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On Inverse Coefficients

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Abstract. Normalized functions f analytic in the open unit disc around the origin and nonvanishing outside the origin can be expressed in the form $z/g(z)$ where $g(z)$ has Taylor coefficients b_n's. These b_n's are called Inverse coefficients. Necessary conditions in terms of some initial b_n's are derived for some classes of analytic functions.

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1. Introduction

Let A_1 be the class of functions f analytic in $U = \{z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = 0,\ f'(0) = 1$ where $\mathbb{C}$ is the set of complex numbers. An f in A_1 with $f(z) \neq 0$ in the punctured disc $U\backslash\{0\}$, may be expressed as

$$f(z) = \psi(g) = \frac{z}{g(z)}$$

in U, where

$$g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$$

in U. We call the coefficients b_n's the inverse coefficients of $f(z)$. Mitrinovic [1], Reade et. al [2] worked on these coefficients.

Mitrinovic [1] obtained estimates for the radius of univalence of certain rational functions. In particular, he found sufficient conditions for functions of the form

$$\frac{z}{1 + b_1 z + b_2 z^2 + \cdots + b_n z^n}, \tag{1.1}$$

$b_n \neq 0$, to be univalent in the unit disk U. A function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.2}$$

analytic in U is said to be starlike of order $\alpha,\ 0 \le \alpha < 1$, if

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha$$

in U. The set of all such functions is denoted by $\mathcal{S}^*(\alpha)$. The functions in $\mathcal{S}^* \equiv \mathcal{S}^*(0)$ are called starlike functions . The function $f(z)$ is said to be convex, if,

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$$

in U.

In the note [2], Reade et. al., showed that the Mitrinovic criterion for univalence of functions of the form (1.1) does not guarantee starlikeness and gave sufficient conditions for such functions to be (i) starlike of order α and (ii) convex, as $n \to \infty$.

Definition 1.1. Let $K > 0$ and f be regular and locally univalent in U. Then f is said to belong to the class $\mathcal{C}_\alpha(K),\ \alpha < 1$, if and only if

$$\liminf_{|z|\to 1^-} k_\alpha(f;z) \ge K.$$

Here $k_\alpha(f;z)$ is a generalization, called α-curvature of the Euclidean curvature of $f(|z| = r)$ at the point $f(z)$ and is given by

$$k_\alpha(f;z) = \frac{\operatorname{Re}\left(1 + \frac{zf''(z)}{(1-\alpha)f'(z)}\right)}{|z|\,|f'(z)|^{1/(1-\alpha)}}$$

where, $z = re^{i\theta}$ and $0 < r < 1$ (see [4]).

The class $\mathcal{C}_0(K)$ was studied in [5].

Definition 1.2. A function $f \in A_1$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n$$

in U, with $a_n \ge 0$ is said to be in the class $\mathcal{C}(\alpha),\ 0 \le \alpha < 1$, if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$$

in U.

2. Main Results

First we derive a necessary condition on the coefficients b_1 and b_2 for the functions in $\mathcal{C}_\alpha(K)$:

Theorem 2.1. *For* $0 \le \alpha < 1,\ 0 < K < 1,$ *and*

$$\psi(g) = \frac{z}{1 + \sum_{n=1}^{\infty} b_n z^n} \in \mathcal{C}_\alpha(K),$$

we have

$$|b_1| \le (1-\alpha)^2(1-K) \tag{2.1}$$

and

$$\frac{3}{1-\alpha}\,|b_2| \le \frac{1 - K - |b_1|^2\,\{1-\alpha(1-\frac{K}{2})\}(1-\alpha)^{-2}}{1-\frac{K}{2}}. \tag{2.2}$$

Both the inequalities are sharp.

Theorem 2.2. *If*

$$\psi(g) = \frac{z}{g(z)} = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n} \in \mathcal{C}(\alpha), \quad 0 \le \alpha < 1$$

in U, *then*

$$|b_n| \le \frac{1-\alpha}{(n+1)(n+1-\alpha)}, \quad n = 0, 1, 2, \ldots. \tag{2.3}$$

The inequality is sharp for

$$g_n(z) = 1 + \sum_{k=1}^{\infty} \left[\frac{1-\alpha}{(n+1)(n+1-\alpha)}\right]^k z^{nk}$$

in U *and*

$$\psi(g) = \frac{z}{g_n(z)} = z - \left[\frac{1-\alpha}{(n+1)(n+1-\alpha)}\right] z^{n+1}, \quad z \in U.$$

3. Propositions and proofs of theorems

Proposition 3.1. *Let* $f(z)$ *be analytic in* U. *Then* $f \in \mathcal{C}_\alpha(K)$, *if and only if,*

$$g(z) = \int_0^z (f'(\tau))^{\frac{1}{1-\alpha}}\, d\tau \in \mathcal{C}_0(K)$$

where $\alpha < 1$ *and* $K > 0$.

