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Some Properties of Prestarlike and Universally Prestarlike Functions

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Abstract. The classes $\mathcal{R}_\alpha^{\mathrm{u}}$ of normalized *universally prestarlike* functions of order $\alpha \le 1$ (in the slit domain $\mathbb{C} \setminus [1, \infty]$) have recently been introduced in [5]. In this note we show that, except for certain Moebius transforms, there are no rational functons in $\mathcal{R}_\alpha^{\mathrm{u}}, \alpha < 1$. A consequence of this is that there are no numbers $t = t_u(\alpha) > 0$ such that $f \in \mathcal{R}_1^{\mathrm{u}}$ implies that $f(tz)/t \in \mathcal{R}_\alpha^{\mathrm{u}}$. This is in sharp contrast to the situation with the (classical) prestarlike functions in the unit disc where such numbers $t(\alpha) > 0$ exist (and will be determined in this paper).

Keywords. Universally prestarlike functions, prestarlike functions, radius of prestarlikeness.

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1. Introduction

1.1. Prestarlike functions. Let $\mathcal{H}(\Omega)$ denote the set of analytic functions in a domain Ω. For domains Ω containing the origin $\mathcal{H}_0(\Omega)$ stands for the set of functions $f \in \mathcal{H}(\Omega)$ with $f(0) = 1$. We also use the notation $\mathcal{H}_1(\Omega) := \{zf : f \in \mathcal{H}_0(\Omega)\}$. In the special case that Ω is the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ we use the abbreviations $\mathcal{H}, \mathcal{H}_0, \mathcal{H}_1$, respectively.

A function $f \in \mathcal{H}_1$ is called *starlike of order* α (with $\alpha < 1$) if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \ge \alpha, \quad z \in \mathbb{D},$$

and the set of such functions is denoted by $\mathcal{S}_\alpha$. Then, finally, a function $f \in \mathcal{H}_1$ is called *prestarlike of order* α if

$$\frac{z}{(1-z)^{2-2\alpha}} * f(z) \ \in \ \mathcal{S}_\alpha, \tag{1.1}$$

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where '$*$' stands for the Hadamard product of two functions in $\mathcal{H}$:

$$g(z)=\sum_{k=0}^{\infty} g_k z^k,\ h(z)=\sum_{k=0}^{\infty} h_k z^k \ \Rightarrow (g*h)(z):=\sum_{k=0}^{\infty} g_k h_k z^k.$$

The sets of these functions are denoted by $\mathcal{R}_\alpha$. For certain reasons one also introduces the set $\mathcal{R}_1$ to consist of the functions $f\in\mathcal{H}_1$ with

$$\operatorname{Re}\frac{f(z)}{z}\geq\frac{1}{2},\quad z\in\mathbb{D}.$$

Prestarlike functions have a number of interesting geometric properties. For instance, the set $\mathcal{C}$ of univalent functions in $\mathcal{H}_1$ which map $\mathbb{D}$ onto convex domains equals $\mathcal{R}_0$, and obviously we also have $\mathcal{R}_{1/2}=\mathcal{S}_{1/2}$. We refer to Ruscheweyh [3] and Sheil-Small [7] for a description of the essentials of the theory of prestarlike functions. A non-obvious and crucial property is given in the following lemma.

Lemma 1.1. *For $\alpha<\beta\leq 1$ we have $\mathcal{R}_\alpha\subset\mathcal{R}_\beta$.*

To define prestarlike functions intrinsically we use the operators

$$(D^\beta f)(z):=\frac{z}{(1-z)^\beta}*f,\quad \beta\geq 0.$$

Then one can see that a function $f\in\mathcal{H}_1$ is prestarlike of order $\alpha\leq 1$ if and only if

$$z\frac{D^{3-2\alpha}f}{D^{2-2\alpha}f}\in\mathcal{R}_1. \tag{1.2}$$

1.2. Universally prestarlike functions. In [5] the notion of prestarlike functions has been extended from the unit disc to other discs and half-planes containing the origin. Let Ω be one such disc or half-plane. Then there are two unique parameters $\gamma\in\mathbb{C}\setminus\{0\}$ and $\rho\in[0,1]$ such that

$$\Omega=\{w_{\gamma,\rho}(z)\ :\ z\in\mathbb{D}\}=:\Omega_{\gamma,\rho},$$

where

$$w_{\gamma,\rho}(z):=\frac{\gamma z}{1-\rho z}.$$

Note that $1\notin\Omega_{\gamma,\rho}$ if and only if $|\gamma+\rho|\leq 1$.

