^{1/n}tz)}{s^2}\,ds\,dt.\end{aligned} \tag{3.10}$$

From (3.8) and (3.10), we can see easily that

$$\frac{zP'(z)}{P(z)} - 1 =$$

$$-\frac{n^2a_{n+1}z^n\int_0^1 t^n\varphi(t)\,dt + \lambda n\int_0^1 \varphi(t)w(tz)\,dt + \lambda n\int_0^1\int_0^1 \varphi(t)[w(s^{1/n}tz)/s^2]\,ds\,dt}{1 - na_{n+1}z^n\int_0^1 t^n\varphi(t)\,dt - \lambda\int_0^1\int_0^1 \varphi(t)[w(s^{1/n}tz)/s^2]\,ds\,dt}.$$

Therefore, using the fact that $|w(z)| \leq |z|^{n+1}$, we get

$$\begin{aligned}\frac{1}{n}\left|\frac{zP'(z)}{P(z)} - 1\right| &< \frac{n|a_{n+1}|\int_0^1 t^n\varphi(t)\,dt + (n+1)\lambda\int_0^1 t^{n+1}\varphi(t)\,dt}{1 - n|a_{n+1}|\int_0^1 t^n\varphi(t)\,dt - \lambda n\int_0^1 t^{n+1}\varphi(t)\,dt}\\ &\leq 1 - \alpha, \quad \text{by (2.7).}\end{aligned}$$

By (3.9), the proof is complete. $\square$

4. Special Integral Transforms

Our applications in Section 4 are stated in terms of the Hadamard product

$$(\phi * \psi)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \quad (|z| < 1)$$

of analytic functions $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\psi(z) = \sum_{n=0}^{\infty} b_n z^n$ in the unit disk Δ. We consider certain special integral transforms that can be represented as a Hadamard product with Gaussian hypergeometric functions, polylogarithms and Hurwitz functions. These functions are defined as follows:

(i) For $a > -1$, $b > -1$ and $b > a$, define

$$G(a, b; z) = \sum_{n=1}^{\infty} \frac{(1+a)(1+b)}{(n+a)(n+b)} z^n = z \int_0^1 \frac{\varphi(t)}{1-tz}\, dt$$

where

$$\varphi(t) = \begin{cases} (a+1)(b+1)\dfrac{t^a - t^b}{b-a}, & \text{for } b \neq a \\ (a+1)^2 t^a \log(1/t), & \text{for } b = a. \end{cases} \tag{4.1}$$

Because of the symmetry in (4.1), without loss of generality, we may assume that $b > a$ in this case.

(ii) For $p \geq 0$ and $a > -1$, define

$$\Phi_p(a; z) = \sum_{n=1}^{\infty} \frac{(1+a)^p}{(n+a)^p} z^n = z \int_0^1 \frac{\varphi(t)}{1-tz}\, dt$$

where

$$\varphi(t) = \frac{(1+a)^p}{\Gamma(p)} \left(\log \frac{1}{t}\right)^{p-1} t^a. \tag{4.2}$$

(iii) For $a > 0$, $b > 0$ and $c+1 > a+b$, we consider the integral representation proved in [2] for the classical Hypergeometric function $F(a, b; c; z)$:

$$zF(a, b; c; z) = z \int_0^1 \frac{\varphi(t)}{1-tz}\, dt$$

where

(4.3)

$$\varphi(t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} t^{b-1}(1-t)^{c-a-b} F\left(\begin{matrix} c-a, 1-a \\ c-a-b+1 \end{matrix}; 1-t\right).$$

We note that each $\varphi(t)$ satisfies the desired normalization conditions of Problem 1.5. Also, our choices of $\varphi(t)$ above imply that for $f \in \mathcal{A}$,

$$V_\varphi(f(z)) = z\int_0^1 \varphi(t)\frac{tz}{f(tz)}\,dt = \frac{z^2}{f(z)} * \Theta(z),$$

where Θ is either $G(a,b;z)$ or $\Phi_p(a;z)$ or $zF(a,b;c;z)$. Thus, we have the following special transformations in the settings of Problem 1.5.

Now we are in a position to use these three special transformations and state the following results without proof.

Theorem 4.4. *Let $f \in \mathcal{U}_n(\lambda,n)$ $(n \geq 1)$. If $a,b > -1$, $b \geq a$ and where $\varphi(t)$ is given by* (4.1)*, then G defined by*

$$G(z) := G(a,b;z) * \frac{z^2}{f(z)} = z\int_0^1 \varphi(t)\frac{tz}{f(tz)}\,dt \tag{4.5}$$

belongs to $\mathcal{S}^*_\alpha$ whenever λ, a, b and $\alpha < 1$ are related by the inequality

$$0 < \lambda \leq \frac{(1-\alpha) - n(2-\alpha)|a_{n+1}|\,|A_{n+1}|}{(2n+1-n\alpha)|A_{n+2}|}$$