Proof. We observe that $f(z)$ is locally univalent in U, if and only if, $g(z)$ is so. This, the fact

$$k_0(g;z) = k_\alpha(f;z)$$

and the definitions of the classes $\mathcal{C}_\alpha(K)$ and $\mathcal{C}_0(K)$ together give Proposition 3.1. ■

Proposition 3.2. *Let* $K > 0,\ \alpha < 1,\ \alpha \neq 0,$ *and*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{C}_\alpha(K).$$

Then the inequality

$$\left| \frac{f''(z)(1-|z|^2)}{f'(z)2(1-\alpha)} - \bar{z} \right|^2 \leq 1 - K \left| f'(z) \right|^{\frac{1}{1-\alpha}} (1-|z|^2) \tag{3.1}$$

is true for z in U. The inequality (3.1) *is sharp.*

Proof. For the following functional $u_\alpha(z)$ and z in U we have

$$\begin{aligned} u_\alpha(z) &= \frac{1 - \left| \frac{f''(z)(1-|z|^2)}{f'(z)2(1-\alpha)} - \bar{z} \right|^2}{\left| f'(z) \right|^{\frac{1}{1-\alpha}} (1-|z|^2)} \\ &\geq \liminf_{|z|\to 1^-} u_\alpha(z) \\ &= \liminf_{|z|\to 1^-} \frac{1}{\left| f'(z) \right|^{\frac{1}{1-\alpha}}} \left[\mathrm{Re}\left(1 + \frac{z f''(z)}{(1-\alpha) f'(z)} \right) - \frac{(1-|z|^2)}{4(1-\alpha)^2} \left| \frac{f''(z)}{f'(z)} \right|^2 \right] \\ &= \liminf_{|z|\to 1^-} k_\alpha(f;z) \geq K, \end{aligned}$$

in view of Proposition 3.1, the equation in its proof and [5]. This gives (3.1).

The inequality (3.1) is sharp for

$$f(z) = \begin{cases} \dfrac{e^{i\varphi}(1-|a|^2)^{1-\alpha}}{K^{1-\alpha} a(2\alpha-1)(1+az)^{1-2\alpha}} + b \ \text{ or } \ \dfrac{e^{i\varphi}}{K^{1-\alpha}} z + b; & \alpha \neq \frac{1}{2} \\ \dfrac{e^{i\varphi}}{a} \left[\dfrac{1-|a|^2}{K} \right]^{1/2} \log(1+\bar{a}z) + b \ \text{ or } \ \dfrac{e^{i\varphi}}{K^{1-\alpha}} z + b; & \alpha = \frac{1}{2} \end{cases} \tag{3.2}$$

for $a \in U \backslash \{0\},\ b \in \mathbb{C}$ and $\varphi \in \mathbb{R}$. This proves Proposition 3.2. ■

Next, using Proposition 3.2, we obtain:

Lemma 3.3. *Let* $K > 0,\ \alpha < 1,\ \alpha \neq 0$ *and* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{C}_\alpha(K)$. *Then*

$$\left| \frac{a_2}{a_1(1-\alpha)} \right|^2 \leq 1 - K\,|a_1|^{\frac{1}{1-\alpha}}\,. \tag{3.3}$$

The inequality (3.3) is sharp only for the functions of the form (3.2).

