Proceedings of the International Conference on Geometric Function Theory,
Special Functions and Applications (ICGFT)
Editors: R. W. Barnard and S. Ponnusamy

J. Analysis
Volume 15 (2007), 111–143

Vector Bundles over Holomorphic Manifolds on Locally Convex Spaces

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Dedicated to a lost child of cold war

Abstract. Our vector bundles are complex quasi-complete locally convex spaces indexed holomorphically by points in holomorphic manifolds modelled on complex quasi-complete locally convex spaces. Vector bundle maps are locally holomorphic perturbations of continuous linear maps. Various natural constructions of new vector bundles from old vector bundles are presented.

Keywords. Vector bundles, holomorphic manifolds, infinite dimensional complex analysis.

2000 MSC. Primary 55R10, Secondary 58B99.

1. Introduction

1.1. Polynomials are probably the simplest intrinsically nonlinear functions. Analytic functions locally defined by power series expansions establish complex analysis as one of the most beautiful and richest branches of pure mathematics. Subtle *computational methods* of infinite dimensional *function theory* offer a natural umbrella [13] for products of distributions which are traditionally regarded as real analysis [5]. There are several candidates of differentiability on complex locally convex spaces such as those listed in [6, pp 57,59,61] but we commit ourselves to the well-known directional derivatives defined in most undergraduate textbooks in advanced calculus. All our holomorphic maps must be locally bounded. To compensate this restriction, our morphisms are locally holomorphic perturbations of continuous linear maps. With coordinate transformations based on holomorphic locally compact perturbations of identity maps, a theory [14], [15] of infinite dimensional holomorphic manifolds is established within the conventional complex locally convex spaces in contrast to the convenient spaces [9]. Examples of holomorphic manifolds in our sense constructed by level sets of regular values are given in [16]. We hope that infinite dimensional

Research repeatedly interrupted by a protracted war of more than thirty years since SIROMATH, DingoBabyAffair, GoldMintSwindle and Caesars.

complex analysis will be accepted as a substantial part of nonlinear functional analysis in addition to the traditional topological methods.

1.2. In this paper, we set up a theory of vector bundles in parallel to Banach manifolds [1], [10] and [4]. Even restricted to the finite dimensional case, our approach is different to others because we separate the parameters from the objects carefully. We start with a quick review of tangent bundles and notations within our framework in §2. Our tags in §3 are variants of local vector bundle maps [1, p167] in Banach manifolds. In §§4-6, we develop vector bundles, vector bundle maps, restrictions, subbundles, quotient bundles, ranges, kernels and product bundles. Philosophically, we consider vector bundles as functional analysis parameterized by points in manifolds. Because vector bundles over the same manifold can be parameterized by the same parameter in §7, direct sums are constructed accordingly in §8. In order to apply Ascoli's Theorem, the spaces $L_k(E, F)$ of compact linear maps in §8 are equipped with the compact-open topology which is another departure from the traditional treatment in Banach manifolds. A well-known obstacle against the development of manifolds modelled on locally convex spaces is the discontinuity of composites of continuous linear maps but we can get around this in §8.7 with compactness and equicontinuity. Cotangent bundles are introduced at the end. This paper together with [12] prepares ground for future development of various derivatives on holomorphic manifolds. For similar or related results, see [2], [3], [8] and [17].

2. Review of Tangent Bundles

2.1. Throughout this paper, a locally convex space means a separated locally convex space over the complex field $\mathbb{C}$. Here we give a quick review of the background from [11] and [14]. Let E, E_2 be quasi-complete locally convex spaces. A map f_k from an open subset X of E into E_2 is (directionally) *differentiable* if for every $a \in X$ and $x \in E$, the map $t \to f_k(a + tx)$ is differentiable on the open subset $\{t \in \mathbb{C} : a + tx \in X\}$ of $\mathbb{C}$. The *derivative* $Df_k(a) : E \to E_2$ at $a \in X$ is a linear map given by

$$Df_k(a)x = \frac{d}{dt} f_k(a + tx)\bigg|_{t=0} \quad \text{for each } x \in E.$$

The map f_k is *locally bounded* if every point $a \in X$ has a neighborhood $\mathbb{V} \subset X$ such that $f_k(\mathbb{V})$ is a bounded subset of E_2; and *locally compact* if $f_k(\mathbb{V})$ is relatively compact in E_2. A map is *holomorphic* if it is differentiable and locally bounded. As a result, holomorphic maps are continuous and their derivatives are continuous linear maps.

