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On An Eigenvalue Problem Of Hydrodynamic Stability

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Abstract. We consider the second order ordinary differential equation

$$W'' + \left[\frac{N^2}{(U_0 - c)^2} - \frac{U_0''}{(U_0 - c)} - k^2\right] W + \frac{1}{(U_0 - c)} \left[T\,(U_0 - c)\,W\right]' = 0,$$

with boundary conditions $W(0) = 0 = W(D)$, which arises in hydrodynamic stability (cf. [1]). Here $c = c_r + ic_i$ is the complex eigenvalue, $W(z)$ is the eigen function, $U_0(z)$ is the basic velocity, $N^2(z) \geq 0$ is a stratification parameter, $T(z)$ measures the variability in the domain and $k > 0$ is the wave number of a normal mode disturbance to a channel flow in a domain between $z = 0$ and $z = D$ and a prime denotes differentiation with respect to z. For this problem we present some results on the location of complex eigenvalues with $c_i > 0$ (corresponding to unstable modes) and an estimate for the growth rate kc_i of any unstable mode.

Keywords. Hydrodynamic stability, shear flows, variable bottom.

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1. Introduction

The Taylor-Goldstein problem in hydrodynamic stability deals with the stability of shear flows of an inviscid, incompressible but density stratified fluid to infinitesimal normal mode disturbances. This problem has been extensively studied (see [2, 5]). Recently Deng et. al. [1] has extended this problem to include the variable bottom of the flow domain as this is necessary in the study of shear flows in sea straits. Deng et. al. [1] found two general results for their problem. In their first result they proved that the basic shear flow is stable if the minimum Richardson number J_0 is larger than or equal to one quarter. In their second result they proved that the unstable eigenvalues lie inside a semicircle in the upper half plane, whose diameter coincides with the range of the basic velocity profile. In this paper we extend their results by proving that the unstable modes lie inside a semi-elliptical region which depends on J_0, the depth of the fluid layer and the wave number k. Further, we also obtain an estimate for the growth rate of any unstable mode. Also the instability region is further improved for a special class of flows which includes the hyperbolic tangent velocity profile.

Our results reduce to the results of Makov and Stepanyants [4] when the bottom parameter $b(z)$ is taken to be equal to 1.

2. The Eigenvalue Problem

The eigenvalue problem considered in this paper is given by the equation

$$W'' + \left[\frac{N^2}{(U_0 - c)^2} - \frac{U_0''}{(U_0 - c)} - k^2\right] W + \frac{1}{U_0 - c}\left[T(U_0 - c)W\right]' = 0. \tag{2.1}$$

with boundary conditions

$$W(0) = 0 = W(D). \tag{2.2}$$

Here the real part of $W(z)e^{ik(x-ct)}$ is the vertical velocity of a normal mode disturbance with k the wave number and c the complex phase velocity. A prime denotes differentiation with respect to the vertical coordinate z varying over $(0, D)$, $U_0(z)$ is the basic velocity, $N^2(z) \geq 0$ is the square of Brunt-Väisäla frequency and $T(z)$ measures the variability in the domain and $T(z) = [\log b(z)]'$, where $b(z)$ gives the width of the channel. If T is a constant then $b(z) = b(0)e^{Tz}$, i.e., the width is an exponential function of z.

3. A General Instability Region

Now we shall prove that the complex eigenvalues of our eigenvalue problem with positive imaginary part lie inside a semi-ellipsetype region.

Theorem 3.1. *The complex eigenvalues of our eigenvalue problem with positive imaginary part lie inside a region given by*

$$\left[c_r - \frac{U_{0\,\min} + U_{0\,\max}}{2}\right]^2 + \left[1 + \frac{2J_0}{1 - 2J_0 + \left(1 - 4J_0 - \left[\frac{4k^2 c_i^2}{\left(U_0'\right)^2_{\max}}\right]\right)^{\frac{1}{2}}}\right] c_i^2 \leq \left[\frac{U_{0\,\max} - U_{0\,\min}}{2}\right]^2,$$

where $J_0 = \min[N^2(z)/(U_0'(z))^2]$ *is the minimum Richardson number.*

Proof. With the transformation $W = (U_0 - c)F$, (2.1) and (2.2) can be written as

$$\left[(U_0 - c)^2 \frac{(bF)'}{b}\right]' + \left[N^2 - k^2(U_0 - c)^2\right] F = 0, \tag{3.1}$$

with boundary conditions

$$F(0) = 0 = F(D). \tag{3.2}$$

From

$$G = [U_0 - c]^{\frac{1}{2}} F, \tag{3.3}$$

we have

$$(bG)' = (U_0 - c)^{\frac{1}{2}} (bF)' + \frac{1}{2}(U_0 - c)^{\frac{-1}{2}} U_0'(bF).$$