Proof. Taking z=0 in Proposition 3.2, gives Lemma 3.3. ■

Proposition 3.4. *Let* $K > 0,\ \alpha < 1,\ \alpha \neq 0$ *and* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{C}_\alpha(K)$. *Then the inequality*

$$\begin{aligned} &\frac{1}{2}\left| \frac{1}{1-\alpha}\left\{ \frac{\alpha}{1-\alpha}\frac{(f''(z))^2}{f'(z)} + f'''(z) \right\} \frac{1}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{(1-\alpha)f'(z)} \right)^2 \right| \\ &\quad \left(1-|z|^2\right)^2 \left(1 - \frac{K}{2}\,|f'(z)|^{\frac{1}{1-\alpha}}\,(1-|z|^2)\right) \\ &\quad \leq 1 - \left| \frac{f''(z)(1-|z|^2)}{f'(z)2(1-\alpha)} - \bar{z} \right|^2 - K\,|f'(z)|^{\frac{1}{1-\alpha}}\,(1-|z|^2) \end{aligned} \tag{3.4}$$

holds for z *in* U. *The inequality is sharp for the function* $f(z)$ *of the form (3.2).*

Proof. Proposotion 3.1 and the inequality [5]

$$\begin{aligned} &\frac{1}{2}\,|[g]_z(z)|\left(1-|z|^2\right)^2 \left(1 - \frac{K}{2}\,|g'(z)|\,(1-|z|^2)\right) \\ &\quad \leq 1 - \left| \frac{g''(z)(1-|z|^2)}{2g'(z)} - \bar{z} \right|^2 - K\,|g'(z)|\,(1-|z|^2) \end{aligned} \tag{3.5}$$

give the inequality (3.4). Here

$$[f]_z(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

■

Lemma 3.5. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{C}_\alpha(K)$ *with* $K > 0,\ \alpha < 1,\ \alpha \neq 0$. *Then*

$$\left| \frac{a_3}{a_1} \right| \leq \left(1 - K\,|a_1|^{\frac{1}{1-\alpha}}\right)\left(1 - \frac{2}{3}\alpha\right)(1-\alpha). \tag{3.6}$$

The inequality (3.6) is sharp only for the functions of the form (3.2).

Proof. For $z=0$ inequality (3.4) gives that

$$\frac{3}{1-\alpha}\left|\frac{a_3}{a_1}-\frac{1-\frac{2\alpha}{3}}{1-\alpha}\left(\frac{a_2}{a_1}\right)^2\right|\leq\frac{1-\left|\frac{a_2}{a_1(1-\alpha)}\right|^2-K\left|a_1\right|^{\frac{1}{1-\alpha}}}{1-\frac{K}{2}\left|a_1\right|^{\frac{1}{1-\alpha}}}. \tag{3.7}$$

From this by applying the triangle inequality and the inequality (3.3), the required inequality is obtained. This completes the proof of Lemma 3.5. ■

Proof of Theorem 2.1. For $f(z)=\sum_{n=0}^{\infty}a_nz^n=\psi(g)$, we have $b_1=-a_2$ and $b_2=a_2^2-a_3$. By Lemma 3.3, we have that

$$\left|a_2^2\right|\leq(1-\alpha)^2(1-K).$$

By substituting $b_1=-a_2$ in this, we obtain the inequality (2.1).

By (3.7), we have

$$\frac{3}{1-\alpha}\left|a_3-\frac{1-\frac{2\alpha}{3}}{1-\alpha}a_2^2\right|\leq\frac{1-\left|\frac{a_2}{1-\alpha}\right|^2-K}{1-\frac{K}{2}}.$$

Hence,

$$\frac{3}{1-\alpha}\left|a_3-a_2^2\right|-\frac{\alpha}{(1-\alpha)^2}\left|a_2\right|^2\leq\frac{1-\left|a_2\right|^2\frac{1}{(1-\alpha)^2}-K}{1-\frac{K}{2}}$$

and so

$$\frac{3}{1-\alpha}\left|a_3-a_2^2\right|\leq\frac{1-K-\left|a_2\right|^2\left(1-\alpha\left(1-\frac{K}{2}\right)\right)\frac{1}{(1-\alpha)^2}}{1-\frac{K}{2}}.$$

Now by substituting b_2 for $a_2^2-a_3$ and b_1 for $-a_2$ in the inequality we obtain the inequality (2.2). The functions

$$\frac{\left((1+az)^{2\alpha-1}-1\right)}{(2\alpha-1)a}\quad\text{for } \alpha\neq\frac{1}{2};$$

$$\frac{e^{i\varphi}}{a}\log\left(1+\bar{a}z\right)+b\quad\text{for } \alpha=1/2$$

and

$$e^{i\varphi}(1-|a|^2)^{\alpha-1}z+b,$$

with $K=1-|a|^2$, $a\in U\backslash\{0\}$, $b\in\mathbb{C}$ and $\varphi\in R$, the set of real numbers, give sharpness in the inequalities (2.1) and (2.2). This completes the proof of the theorem.