2.2. A map $f : X \to E_2$ is called a *morphism* if at each point $a_0 \in X$, there is a *representation* $f = f_j + f_k$ on some open neighborhood $\mathbb{V} \subset X$ where $f_j : E \to E_2$ is a continuous linear map and $f_k : \mathbb{V} \to E_2$ is a holomorphic map. In this case, f_k is called the *holomorphic part* and f_j the *linear part* of f on $\mathbb{V}$. A morphism f is *locally compact* if every point $a_0 \in X$ has a representation $f = f_j + f_k$ on an open neighborhood $\mathbb{V} \subset X$ such that $f_k(\mathbb{V})$ is relatively compact in E_2. Let X, Y be open subsets of E, E_2 respectively. A morphism $f : X \to E$ is *special* (respectively *special locally compact*) if every point $a_0 \in X$ has a representation $f = f_j + f_k$ on a neighborhood $\mathbb{V}$ where f_j is the identity map on $E = E_2$ (and respectively f_k is locally compact on $\mathbb{V}$).

Although it was stated in its introduction and was included in every proof, it was an obvious but unforgivable hiccup that the definition [14, 2.4] included local compactness as part of special morphisms but failed to mention it explicitly. Both [15], [16] followed the same definition in this paper that local compactness is *no longer* part of special morphisms in order to emphasize its role but unfortunately both articles declared the notations of [14, 2.4].

A bijection $f : X \to Y$ is a *bi-morphism* if both f and f^{-1} are morphisms. The following lemma fills in a small gap of the theory.

2.3. **Lemma**. Let $f : X \to X_2$ be a bi-morphism. If f is special or locally compact or both jointly, then so is f^{-1}.

Proof. Take any $a_0 \in X$. Let $f = f_j + f_k$ where f_j, f_k are the linear and holomorphic parts on some open neighborhood $\mathbb{V}$ of a_0 respectively. Take any $a \in \mathbb{V}$ and write $b = f(a)$. Firstly, suppose that f is special. We may assume that f_j is the identity map on $E = E_2$. From $f^{-1}(b) = a = b - f_k f^{-1}(b)$, f^{-1} is also a holomorphic perturbation of the identity map, that is a special morphism. Next, suppose that f is a locally compact morphism or a special locally compact morphism. We may assume that $f_k(\mathbb{V})$ is contained in some compact subset S of E_2. For the first case, by [11, 2.8] we may assume that $f_j = Df(a_0)$ is the derivative of f at a_0 which is a topological isomorphism from E onto E_2 as a result of the Chain Rule. For the second case we may assume that f_j is the identity map on $E = E_2$. From

$$f^{-1}(b) = a = f_j^{-1}(b) - f_j^{-1} f_k f^{-1}(b),$$

the image of the holomorphic part $-f_j^{-1} f_k f^{-1}$ is contained in the compact set $-f_j^{-1}(S)$. Therefore f^{-1} is also a locally compact morphism. □

2.4. Let M be a nonempty set. A *patch* on M *modelled on* E is a pair (V, φ) where V is a subset of M and $\varphi : V \to E$ is an injection. Two patches $(V, \varphi), (W, \psi)$ on M are *compatible* if both $\varphi(V \cap W)$, $\psi(V \cap W)$ are open in E and both coordinate transformations

$$\psi\varphi^{-1} : \varphi(V \cap W) \to \psi(V \cap W)$$

and

$$\varphi\psi^{-1} : \psi(V \cap W) \to \varphi(V \cap W)$$

are *special locally compact* morphisms. A cover $\mathscr{A}$ of M by patches is called an *atlas* if every two members in $\mathscr{A}$ are compatible. In this case, the family $\mathscr{T}$ of subsets B of M such that for every $(V, \varphi) \in \mathscr{A}$, the set $\varphi(B \cap V)$ is open in E is a topology on M called the *manifold topology* induced by $\mathscr{A}$. A patch on M is called a *chart* if it is compatible with every patch in $\mathscr{A}$. To characterize $\mathscr{T}$ in terms of charts, a subset B of M is open iff for every $m \in B$, there is a chart (V, φ) at m with $V \subset B$. A set M with an atlas $\mathscr{A}$ is called a *holomorphic manifold* if its manifold topology is separated. Locally compact maps and morphisms between manifolds are defined in terms of charts in the standard way.