This implies that

$$|(bG)'|^2 \geq |U_0 - c|\, |(bF)'|^2 + \frac{|U_0'^2|\, |bF|^2}{4|U_0 - c|} - |U_0'|\, |(bF)'|\, |bF|.$$

This implies that

$$\left[\frac{|(bG)'|^2}{b} + k^2 b|G|^2\right] \geq |U_0 - c| \left[\frac{|(bF)'|^2}{b} + k^2 b|F|^2\right] + \frac{|U_0'^2|\, b|F|^2}{4|U_0 - c|} - |U_0'|\, |(bF)'|\, |F|.$$

The use of Cauchy-Schwarz inequality gives,

$$\int |U_0'||F|\, |(bF)'|\, dz \leq \left[\int \frac{|U_0'^2|\, b|F|^2}{4|U_0 - c|}\, dz\right]^{\frac{1}{2}} \left[\int \frac{4|U_0 - c|\, |(bF)'|^2}{b}\, dz\right]^{\frac{1}{2}} = 2BE,$$

where

$$B^2 = \int \frac{|U_0'^2|\, b|F|^2}{4|U_0 - c|}\, dz, \quad E^2 = \int \frac{|U_0 - c|\, |(bF)'|^2}{b}\, dz, \text{ and } D^2 = \int |U_0 - c| b|F|^2\, dz.$$

Therefore, we have

$$\int \left[\frac{|(bG)'|^2}{b} + k^2 b|G|^2\right] dz \geq B^2 + E^2 - 2BE + k^2 D^2. \tag{3.4}$$

Using (3.3) in (3.1) and equating imaginary parts, we get

$$\int \left[\frac{|(bG)'|^2}{b} + k^2 b|G|^2\right] dz = \int \left[\frac{[U_0']^2}{4} - N^2\right] \frac{b|G|^2}{|U_0 - c|^2}\, dz; \tag{3.5}$$

that is

$$\int \left[\frac{|(bG)'|^2}{b} + k^2 b|G|^2\right] dz \leq (1 - 4J_0) B^2.$$

Therefore we have

$$B^2 + E^2 - 2BE + k^2 D^2 \leq (1 - 4J_0) B^2;$$

that is

$$E^2 - 2BE + 4J_0B^2 + k^2D^2 \leq 0. \tag{3.6}$$

Solving this inequality with respect to E, we obtain

$$B - \sqrt{B^2 - 4J_0B^2 - k^2D^2} \leq E \leq B + \sqrt{B^2 - 4J_0B^2 - k^2D^2}. \tag{3.7}$$

From (3.6), we have

$$E^2 + k^2D^2 \leq 2BE - 4J_0B^2.$$

Using (3.7), we have

$$E^2 + k^2D^2 \leq 2B^2\left[1 - 2J_0 + \left(1 - 4J_0 - \frac{k^2D^2}{B^2}\right)^{\frac{1}{2}}\right]. \tag{3.8}$$

Now,

$$\frac{D^2}{B^2} \geq \frac{4c_i^2}{(U_0')^2_{\max}}. \tag{3.9}$$

Also it is easy to see that

$$E^2 + k^2D^2 \geq c_i\left[\int\left[\frac{|(bF)'|^2}{b} + k^2b|F|^2\right]dz\right], \tag{3.10}$$

and

$$B^2 \leq \frac{1}{4c_i}\int |U_0'|^2\, b|F|^2\, dz. \tag{3.11}$$

From (3.8) and (3.10), we have

$$c_i\left[\int\left[\frac{|(bF)'|^2}{b} + k^2b|F|^2\right]dz\right] \leq 2B^2\left[1 - 2J_0 + \left(1 - 4J_0 - \frac{k^2D^2}{B^2}\right)^{\frac{1}{2}}\right].$$

Using (3.9) and (3.11) in the above equation we get

$$\frac{2c_i^2}{\left[1 - 2J_0 + \left(1 - 4J_0 - \frac{4k^2c_i^2}{(U_0')^2_{\max}}\right)^{\frac{1}{2}}\right]}\int\left[\frac{|(bF)'|^2}{b} + k^2b|F|^2\right]dz \leq \int |U_0'|^2\, b|F|^2\, dz. \tag{3.12}$$