where

$$A_{n+1} = \frac{(a+1)(b+1)}{(a+n+1)(b+n+1)}.$$

Theorem 4.6. *Let $f \in \mathcal{U}_n(\lambda,n)$ $(n \geq 1)$, $\alpha < 1$. If $p \geq 0$, and $\varphi(t)$ is given by* (4.2)*, then Φ defined by*

$$\Phi(z) := \Phi_p(a;z) * \frac{z^2}{f(z)} = z\int_0^1 \varphi(t)\frac{tz}{f(tz)}\,dt \tag{4.7}$$

belongs to $\mathcal{S}^*_\alpha$ whenever λ, a, p and $\alpha < 1$ satisfy the inequality

$$\lambda \leq \frac{(1-\alpha) - n(2-\alpha)|a_{n+1}|\,|A_{n+1}|}{(2n+1-n\alpha)|A_{n+2}|},$$

where

$$A_{n+1} = \frac{(a+1)^p}{(a+n+1)^p}.$$

Theorem 4.8. *Let $f \in \mathcal{U}_n(\lambda,n)$ $(n \geq 1)$. If $a > 0$, $b > 0$, $c+1 > a+b$, and $\varphi(t)$ is given by (4.3), then H defined by*

$$H(z) := zF(a,b;c;z) * \frac{z^2}{f(z)} = z\int_0^1 \varphi(t)\frac{tz}{f(tz)}\,dt \tag{4.9}$$

belongs to $\mathcal{S}^*_\alpha$ whenever λ, a, b, c and $\alpha < 1$ satisfy the inequality

$$\lambda \leq \frac{(1-\alpha) - n(2-\alpha)|a_{n+1}|\,|A_{n+1}|}{(2n+1-n\alpha)|A_{n+2}|}$$

where

$$A_{n+1} = \frac{(a)_n(b)_n}{(c)_n n!}.$$

Remark 4.10. It is interesting to note that all the operators coincide in some special cases. For example $p = 1$ in Case (ii) is the same as Case (iii) if we set $a = 1$ and then replace b and c by $a + 1$ and $a + 2$, respectively. Similarly, the limiting case $b \to \infty$ in Case (i) is equivalent to $p = 1$ in Case (ii). □

As indicated in Remark 4.10, the following corollary follows from either of these three Theorems.

Corollary 4.11. [13] *Let $f \in \mathcal{U}_n(\lambda, n)$ with $\alpha < 1$, $\lambda > 0$, $c > 0$, $n \geq 1$ and let F_c be defined by*

$$F_c(z) = \frac{c}{z^{c-1}} \int_0^z \frac{\zeta^c}{f(\zeta)}\, d\zeta. \tag{4.12}$$

Then $F_c \in \mathcal{S}^*_\alpha$, in particular, $F_c \in \mathcal{S}^*(\alpha)$, whenever c, λ are related by

$$0 < \lambda \leq \frac{c+n+1}{c+n} \left(\frac{(1-\alpha)(c+n) - |a_{n+1}|n(2-\alpha)c}{c(2n+1-n\alpha)} \right).$$

The case $n = 1$ of Corollary 4.11 gives a result of the authors in [17].

Similarly, for $\mu = 1$ and $\varphi(t)$ defined by (4.1), (4.2) and (4.3), we have the following special case of Theorem 2.8. These theorems can also be obtained from Theorems 4.4, and 4.8, respectively.

Theorem 4.13. *Let $\lambda > 0$, $a > -1$, $b > -1$, $\alpha < 1$ and $b \geq a$. If $f \in \mathcal{U}(\lambda)$, then G defined by (4.5) belongs to $\mathcal{S}^*_\alpha$ whenever λ, a, b and $\alpha < 1$ are related by the inequality $0 < \lambda \leq \lambda_0$, where*

$$\lambda_0(a, b) = \frac{[(1-\alpha) - (2-\alpha)|A_2|](a+3)(b+3)}{(3-\alpha)(a+1)(b+1)}.$$

Here $A_2 = a_2 \dfrac{(a+1)(b+1)}{(a+2)(b+2)}$.

In particular, we have

(i) $G \in \mathcal{S}^*$ *if* $f \in \mathcal{U}(\lambda)$ *with* $0 < \lambda \leq \dfrac{(1-2|A_2|)(a+3)(b+3)}{3(a+1)(b+1)}$.

(ii) $G \in \mathcal{S}^*$ *if* $f \in \mathcal{U}_2(\lambda) := \mathcal{U}(\lambda) \cap \mathcal{A}_2$ *with* $0 < \lambda \leq \dfrac{(a+3)(b+3)}{3(a+1)(b+1)}$.

Theorem 4.14. *Let $\lambda > 0$, $a > -1$ and $p > 0$. If $f \in \mathcal{U}(\lambda)$, then Φ defined by* (4.7) *belongs to $\mathcal{S}^*_\alpha$ whenever λ, a, p and $\alpha < 1$ satisfy the inequality*

$$\lambda \leq \frac{[(1-\alpha)-(2-\alpha)|A_2|](a+3)^p}{(3-\alpha)(a+1)^p}, \quad \text{where} \quad A_2 = a_2\left(\frac{a+1}{a+2}\right)^p.$$

In particular,

(i) $\Phi \in \mathcal{S}^*$ *if* $f \in \mathcal{U}(\lambda)$ *with* $0 < \lambda \leq \dfrac{(1-2|A_2|)(a+3)^p}{3(a+1)^p}$.

