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Neighbourhoods of a certain subclass of $SP(\beta)$

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Abstract. The aim of this paper is to introduce the class $SP_s(\beta)$ which is a subclass of $SP(\beta)$ $(0 < \beta < \infty)$ satisfying the condition

$$\left|\frac{2zf'(z)}{f(z)-f(-z)} - \beta\right| \le \mathrm{Re}\left\{\frac{2zf'(z)}{f(z)-f(-z)}\right\} + \beta,$$

for all $z \in E = \{z : |z| < 1\}$. We study neighbourhoods of this class and also prove a necessary and sufficient condition in terms of convolutions for a function f to be in $SP_s(\beta)$. Further more, it is shown that the class $SP_s(\beta)$ is closed under convolution with functions f which are convex univalent in E.

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1. Introduction

Let $\mathcal{A}$ denote the class of function f analytic in the unit disc $E = \{z : |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$. Any $f \in \mathcal{A}$ has the Taylor's expansion $f(z) = z + a_2z^2 + \cdots$ in E. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ that are univalent in E. Let CV and ST denote the subclasses of $\mathcal{S}$ consisting of convex and starlike functions respectively. The convolution or Hadamard product of

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

It is observed that

$$f(z) * \frac{z}{(1-z)^2} = zf'(z) \quad \text{and} \quad f(z) * \frac{z}{(1-z^2)} = \frac{[f(z)-f(-z)]}{2}.$$

Goodman [1, 2] defined the following subclasses of CV and ST as follows.

Definition A. A function f is uniformly convex (starlike) in E if f is in $CV(ST)$ and has the property that for every circular arc γ with centre at ς contained in E then the arc $f(\gamma)$ is convex (starlike w.r.t. $f(\varsigma)$).

Goodman [1, 2] gave the following two variable analytic characterizations of these classes denoted respectively by UCV and UST.

Theorem A. A function f in $\mathcal{A}$ is in UCV if and only if

$$\operatorname{Re}\left\{1+(z-\varsigma)\frac{f''(z)}{f'(z)}\right\}>0, \quad z\neq\varsigma$$

for all z and ς is in E.

Theorem B. A function f in A is in UST if and only if

$$\operatorname{Re}\left\{\frac{f(z)-f(\varsigma)}{(z-\varsigma)f'(z)}\right\}>0$$

for every pair z, ς lying in E.

It is clear that if $f\in UST$, then $f\in ST$, the class of starlike univalent functions and if $f\in UCV$, then $f\in CV$, the class of convex univalent functions. However, the classical Alexander Theorem that $f\in CV$ if and only if $zf'(z)$ is in ST is no longer hold between the classes UCV and UST. Ronning [6] defined a subclass of starlike functions S_p with the property that a function $f\in UCV$ if and only $zf'\in S_p$.

Ma and Minda [3] and Ronning [6], independently found a more applicable one variable characterization for UCV.

Theorem C. A function f in $\mathcal{A}$ is in UCV if and only if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\geq\left|\frac{zf''(z)}{f'(z)}\right|, \quad z\in E.$$

Ronning [6] proved a one-variable characterization for S_p as follows.

Theorem D. A function f in $\mathcal{A}$ is in S_p if and only if

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}, \quad z\in E.$$

Ronning [7] generalized the class S_p by introducing a parameter β and defined the class $S_p(\beta)$ by

$$\left|\frac{zf'(z)}{f(z)}-\beta\right|\leq\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}+\beta, \quad 0<\beta<\infty.$$

The notion of δ-neighbourhood was first introduced by Ruscheweyh [8].

Definition B. For $\delta\geq 0$ the δ-neighbourhood of $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ analytic in E is defined by

$$N_\delta(f)=\left\{g(z)=z+\sum_{n=2}^{\infty}b_nz^n: \quad \sum_{n=2}^{\infty}n|a_n-b_n|\leq\delta\right\}.$$

T-δ neighborhood was given by Sheil-Small and Silvia [10] as.