Also it is seen that

$$\int N^2b|F|^2\, dz \geq J_0\int (U_0')^2\, b|F|^2\, dz. \tag{3.13}$$

From [1] we know that the complex wave velocity c for any unstable mode lies inside the region

$$\left[\left[c_r - \frac{U_{0\min} + U_{0\max}}{2}\right]^2 + c_i^2 - \left[\frac{U_{0\max} - U_{0\min}}{2}\right]^2\right] \cdot \int Q\, dz + \int N^2 b|F|^2\, dz \leq 0,$$

where

$$Q = \frac{|(bF)'|^2}{b} + k^2 b|F|^2.$$

Using (3.13), we get

$$\left[\left[c_r - \frac{U_{0\min} + U_{0\max}}{2}\right]^2 + c_i^2 - \left[\frac{U_{0\max} - U_{0\min}}{2}\right]^2\right] \cdot \int \left[\frac{|(bF)'|^2}{b} + k^2 b|F|^2\right] dz$$
$$+ J_0 \int (U_0')^2\, b|F|^2\, dz \leq 0.$$

Using (3.12), we get

$$\left[c_r - \frac{U_{0\min} + U_{0\max}}{2}\right]^2 + \left[1 + \frac{2J_0}{1 - 2J_0 + \left(1 - 4J_0 - \left[\frac{4k^2 c_i^2}{(U_0')^2_{\max}}\right]\right)^{\frac{1}{2}}}\right] c_i^2$$
$$\leq \left[\frac{U_{0\max} - U_{0\min}}{2}\right]^2,$$

which completes the proof of the theorem. ■

4. Instability Region For Specific Velocity Profiles

The instability region given in section 3 is valid for an arbitrary velocity profile $U_0(z)$ and the stratification parameter $N^2(z)$. Now we shall improve upon this result for a specific class of velocity profiles.

Multiplying (3.1) by bF^* (where F^* is the complex conjugate of F), integrating over $(0, D)$ and using (3.2), we get

$$\int (U_0 - c)^2 \left[\frac{|(bF)'|^2}{b} + k^2 b|F|^2\right] dz - \int N^2 b\,|F|^2\, dz = 0.$$

Equating imaginary parts, we get

$$\int U_0 Q\, dz = c_r \int Q\, dz. \tag{4.1}$$

Equating real parts and using (4.1), we get

$$\int U_0^2 Q\,dz = \left(c_r^2 + c_i^2\right)\int Q\,dz + \int N^2 b\,|F|^2\ dz. \tag{4.2}$$

Now,

$$Q - 2k\frac{\left|(bF)'\right|}{b}b\,|F| = \left[\frac{|(bF)|'}{b^{\frac{1}{2}}} - kb^{\frac{1}{2}}\,|F|\right]^2.$$

Now consider the apparent inequality

$$\int \left(U_0 - U_{0\,\min}\right)\left(U_0 - U_{0\,\max}\right)\left[Q - 2k\left|(bF)'\right|\,|F|\right]\ dz \le 0;$$

that is

(4.3)

$$\int U_0^2 Q\,dz - \left(U_{0\,\min} + U_{0\,\max}\right)\int U_0 Q\,dz + \left(U_{0\,\min}U_{0\,\max}\right)\int Q\,dz$$
$$+ 2k\int\left[\left(\frac{U_{0\,\max} - U_{0\,\min}}{2}\right)^2 - \left(U_0 - \frac{U_{0\,\min} + U_{0\,\max}}{2}\right)^2\right]\left|(bF)'\right|\,|F|\ dz$$
$$\le 0.$$

Substituting (4.1) and (4.2) in (4.3) we have

$$\left(c_r^2 + c_i^2\right)\int Q\,dz + \int N^2 b\,|F|^2\ dz - \left(U_{0\,\min} + U_{0\,\max}\right)c_r\int Q\,dz$$
$$+ 2k\int\left[\left(\frac{U_{0\,\max} - U_{0\,\min}}{2}\right)^2 - \left(U_0 - \frac{U_{0\,\min} + U_{0\,\max}}{2}\right)^2\right]\left|(bF)'\right|\,|F|\ dz \le 0;$$

that is

(4.4)