Definition 1.2. Let $\alpha\leq 1$ and $\Omega=\Omega_{\gamma,\rho}$ for some admissible pair (γ,ρ). A function $f\in\mathcal{H}_1(\Omega)$ is called *prestarlike of order α in Ω* if

$$f_{\gamma,\rho}(z):=\frac{1}{\gamma}f(w_{\gamma,\rho}(z))\in\mathcal{R}_\alpha.$$

The set of these functions f is denoted by $\mathcal{R}_\alpha(\Omega)$.

Let $\Lambda := \mathbb{C} \setminus [1, \infty]$. In this paper we are mainly concerned with the following objects (see [5, Def. 1.2] and [4] for further results).

Definition 1.3. Let $\alpha < 1$. A function $f \in \mathcal{H}_1(\Lambda)$ is called *universally prestarlike of order* α if and only if f is prestarlike of order α in all sets $\Omega_{\gamma,\rho}$ for which $1 \notin \Omega_{\gamma,\rho}$. The set of these functions is denoted by $\mathcal{R}^{\mathrm{u}}_\alpha$.

Let $\mathcal{M}[a, b]$ denote the set of probability measures on the interval $[a, b]$, and set

$$\mathcal{T} := \left\{ \int_0^1 \frac{d\mu(t)}{1 - tz} \; : \; \mu \in \mathcal{M}[0, 1] \right\}. \tag{1.3}$$

Note that $\mathcal{T} \subset \mathcal{H}_0(\Lambda)$ and that $z \cdot \mathcal{T} \subset \mathcal{R}_1$.

The main result in [5] was

Theorem 1.4. *Let* $\alpha \leq 1$ *and* $f \in \mathcal{H}_1(\Lambda)$. *Then* $f \in \mathcal{R}^{\mathrm{u}}_\alpha$ *if and only if*

$$\frac{D^{3-2\alpha} f}{D^{2-2\alpha} f} \in \mathcal{T}. \tag{1.4}$$

From this and Lemma 1.1 it follows immediately that for $\alpha < 1$

$$f \in \mathcal{R}^{\mathrm{u}}_\alpha \quad \Rightarrow \quad \frac{f(z)}{z} \in \mathcal{T}. \tag{1.5}$$

Note that the functions

$$\frac{z}{1 - tz}, \; t \in [0, 1], \tag{1.6}$$

belong to $\mathcal{R}^{\mathrm{u}}_\alpha$ for all $\alpha \leq 1$ (actually one can show that they are the only functions with this property). Our first result in this note is

Theorem 1.5. *Let* $\alpha < 1$. *Then the functions* (1.6) *are the only rational functions in* $\mathcal{R}^{\mathrm{u}}_\alpha$.

For $\alpha \leq \frac{1}{2}$ this result has already been established in [5]. Note that $\mathcal{R}^{\mathrm{u}}_1$ has many additional rational members (for instance $f_0(z) := \frac{1}{2}(z + \frac{z}{1-z})$).

Theorem 1.5 has an interesting general consequence. From Theorem 1.4 it follows immediately that for $t \in [0, 1]$ we have

$$f \in \mathcal{R}^{\mathrm{u}}_\alpha \quad \Rightarrow \quad \frac{1}{t} f(tz) \in \mathcal{R}^{\mathrm{u}}_\alpha.$$

However, there is no positive 'radius of universal prestarlikeness of order $\alpha < 1$' in $\mathcal{R}^{\mathrm{u}}_1$:

Corollary 1.6. *Let* $\alpha < 1$. *Then there is no* $t = t(\alpha) \in (0, 1]$ *such that* $f \in \mathcal{R}^{\mathrm{u}}_1$ *implies* $f(tz)/t \in \mathcal{R}^{\mathrm{u}}_\alpha$.