Definition C. For $\delta \geq 0$, and $T = \{t_n\}_{n=2}^{\infty}$ a sequence of non negative reals a T-δ neighbourhood of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in E is defined by

$$TN_\delta(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n : \quad \sum_{n=2}^{\infty} t_n |a_n - b_n| \leq \delta \right\}.$$

Padmanabhan [4] has introduced the neighbourhoods of functions in the class S_p. Parvatham and Premabai [5] has introduced the following class of functions SP_s and studied T-δ, neighborhoods in this class

Definition D. Any function $f \in \mathcal{A}$ is said to be in the class SP_s of uniformly starlike with respect to symmetric points if for all $z \in E$.

$$\left| \frac{zf'(z)}{f(z) - f(-z)} - 1 \right| \leq \text{Re } \left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\}.$$

In this paper we introduce a new class of functions and study properties of T-δ neighbourhoods of functions in this class.

Definition 1.1. A function $f(z)$ in $\mathcal{A}$ is said to be in the class $SP_s(\beta)$ if for all $z \in E$,

$$\left| \frac{2zf'(z)}{f(z) - f(-z)} - \beta \right| \leq \text{Re } \left\{ \frac{2zf'(z)}{f(z) - f(-z)} \right\} + \beta \quad \text{for } 0 < \beta < \infty.$$

This implies that $f \in SP_s(\beta)$ if and only if $(2zf'(z))/(f(z) - f(-z))$ lies in the region Ω bounded by a parabola with vertex at the origin and parameterized by $(t^2 + 4i\beta t)/4\beta$ for any real t. It is known. Ronning [7] shown that the function

$$Q_\beta(z) = \beta \left[1 + \frac{4}{\pi^2} \left(\log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2 \right]$$

maps the unit disc E onto the parabolic region Ω. (The branch of square root is chosen so that $\text{Im} \sqrt{z} \geq 0$). Then from the above definition $f \in \mathcal{A}$ is in the class $SP_s(\beta)$ if and only if

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec Q_\beta(z),$$

where $\prec$ denotes subordination.

First let us state a Lemma due to Ruscheweyh and Sheil-Small [9], which is needed to establish our results.

Lemma A. If ϕ is convex univalent function with $\phi(0) = 0 = \phi'(0) - 1$ in the unit disc E and g is starlike univalent in E, then for any analytic function F in E, the image of E under $\frac{\phi * Fg(z)}{\phi * g(z)}$ is a subset of the convex hull of $F(E)$.

Definition 1.2. Let $SP'_s(\beta)$ be the class of all functions $h_\beta(z)$ in $\mathcal{A}$ of the form

$$h_\beta(z) = \frac{4\beta}{4\beta - 4i\beta t - t^2}\left[\frac{z}{(1-z)^2} - \frac{t^2+4i\beta t}{4\beta}\frac{z}{(1-z^2)}\right].$$

We now give the characterization for function f in $\mathcal{A}$ to be in $SP_s(\beta)$.

Theorem 1.3. *A function f in $\mathcal{A}$ is in $SP_s(\beta)$ if and only if for all z in E for all*

$$h_\beta \in SP's(\beta) \quad and \quad \frac{f * h_\beta(z)}{z} \neq 0.$$

Proof. Let us assume that for $f \in \mathcal{A}$ and $\frac{f*h_\beta(z)}{z} \neq 0$, for all $h_\beta \in SP'_s(\beta)$ and for $z \in E$. From the definition of $h_\beta(z)$ it follows that

$$\frac{f*h_\beta(z)}{z} = \frac{4\beta}{4\beta - 4\beta it - t^2}\left[zf'(z) - \frac{t^2+4\beta it}{4\beta}\left[\frac{f(z)-f(-z)}{2}\right]\right] \neq 0$$

or equivalently

$$\frac{2zf'(z)}{f(z)-f(-z)} \neq \frac{t^2+4\beta it}{4\beta}, \text{ for } t \in \mathbb{R}.$$

This means that $\frac{2zf'(z)}{f(z)-f(-z)}$ lies completely either inside Ω_β or complement of Ω_β for all z in E. At $z=0$, $\frac{2zf'(z)}{f(z)-f(-z)}$ $=1 \in \Omega_\beta$. So that $\frac{2zf'(z)}{f(z)-f(-z)} \subset \Omega_\beta$, which shows that $f \in SP_s(\beta)$.