$$\left[\left[c_r - \frac{U_{0\,\min} + U_{0\,\max}}{2}\right]^2 + c_i^2 - \left[\frac{U_{0\,\max} - U_{0\,\min}}{2}\right]^2\right]\cdot\int Q\,dz$$
$$+ \int N^2 b\,|F|^2\ dz + 2k\left(\frac{U_{o\,\max} - U_{0\,\min}}{2}\right)$$
$$\int\left[\left(\frac{U_{0\,\max} - U_{0\,\min}}{2}\right) - \frac{\left(U_0 - \frac{U_{0\,\min} + U_{0\,\max}}{2}\right)^2}{\left(\frac{U_{o\,\max} - U_{0\,\min}}{2}\right)}\right]\left|(bF)'\right|\,|F|\ dz \le 0.$$

Substituting (3.3) in (3.5) we get

$$\int |U_0 - c|\left[\frac{\left|(bF)'\right|^2}{b} + k^2 b\,|F|^2\right]\ dz + \int \frac{N^2 b\,|F|^2}{|U_0 - c|}\,dz \le \int \left|U_0'\right|\,\left|(bF)'\right|\,|F|\ dz;$$

that is

$$\int |U_0 - c|\, Q\, dz + \int \frac{N^2 b\, |F|^2}{|U_0 - c|}\, dz \le \int \left|U_0'\right| \left|(bF)'\right| |F|\; dz. \tag{4.5}$$

By the semi-circle theorem it is obvious that

$$|U_0 - c| \le U_{0\,\max} - U_{0\,\min}, \text{ and so } \quad \frac{1}{|U_0 - c|} \ge \frac{1}{(U_{0\,\max} - U_{0\,\min})}$$

and also we have $|U_0 - c| \ge c_i$. Therefore (4.5) can be written as

$$c_i \int Q\, dz + \frac{1}{(U_{0\,\max} - U_{0\,\min})} \int N^2 b\, |F|^2\; dz \le \int \left|U_0'\right| \left|(bF)'\right| |F|\; dz. \tag{4.6}$$

Now when the condition

$$\int \left[\frac{U_{0\,\max} - U_{0\,\min}}{2} - \frac{\left(U_0 - \left(\frac{U_{0\,\min} + U_{0\,\max}}{2}\right)\right)^2}{\frac{U_{0\,\max} - U_{0\,\min}}{2}}\right] \left|(bF)'\right| |F|\; dz \ge \int h \left|U_0'\right| \left|(bF)'\right| |F|\; dz. \tag{4.7}$$

is fulfilled, where h is a certain constant with the dimensions of length, the inequality (4.4) can be strengthened by expressing the last component in terms of the integral $\int Q\; dz$ with the help of (4.6). Unfortunately, the validity of (4.7) in the general case has not been proved. However, one can readily distinguish the flow profiles at which relation (4.7) is fulfilled. In particular, equating the integrands on the left- and right-hand sides of (4.7), we obtain a differential equation for $U_0(z)$ namely,

$$h \left|U_0'\right| = \left[\frac{U_{0\,\max} - U_{0\,\min}}{2}\right] - \frac{\left[U_0 - \left(\frac{U_{0\,\min} + U_{0\,\max}}{2}\right)\right]^2}{\left(\frac{U_{0\,\max} - U_{0\,\min}}{2}\right)};$$

that is

$$\left|U_0'\right| = \frac{\pm 2}{h\,[U_{0\,\max} - U_{0\,\min}]} \left[\left(\frac{U_{0\,\max} - U_{0\,\min}}{2}\right)^2 - \left(U_0 - \left(\frac{U_{0\,\min} + U_{0\,\max}}{2}\right)\right)^2\right]. \tag{4.8}$$

Therefore

$$U_0(z) = \pm \left[\frac{U_{0\,\min} + U_{0\,\max}}{2} + \left(\frac{U_{0\,\max} - U_{0\,\min}}{2}\right) \tanh\left(\frac{z}{h} - \frac{1}{2}\right)\right]. \tag{4.9}$$

Here h is used in the sense of the characteristic thickness of a shear layer. This profile is often used in numerical computations of shear-flow stability, since its form provides a good approximation to actually observed flows [3]. We may state that (4.7) is fulfilled for the profiles (4.9) or smoother ones, i.e. profiles for which

$$\left|U_0'\right| \leq \frac{2}{[U_{0\,\max} - U_{0\,\min}]}\left[\left(\frac{U_{0\,\max} - U_{0\,\min}}{2}\right)^2 - \left(U_0 - \left(\frac{U_{0\,\min} + U_{0\,\max}}{2}\right)\right)^2\right].$$