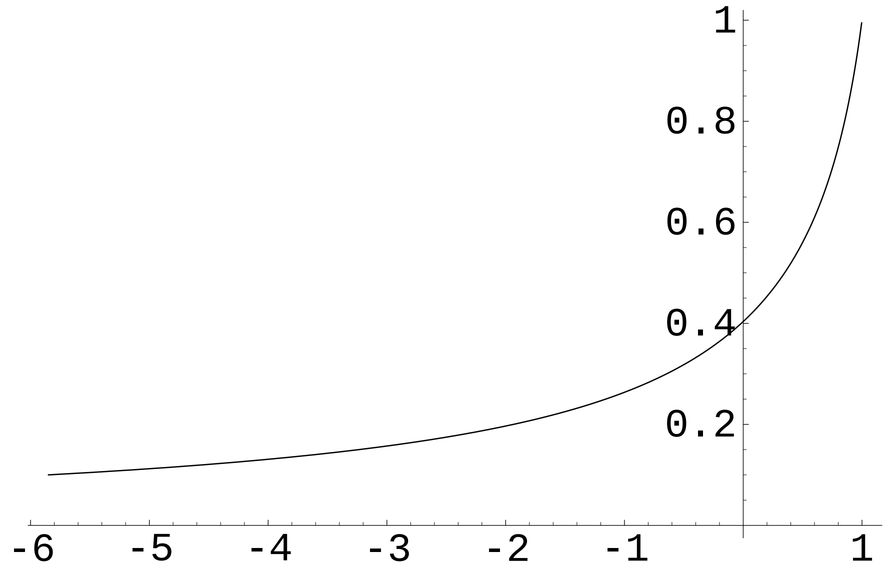

FIGURE 1. $t(\alpha)$, $\alpha = -6, \ldots, 1$.

For a proof just recall that for the function f_0 from above the function $f_0(tz)/t$, with $t \in (0, 1]$, is also rational, but certainly not of the form (1.6). Theorem 1.5 completes the proof.

Corollary 1.6 in its 'radius of...' interpretation sounds a bit unusual at first glance, in particular when compared with the following unit disc situation. However, a closer look at the geometry of the domain Λ sort of explains the difference.

We introduce the function $\alpha : [0, 1] \to [-\infty, 1]$ by

$$\alpha(t) := 1 - \inf_{\varphi \in (0,\pi/2]} \frac{\arctan\left(\frac{1-t^2}{2t\sin(\varphi)}\right)}{2\arctan\left(\frac{t\sin(\varphi)}{1-t\cos(\varphi)}\right)} .$$

$\alpha(t)$ is surjective, concave and strictly increasing. Let $t(\alpha)$ denote its inverse function.

Theorem 1.7. *Let $f \in \mathcal{R}_1$. Then, for $\alpha \in (-\infty, 1)$ we have $f(t(\alpha)z)/t(\alpha) \in \mathcal{R}_\alpha$. This result is sharp for each α.*

Fig. 1 gives a partial graph of $t(\alpha)$. We note in passing that the value $t(0)$ is the radius of convexity in $\mathcal{R}_1$, and a numerical evaluation gives

$$t(0) = 0.4035150 \ldots .$$

For similar results we refer to Jankovics [1].

A simple but perhaps useful corollary of Theorem 1.7 is as follows. Let

$$\mathcal{V}_\beta(t) := \left\{ \int_0^{2\pi} \frac{z\, d\mu(\varphi)}{(1 - t\, e^{i\varphi}\, z)^\beta} \;:\; \mu \in \mathcal{M}[0, 2\pi] \right\}.$$

Corollary 1.8. *For $\alpha < 1$ we have*

$$\mathcal{V}_1(t(\alpha)) \subset \mathcal{R}_\alpha \subset \mathcal{V}_1(1), \tag{1.7}$$

and

$$\mathcal{V}_{2-2\alpha}(t(\alpha)) \subset \mathcal{S}_\alpha \subset \mathcal{V}_{2-2\alpha}(1). \tag{1.8}$$

$t(\alpha)$ is the largest possible number, for each α, in both, (1.7) and(1.8) .

Note that (1.8) follows from (1.7) by convolution with $\frac{z}{(1-z)^{2-2\alpha}}$.