Conversely let $f \in SP'_s(\beta)$. Hence $\frac{2zf'(z)}{f(z)-f(-z)}$ lies with in the parabola with vertex at the origin and where boundary is given by $\frac{t2+4i\beta t}{4\beta}$, for $t \in \mathbb{R}$.

$$f \in SP_s(\beta) \text{ only if } \frac{2zf'(z)}{f(z)-f(-z)} \neq \frac{t^2+4\beta it}{4\beta},$$

or equivalently

$$f(z) * \left[\frac{z}{(1-z)^2} - \frac{t^2+4\beta it}{4\beta}\left(\frac{z}{1-z^2}\right)\right] \neq 0, \ \forall\ z \in E - \{0\}.$$

Normalizing the function with in the brackets we get

$$\frac{f(z)*h_\beta(z)}{z} \neq 0, \quad z \in E$$

where $h_\beta(z)$ is the function in Definition 1.2. ∎

In order to establish the T-δ neighbourhoods of functions belonging to the class $SP_s(\beta)$ we need the following Lemmas.

Lemma 1.4. *Let $h_\beta(z) = z + \sum_{k=2}^{\infty} c_k z^k \in SP'_s(\beta)$, then $|c_k| \leq k, k = 2, 3, \ldots$.*

Proof. Let $h_\beta(z) \in SP_s'(\beta)$, then for any real t.

$$\begin{aligned} h_\beta(z) &= \frac{4\beta}{4\beta - 4\beta it - t^2}\left[\frac{z}{(1-z)^2} - \frac{t^2+4\beta it}{4\beta}\left(\frac{z}{1-z^2}\right)\right] \\ &= \frac{4\beta}{4\beta - 4\beta it - t^2}\left[(z+2z^2+\cdots) - \frac{t^2+4\beta it}{4\beta}\left(z+z^3+\cdots\right)\right] \\ &= z + \sum_{k=2}^{\infty} c_k z^k, \end{aligned}$$

where

$$c_k = \begin{cases} \dfrac{4\beta k}{4\beta - 4\beta it - t^2} & \text{when } k \text{ is even,} \\ \dfrac{4\beta k - 4\beta it - t^2}{4\beta - 4\beta it - t^2} & \text{when } k \text{ is odd.} \end{cases}$$

and so

$$\begin{aligned} |c_k|^2 &\le \frac{(4k\beta - t^2)^2 + 16\beta^2 t^2}{(4\beta+t^2)^2} \qquad \textit{if } \beta \ge 1 \\ &= \frac{16\beta^2k^2 - 8k\beta t^2 + t^4 + 16\beta^2t^2}{(4\beta+t^2)^2} \\ &= \frac{16\beta^2(k+1)(k-1) - 8\beta t^2)k + 1 - 2\beta)}{(4\beta+t^2)^2} + 1 \\ &\le \frac{16\beta^2(k-1)(k+1-2\beta) + 32\beta^3(k-1)}{16\beta^2} + 1 \\ &= k^2. \end{aligned}$$

Hence, $|c_k| \le k \quad \forall\, k \ge 2$. ■

Lemma 1.5. *For $f \in \mathcal{A}$ and for every $\epsilon \in \mathbb{C}$ such that $|\epsilon| < \delta$ if*

$$F_\epsilon(z) = \left\{\frac{f(z)+\epsilon z}{1+\epsilon}\right\} \in SP_s(\beta),$$

then for every $h_\beta \in SP_s'(\beta)$ implies $\left|\dfrac{f * h_\beta(z)}{z}\right| \neq \delta, \quad \forall\, z \in E.$

Proof. Let $F_\epsilon(z) \in SP_s(\beta)$, then by Theorem 1.3

$$(f * h_\beta(z))/z \neq 0, \quad \forall\, h_\beta \in SP_s'(\beta)$$

and $z \in E$. Equivalently

$$\frac{f * h_\beta(z) + \varepsilon z}{(1+\varepsilon)(z)} \neq 0 \ \text{ or } \ \frac{f * h_\beta(z)}{z} \neq -\varepsilon.$$