2.5. Let M be a holomorphic manifold modelled on E. A (complex) *local curve* at the *base point* $m \in M$ is a quadruple $(p, \alpha, \mathbb{P}, m)$ where $\mathbb{P}$ is an open neighborhood of $\alpha \in \mathbb{C}$ and $p : \mathbb{P} \to M$ is a holomorphic map satisfying $p(\alpha) = m$. We may simply write p, (p, α) or (p, α, m) if there is no ambiguity. Two local curves (p, α, m), (q, β, n) are *equivalent*, denoted by $p \sim q$, if $m = n$ and for some chart (V, φ) at m we have $(\varphi p)'(\alpha) = (\varphi q)'(\beta)$. The equivalent classes induced by the equivalence relation $\sim$ are called *tangents* of M. The set $T_m M$ of all tangents at m is called the *tangent space* at m. The tangent containing a local curve p is denoted by $[p]$. The map φ_m from $T_m M$ into E given by $\varphi_m([p]) = (\varphi p)'(\alpha)$ is a bijection which turns $T_m M$ into a quasi-complete locally convex space topologically isomorphic to E independent of the choice of (V, φ). The rule of coordinate transformation from a chart (V, φ) to a chart (W, ψ) for tangents at m is given by $\psi_m(p) = D(\psi\varphi^{-1})(a)\varphi_m(p)$ where $a = \varphi(m)$.

2.6. Take any $a_0 \in \varphi(V \cap W)$. We have $\psi\varphi^{-1} = I + K$ on some neighborhood $\mathbb{V} \subset \varphi(V \cap W)$ of a_0 where I is the identity map on E and $K : \varphi(V \cap W) \to E$ is a locally compact holomorphic map. It follows from the Generalized Hartogs' Theorem that the map $(a, x) \to DK(a)x$ from $\mathbb{V} \times E$ into E is a holomorphic locally compact map. This completes the motivation for the definitions later where $\Phi(a)$ corresponds to $D(\psi\varphi^{-1})(a)$ and Ω_m^φ corresponds to φ_m.

3. Locally Compact Tags

3.1. Let E, E_2, F, F_2 be quasi-complete locally convex spaces and let $L(F, F_2)$ be the set of all continuous linear maps from F into F_2. We may write $L(F) = L(F, F)$. Suppose that X, X_2 are open subsets of E, E_2 respectively. A map

$$G : X \times F \to X_2 \times F_2$$

is called a *parameterized linear map* if there exist a map $g : X \to X_2$ and a map $\Phi : X \to L(F, F_2)$ such that $G(a, x) = \big(g(a), \Phi(a)x\big)$ for all $(a, x) \in X \times F$. In this case, g is called the *parameter* part and Φ the *main* part of G.

3.2. A parameterized linear map $G : X \times F \to X_2 \times F_2$ is called a *tag* if the parameter part $g : X \to X_2$ is a morphism and if for every $a_0 \in X$, there exist an open neighborhood $\mathbb{V} \subset X$, a continuous linear map $\Phi_j : F \to F_2$ and a map $\Phi_k : \mathbb{V} \to L(F, F_2)$ such that $\Phi(a) = \Phi_j + \Phi_k(a)$ for each $a \in \mathbb{V}$ and that the map $(a, x) \to \Phi_k(a)x$ from $\mathbb{V} \times F$ into F_2 is holomorphic. In this case, Φ_j is called the *linear part* and Φ_k the *holomorphic part* of Φ on $\mathbb{V}$. A tag is *isomorphic* if it is bijective and its inverse map is also a tag. A tag G is *locally compact* if $(a, x) \to \Phi_k(a)x$ is a locally compact map on $\mathbb{V} \times F$. A tag G is *special* if Φ_j is the identity map on $F = F_2$. By a special locally compact tag G, we mean $\Phi(a) = \Phi_j + \Phi_k(a)$ for each a in some neighborhood $\mathbb{V}$ of a_0 where Φ_j is the identity map on $F = F_2$ and at the same time $(a, x) \to \Phi_k(a)x$ is a locally compact map on $\mathbb{V} \times F$. It would be nice if we could prove that the separate occurrences imply the joint occurrence. It would be good if we could have standard representations similar to [11, 2.7].