2. Proof of Theorem 1.5

As mentioned above, the cases of $\alpha \le \frac{1}{2}$ of Theorem 1.5 have already been established in [5]. Therefore we now assume $\frac{1}{2} < \alpha < 1$. Let $f \in \mathcal{R}_\alpha^{\mathrm{u}}$ be rational, and set $F(z) := \frac{f(z)}{z}$, so that by (1.5) we have $F \in \mathcal{T}$. This implies that F can have poles only on $[1, \infty]$. Furthermore, since the functions in $\mathcal{T}$ map the upper (lower) half-plane into itself, it is clear that all poles can be of order 1 only, with positive residues. Hence F is of the form

$$F(z) = \sum_{k=1}^{n} \frac{\mu_k}{1 - t_k z}, \quad 0 < \mu_k, \le 1,\ 0 \le t_k \le 1, \sum_{k=1}^{n} \mu_k = 1.$$

We may assume that $n \ge 2$, $t_1 = \max_{1 \le k \le n} t_k$, and set

$$q_\gamma(z) := \sum_{k=1}^{n} \frac{\mu_k}{\mu_1} \frac{1}{(1 - t_k z)^\gamma}.$$

Then the condition $f \in \mathcal{R}_\alpha^{\mathrm{u}}$ can be rewritten as

$$R(z) := \frac{q_{\beta+1}(z)}{q_\beta(z)} \in \mathcal{T}, \tag{2.1}$$

where $0 < \beta := 2 - 2\alpha < 1$. After a rearrangement we find

$$R(z) = \frac{1}{1 - t_1 z} + \frac{1}{(1 - t_1 z)^{1-\beta}}\, Q(z),$$

where

$$Q(z) := \frac{\sum_{k=2}^{n} \frac{\mu_k}{\mu_1} \frac{(t_k - t_1) z}{(1 - t_k z)^{\beta+1}}}{1 - \sum_{k=2}^{n} \frac{\mu_k}{\mu_1} \frac{(1 - t_1 z)^\beta}{(1 - t_k z)^\beta}}.$$

Now set

$$z := \frac{1}{t_1} + \rho e^{i\varphi},$$

with $\rho > 0$ but small, and $\varphi \in (0, \pi)$. By construction

$$Q(z) = -c + \mathcal{O}(\rho),$$

with $c := -Q(1/t_1) > 0$. Writing $\delta := \pi - \varphi$ we now find

$$R(z) = \frac{1}{\rho} e^{i\delta} - \rho^{\beta} e^{i(1-\beta)\delta}(c + \mathcal{O}(\rho)).$$

We now restrict δ to the interval $(\frac{\pi}{2-2\beta}, \pi)$ and choose $M > 0$ so that for the quantity $\mathcal{O}(\rho)$ from above we have $|\mathcal{O}(\rho)| \le M\rho$, independently of δ. Then

$$\frac{\rho}{\sin((1-\beta)\delta)} \operatorname{Im} R(z) \le \frac{\sin(\delta)}{\sin((1-\beta)\delta)} - \rho^{\beta} c + \frac{\rho^{\beta+1} M}{\sin((1-\beta)\pi)}.$$

Now choose ρ so small that

$$\frac{\rho M}{\sin((1-\beta)\pi)} \le \frac{1}{2} c,$$

and then $\delta < \pi$ (but close to π) so that

$$\frac{\sin(\delta)}{\sin((1-\beta)\delta)} < \frac{1}{2} \rho^{\beta} c.$$

Then we can conclude that for this z, which is in the upper half-plane, we have $\operatorname{Im} R(z) < 0$, a contradiction to (2.1). □

3. Proof of Theorem 1.7

We first note that the convex set $\mathcal{R}_1 = \mathcal{V}_1(1)$ satisfies the condition of the main theorem in [2], which for the present case can be stated as follows:

Lemma 3.1. *Let λ_1, λ_2 be two continuous linear functionals on $\mathcal{R}_1$ and assume that $0 \notin \lambda_2(\mathcal{R}_1)$. Then the range of the functional $\lambda(f) := \frac{\lambda_1(f)}{\lambda_2(f)}$ over $\mathcal{R}_1$ equals the set*

$$\lambda\left(\frac{\rho\, z}{1-xz} + \frac{(1-\rho)\, z}{1-yz}\right), \quad \rho \in [0,1], |x| = |y| = 1.$$