Proof of Theorem 2.2. Since $\psi(g)\in\mathcal{C}(\alpha)$ it has the Taylor series expansion

$$\psi(g)=z-\sum_{k=0}^{\infty}a_nz^n,\quad a_n\geq 0,\ z\in U.$$

By the definition of $g(z)$,

$$b_n = \sum_{k=0}^{n-1} b_k a_{n-k+1} \tag{3.8}$$

for $n \geq 1$ where $b_0 = 1$.

First we show that $\{b_n\}$ is a sequence of nonnegative real numbers. It follows from the equation (3.8) that $b_1 = a_2 \geq 0$. Now assume that $b_k \geq 0,\ 1 \leq k \leq n,\ n \in \mathbb{N}$, the set of natural numbers. Since,

$$b_{n+1} = \sum_{k=0}^{n} b_k a_{n+2-k}$$

and a_k's are nonnegative, we have $b_{n+1} \geq 0$. This proves that $\{b_n\}$ is a sequence of nonnegative real numbers.

By the necessary and sufficient condition [3] for f to be in $\mathcal{C}(\alpha)$:

$$\sum_{n=2}^{\infty} n(n-\alpha)a_n \leq 1-\alpha, \tag{3.9}$$

we have

$$b_1 = a_2 \leq \frac{1-\alpha}{2(2-\alpha)}.$$

This proves the inequality (2.3) for $n = 1$.

Now, let the inequality (2.3) be true for n, satisfying $1 \leq n \leq k,\ k \in \mathbb{N}$. Then,

$$b_{k+1} = \sum_{n=0}^{k} b_n a_{k+2-n} \leq \sum_{n=0}^{k} \frac{1-\alpha}{(n+1)(n+1-\alpha)} a_{k+2-n}. \tag{3.10}$$

Set, for $n \geq 2$,

$$a_n = \lambda_n \frac{1-\alpha}{n(n-\alpha)}.$$

For $\psi(g) = z - \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}(\alpha)$, it is necessary, by (3.9), that

$$\sum_{n=2}^{\infty} n(n-\alpha)a_n \leq 1-\alpha.$$

Thus, $\lambda_n \geq 0$ for $n \geq 2$ and

$$\sum_{n=1}^{k+1} \lambda_{n+1} \leq 1. \tag{3.11}$$

The inequality (3.10) is equivalent to

$$\begin{aligned} b_{k+1} &\leq \sum_{n=0}^{k} \lambda_{k+2-n} \frac{1-\alpha}{(n+1)(n+1-\alpha)} \cdot \frac{1-\alpha}{(k+2-n)(k+2-n-\alpha)} \\ &\leq \frac{1-\alpha}{(k+2)(k+2-\alpha)} \sum_{n=0}^{k} \lambda_{k+2-n} \\ &\leq \frac{1-\alpha}{(k+2)(k+2-\alpha)}. \end{aligned}$$

The second inequality holds since

$$(n+1)(n+1-\alpha)(k+2-n)(k+2-n-\alpha) \geq (1-\alpha)(k+2)(k+2-\alpha)$$

for $0 \leq n \leq k$ and the last inequality holds due to (3.11). This proves the inequality (2.3) for $n = k+1$ and the proof of the theorem is complete by the induction argument. It is easily seen that sharpness of (2.3) is attained for the function $\psi(g_n)$ where g_n is as in the statement of the theorem.

References

[1] S. Mitrinovic Dragoslav, *On the univalence of rational functions*, Univ.Beograd Publ. Elektrotehn, Fak., Ser.Mat.Fiz., No.**634**-No.**677**(1979), 221–227.

[2] M.O. Reade, H. Silverman, and P.G. Todorov, *On the starlikeness and convexity of a class of analytic functions*, Rendiconti del Circolo Matematico di Palermo, serie II, TomoXXXIII(1984), 265–272.

[3] H. Silverman, *Univalent Functions with negative coefficients*, Proc. Amer. Math. Soc. **51**(1975), 109–116.

[4] V. Srinivas, *Convex functions of bounded α-type*, J. Analysis, **2**(1994), 79–85.

[5] K.J. Wirths, *Coefficient bounds for functions of bounded type*, Proc. Amer. Math. Soc. **103**(1988), 525–530.

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