Using (4.6) and (4.7) in (4.4), we get

$$\begin{aligned}(4.10)\qquad &\left[\left[c_r - \frac{U_{0\,\min} + U_{0\,\max}}{2}\right]^2 + \left(c_i + kh\left[\frac{U_{0\,\max} - U_{0\,\min}}{2}\right]\right)^2\right. \\ &\quad \left. - \left(1 + k^2h^2\right)\left[\frac{U_{0\,\max} - U_{0\,\min}}{2}\right]^2\right]\int Q\,dz \\ &\quad + (1 + kh)\int N^2 b\,|F|^2\,dz \leq 0.\end{aligned}$$

Since $N^2 \geq 0$, we substitute for the last term from (3.12) and (3.13) to get the following improved instability region.

Theorem 4.1. *An instability region for the eigenvalue problem is given by*

$$\begin{aligned}&\left[c_r - \frac{U_{0\,\min} + U_{0\,\max}}{2}\right]^2 + \left(c_i + kh\left[\frac{U_{0\,\max} - U_{0\,\min}}{2}\right]\right)^2 \\ &\quad + \frac{2(1 + kh)J_0 c_i^2}{\left[1 - 2J_0 + \left(1 - 4J_0 - \frac{4k^2c_i^2}{\left(U_0'\right)^2_{\max}}\right)^{\frac{1}{2}}\right]} \leq \left[1 + (kh)^2\right]\left[\frac{U_{0\,\max} - U_{0\,\min}}{2}\right]^2.\end{aligned}$$

5. Estimate For Growth Rate

Theorem 5.1. *An estimate for growth rate of an unstable mode is given by*

$$k^2 c_i^2 \leq \frac{\left(U_0'\right)^2_{\max}\left[\frac{1}{4} - J_0\right]}{\left[\frac{b_{\min}\pi^2}{b_{\max}k^2D^2} + 1\right]}.$$

Proof. By the well-known Rayleigh-Ritz inequality, we have

$$\int \frac{\left|(bG)'\right|^2}{b}\,dz \geq \frac{b_{\min}\pi^2}{b_{\max}D^2}\int b\,|G|^2\,dz.$$

Using this in (3.5), we have

$$\left[\frac{b_{\min}\pi^2}{b_{\max}D^2}+k^2\right]\int b\,|G|^2\,dz \le \frac{\left(\frac{\left(U_0'\right)^2}{4}-N^2\right)_{\max}}{c_i^2}\int b\,|G|^2\,dz,$$

which yields the estimate

$$k^2c_i^2 \le \frac{\left(U_0'\right)^2_{\max}\left[\frac{1}{4}-J_0\right]}{\left[\frac{b_{\min}\pi^2}{b_{\max}k^2D^2}+1\right]}.$$

The proof is complete. ■

It is seen that the above estimate for the growth rate depends on the stratification parameter, the wave number k and the depth of the shear layer D. Further, it is seen that $c_i \to 0+$ as $k \to \infty$. An open problem here is to prove or disprove Howard's conjecture, namely $kc_i \to 0+$ as $k \to \infty$.

6. Concluding Remarks

In this paper we study an eigenvalue problem of hydrodynamic stability formulated recently in [1]. For this problem we obtain a general instability region which gives a region inside which all eigenvalues c with positive imaginary part should lie in the $c_r - c_i$ plane. Then we improve upon this region for a special class of velocity profiles. Finally we obtain an estimate for the growth rate of any unstable mode.

References

[1] J. Deng, L. Pratt, L. Howard and C. Jones, *On Stratified Shear Flow In Sea Straits of Arbitrary Cross Section*, Studies in Appl. Math. **111**(2003), 409–434.
[2] P.G. Drazin and W.H. Reid, *Hydrodynamic Stability*, Cambridge 1981.
[3] P. Hazel, *Numerical Studies of the Stability of Inviscid Stratified Shear Flows*, J. Fluid Mech. **51**(1972), 39–61.
[4] Yu.N. Makov and Yu.A. Stepanyants, *Note On The Paper of Kochar and Jain On Howard's Semicircle Theorem*, J. Fluid Mech. **140**(1984), 1–10.
[5] M. Subbiah, *Some Results On The Stability of Shear Flow*, Proc. International Symposium On Advances In Fluid Mechanics, Tata McGraw Hill (2005), 389–397.

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