Taking this lemma into account we have to show that for any $t \in (0,1)$ and $|z| \le t$ we have

$$\operatorname{Re} \frac{D^{\beta+1}\left(\frac{\rho\, z}{1-xz} + \frac{(1-\rho)\, z}{1-yz}\right)}{D^{\beta}\left(\frac{\rho\, z}{1-xz} + \frac{(1-\rho)\, z}{1-yz}\right)} \ge \frac{1}{2}, \quad \rho \in [0,1],\ |x| = |y| = 1,\ \beta \le 2 - 2\alpha(t), \tag{3.1}$$

which is easily seen to be equivalent to

$$\operatorname{Re}\frac{\frac{1}{(1-x)^\beta}\left(\frac{1}{1-x}-\frac{1}{2}\right)+\frac{\lambda}{(1-y)^\beta}\left(\frac{1}{1-y}-\frac{1}{2}\right)}{\frac{1}{(1-x)^\beta}+\frac{\lambda}{(1-y)^\beta}}\geq 0,\quad \lambda\geq 0,\ |x|=|y|=t,$$

and, with $\gamma := \beta + 1$,

$$\frac{1-t^2}{|1-x|^{2\gamma}}+\lambda^2\frac{1-t^2}{|1-y|^{2\gamma}}+2\lambda\operatorname{Re}\frac{1-x\overline{y}}{(1-x)^\gamma(1-\overline{y})^\gamma}\geq 0\quad \lambda\geq 0,\ |x|=|y|=t,$$

and the same range for β (respectively γ). Taking the minimum with respect to λ we arrive at the necessary and sufficient condition

$$\text{(3.2)}\qquad \operatorname{Re}\frac{1-xy}{1-t^2}\left(\frac{|1-x||1-y|}{(1-x)(1-y)}\right)^\gamma\geq -1,\quad |x|=|y|=t,\ \gamma\leq 3-2\alpha(t).$$

Since (3.2) is certainly true if $\operatorname{Re} b(x,y,\gamma)>0$, where

$$b(x,y,\gamma)=\frac{1-xy}{(1-x)^\gamma(1-y)^\gamma},$$

and certainly false for $b(x,y,\gamma)<0$ we conclude that the critical cases for x,y,γ are those with $\arg b(x.y,\gamma)\in(-\pi,0)$. Fixing $z:=xy$ and γ it is not difficult to see that under these restrictions the left hand side of (3.2) is smallest when $\arg(x,z/x,\gamma)$ is smallest, and a simple application of calculus shows that this is the case for $x=z/x$, and hence for $x=y$. We can therefore restrict our attention to this case, and start over again.

Our necessary and sufficient condition is now, after some rewriting, that for $\beta\leq 2-2\alpha(t)$ we have

$$\text{(3.3)}\qquad \operatorname{Re}\frac{(1+x)(1-\overline{x})}{1-t^2}\left(\frac{|1-x|}{1-x}\right)^{2\beta}\geq -1,\quad |x|=t.$$

Let $x=t\,e^{i\varphi}$ and note that the expression on the left of (3.3) is invariant under $x\to\overline{x}$ so that we can restrict our attention to $\varphi\in(0,\pi)$. Then, for $\arg(1-x)=:\mu=\mu(t,\varphi)$ we have $\mu<0$ and we find the equivalent conditions

$$\cos(2\beta\mu)+\frac{2t\sin(\varphi)}{1-t^2}\sin(2\beta\mu)\geq -1,$$

and

$$\tan(\beta\mu)\geq -\frac{1-t^2}{2t\sin(\varphi)}.$$

Using

$$\mu=\arctan\left(\frac{-t\sin(\varphi)}{1-t\cos(\varphi)}\right)$$

we finally arrive at the necessary and sufficient condition

$$\beta \leq \frac{\arctan\left(\frac{1-t^2}{2t\sin(\varphi)}\right)}{\arctan\left(\frac{t\sin(\varphi)}{1-t\cos(\varphi)}\right)} =: g(t,\varphi)$$

for every $\varphi \in (0,\pi)$. Actually, since obviously $g(t,\varphi) \leq g(t,\pi-\varphi)$ for $\varphi \in (0,\frac{\pi}{2}]$, we are left with the condition

$$\beta \leq \inf_{\varphi\in(0,\frac{\pi}{2}]} g(t,\varphi),$$

which is equivalent to our assertion. □

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