Hence

$$\left|\frac{f * h_\beta(z)}{z}\right| \geq \delta, \quad \forall\, z \in E.$$

■

Theorem 1.6. *Let $f \in \mathcal{A}$ and $\epsilon \in \mathbb{C}$ and for $|\epsilon| < \delta < 1$, if $F\epsilon(z) \in SP_s(\beta)$. Then $TN_\delta(f) \subset SP_s(\beta)$.*

Proof. Let $h_\beta \in SP'_s(\beta)$ and $g(z) = z + \sum_{m=2}^{\infty} b_n z^n$ is in $TN_\delta(f)$. Then

$$\begin{aligned}\left|\frac{(g * h_\beta)(z)}{z}\right| &= \left|\frac{(f * h_\beta)(z)}{z} + \frac{(g-f) * h_\beta)(z)}{z}\right| \\ &\geq \left|\frac{(f * h_\beta)(z)}{z}\right| - \left|\frac{(g-f)(z) * h_\beta(z)}{z}\right| \\ &\geq \delta - \left|\sum_{n=2}^{\infty} \frac{(b_n - a_n)c_n z^n}{z}\right|.\end{aligned}$$

Hence, by Lemma 1.5 we have

$$\begin{aligned}\left|\frac{g * h_\beta(z)}{z}\right| &\geq \delta - |z| \sum_{n=2}^{\infty} |c_n|\,|b_n - a_n| \\ &\geq \delta - \sum_{n=2}^{\infty} t_n |b_n - a_n| \\ &> \delta - \delta = 0, \quad \text{by Lemma 1.4.}\end{aligned}$$

Thus

$$(g * h_\beta(z))/z \neq 0,$$

for all $z \in E$ and for all $h_\beta \in SP'_s(\beta)$. Also by Theorem 1.3, we have $g \in SP_s(\beta)$. Hence $TN_\delta(f) \subset SP_s(\beta)$. ■

Next we will prove that the class $SP_s(\beta)$ is closed under convolution with functions of which are convex univalent in E.

Lemma 1.7. *If $g \in SP_s(\beta)$ then $G(z) \in SP_s(\beta) \subset ST$ where*

$$G(z) = \frac{g(z) - g(-z)}{2}.$$

Proof. Since $g \in SP_s(\beta)$, $\frac{2zg'(z)}{g(z)-g(-z)} \in \Omega_\beta$. Now

$$\frac{zG'(z)}{G(z)} = \frac{zg'(z)}{2G(z)} + \frac{(-z)g'(-z)}{2G(-z)} = \frac{\varsigma_1}{2} + \frac{\varsigma_2}{2} = \varsigma_3,$$

where ς_1 and $\varsigma_2 \in \Omega_\beta$.

Since Ω_β is convex $\varsigma_3 \in \Omega_\beta$ and hence $zG'(z)/G(z) \in \Omega_\beta$ it can be easily seen that $SP_s(\beta) \subset ST$. Thus $G(z) \in SP_s(\beta) \subset ST$. ■

Theorem 1.8. *Let $f(z) \in CV$ the class of convex functions and $g(z) \in SP_s(\beta)$. Then $(f * g)(z) \in SP_s(\beta)$.*

Proof. Let $f(z) \in CV$ and $g(z) \in SP_s(\beta)$, $G(z) = (g(z) - g(-z))/2$ and Ω_β is a convex domain. Since $g \in SP_s(\beta)$, $G(z) \in ST$ by Lemma 1.7. Hence by an application of Lemma A we get

$$\frac{z(f*g)'(z)}{(f*G)(z)} = \frac{(f*zg')(z)}{(f*G)(z)} = \frac{f * \frac{zg'(z)}{G(z)}.G(z)}{(f*G)(z)} \subset \overline{Co}\left(\frac{zg'(z)}{G(z)}\right) \subset \Omega_\beta.$$

Since Ω_β is convex and $g \in SP_s(\beta)$. This proves that $(f * g)(z) \in SP_s(\beta)$. ■

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