(ii) $\Phi \in \mathcal{S}^*$ *if* $f \in \mathcal{U}_2(\lambda)$ *with* $0 < \lambda \leq \dfrac{1}{3}\left(\dfrac{a+3}{a+1}\right)^p$.

Theorem 4.15. *Let $\alpha < 1$, $a > 0$, $b > 0$ and $c + 1 > a + b$. If $f \in \mathcal{U}(\lambda)$, then H defined by* (4.9) *belongs to $\mathcal{S}^*_\alpha$ whenever λ, a, b, c and $\alpha < 1$ satisfy the inequality*

$$\lambda \leq \frac{[(1-\alpha)-(2-\alpha)|A_2|]2c(c+1)}{(3-\alpha)a(a+1)b(b+1)},$$

where $A_2 = a_2\,(ab/c)$. In particular, we have the following:

(i) $H \in \mathcal{S}^*$ *if* $f \in \mathcal{U}(\lambda)$ *with* $0 < \lambda \leq \dfrac{(1-2|A_2|)2c(c+1)}{3a(a+1)b(b+1)}$.

(ii) $H \in \mathcal{S}^*$ *if* $f \in \mathcal{U}_2(\lambda)$ *with* $0 < \lambda \leq \dfrac{2c(c+1)}{3a(a+1)b(b+1)}$.

In the special case $n = 1$, $\alpha = 0$ and $\mu = 1$, Theorems 4.4, 4.6, and 4.8 coincide with Theorems 4.13(i), 4.14(i), 4.15(i), respectively.

For $\varphi(t)$ as in (4.1), (4.2) and (4.3), we have the following special case of Theorem 2.1 in case of missing Taylor coefficients.

Theorem 4.16. *For $\mu < n$, let $f \in \mathcal{U}_n(\lambda, \mu)$. Then G defined by* (4.5) *belongs to $\mathcal{S}^*_\alpha$ whenever λ, a, b and $\alpha < 1$ are related by the inequality*

$$\lambda \leq \frac{(1-\alpha)(n-\mu)(a+n+1)(b+n+1)}{(a+1)(b+1)(n+\mu(1-\alpha))}.$$

Theorem 4.17. *For $\mu < n$, let $f \in \mathcal{U}_n(\lambda, \mu)$, then Φ defined by* (4.7) *belongs to $\mathcal{S}^*_\alpha$ whenever λ, a, p and $\alpha < 1$ satisfy the inequality*

$$\lambda \leq \frac{(1-\alpha)(n-\mu)(a+n+1)^p}{(n+\mu(1-\alpha))(a+1)^p}.$$

Theorem 4.18. *For $\mu < n$, let $f \in \mathcal{U}_n(\lambda, \mu)$, then H defined by* (4.9) *belongs to $\mathcal{S}^*_\alpha$ whenever λ, a, b, c and $\alpha < 1$ satisfy the inequality*

$$\lambda \leq \frac{(1-\alpha)(n-\mu)(c)_n n!}{(n+\mu(1-\alpha))(a)_n(b)_n}.$$

In particular, when $n = 2$, $\alpha = 0$ and $\mu = 1$, Theorems 4.16, 4.17 and 4.18 imply Theorems 4.13(ii), 4.14(ii) and 4.15(ii), respectively, but not necessarily the converse.

As indicated in Remark 4.10, the following corollary follows from one of these two Theorems above.

Corollary 4.19. [13] *For $\mu < n$, let $f \in \mathcal{U}_n(\lambda, \mu)$ with $\alpha < 1$, $\lambda > 0$, $c > 0$ and let F_c be defined by* (4.12). *Then, $F_c \in \mathcal{S}^*_\alpha$, and in particular, $F_c \in \mathcal{S}^*(\alpha)$, whenever c, λ are related by*

$$0 < \lambda \leq \frac{(1-\alpha)(c+n)(n-\mu)}{c(n+1-\alpha)}.$$

For $\mu = 1$, one has corresponding results when $f \in \mathcal{U}_n(\lambda)$ with $n > 1$.

5. Conclusion

In this paper the problem of starlikeness of $V^\mu_\varphi(f)$ has been solved when $f \in \mathcal{U}(\lambda, \mu)$. The problem of convexity is still open except for the transformation F_c defined by (4.12), see [16]. Moreover, the transformation $V^\mu_\varphi(f)$ has not been studied for other subfamilies of the family of analytic functions in the unit disk.

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S. Ponnusamy E-mail: samy@iitm.ac.in
Address:
Department of Mathematics,
Indian Institute of Technology Madras
Chennai- 600 036, India

P. Vasundhra E-mail: vasu2kk@yahoo.com
Address:
Department of Mathematics,
Indian Institute of Technology Madras
Chennai- 600 036, India