A linear map $\xi : F \to F_2$ is *compact* if there is a 0-neighborhood $\mathfrak{U}$ of F such that the set $\xi(\mathfrak{U})$ is relatively compact in F_2. A family $\mathbb{F}$ of linear maps from F into F_2 is *collectively compact* if there exist a 0-neighborhood $\mathfrak{U}$ of F and a compact subset C of F_2 such that $\xi(\mathfrak{U}) \subset C$ for all $\xi \in \mathbb{F}$. A map $\Psi_k : X \to L(F, F_2)$ is *locally collectively compact* if every $a_0 \in X$ has a neighborhood $\mathbb{V}$ such that $\Psi_k(\mathbb{V})$ is collectively compact.

3.3. **Lemma**. Let $\Psi_k : X \to L(F, F_2)$ be a map. If $\Lambda : X \times F \to F_2$ given by $\Lambda(a, x) = \Psi_k(a)x$ is a locally compact holomorphic map, then Ψ_k is locally collectively compact. Furthermore the map $\Psi_k : X \to L_k(F, F_2)$ is locally compact if the space $L_k(F, F_2)$ of all compact linear maps is equipped with the compact-open topology.

Proof. Since Λ is locally compact at $(a_0, 0) \in X \times F$, there exist an open neighborhood $\mathbb{V}$ of a_0, an open 0-neighborhood $\mathfrak{U}$ of F and a compact subset C of F_2 such that $\Lambda(\mathbb{V} \times \mathfrak{U}) \subset C$. Hence the set $\Psi_k(\mathbb{V})$ is collectively compact because $\Psi_k(\mathbb{V})(\mathfrak{U}) \subset C$. In particular, we have $\Psi_k(\mathbb{V}) \subset L_k(F, F_2)$. We need to prove that $\Psi_k(\mathbb{V})$ is a relatively compact subset of $L_k(F, F_2)$. Take any $x \in F$. Then $x \in \theta\,\mathfrak{U}$ for some $\theta > 0$. Hence $\Psi_k(\mathbb{V})(x) \subset \Psi_k(\mathbb{V})(\theta\,\mathfrak{U}) \subset \theta C$. As a subset of the compact set $\theta\, C$, the set $\Psi_k(\mathbb{V})(x)$ is relatively compact in F_2. Next take any 0-neighborhood $\mathfrak{W}$ of F_2. Then $C \subset \tau\,\mathfrak{W}$ for some $\tau > 0$. Hence $\Psi_k(\mathbb{V})(\mathfrak{U}/\tau) \subset \mathfrak{W}$. Since $\mathfrak{U}/\tau$ is also a 0-neighborhood of F, the set $\Psi_k(\mathbb{V})$ is equicontinuous. By Ascoli's Theorem, e.g. [7, p34], $\Psi_k(\mathbb{V})$ is relatively compact in $L_k(F, F_2)$ equipped with the compact-open topology. □

3.4. **Theorem**. If $G : X \times F \to X_2 \times F_2$ is an isomorphic tag, then:

(a) the parameter part $g : X \to X_2$ is a bi-morphism,

(b) each $\Phi(a) : F \to F_2$ is a topological isomorphism for every $a \in X$,

(c) if G is special only or special locally compact, then so is G^{-1}.

Proof. Let h be the parameter part and Ψ be the main part of the tag G^{-1}. Then we have $G^{-1}(b, y) = \big(h(b), \Psi(b)y\big)$ for every $(b, y) \in X_2 \times F_2$. Clearly hg and gh are the identity map on X, X_2 respectively. For every $a \in X$, let $b = g(a)$. Both $\Psi(b)\Phi(a)$ and $\Phi(a)\Psi(b)$ are the identity map on F, F_2 respectively. This proves (a) and (b). In particular, $g : X \to X_2$ is a homeomorphism. Next, suppose that G is a special tag. Take any $b_0 \in X_2$. Choose an open neighborhood $\mathbb{V} \subset X$ of $a_0 = h(b_0)$ such that $\Phi(a) = \Phi_j + \Phi_k(a)$ for all $a \in \mathbb{V}$ where Φ_j is the identity map on $F = F_2$ and Φ_k is the holomorphic part of Φ on $\mathbb{V}$. There is an open neighborhood $\mathbb{W} \subset g(\mathbb{V})$ of b_0 such that $\Psi(b) = \Psi_j + \Psi_k(b)$ for all $b \in \mathbb{W}$ where Ψ_j is the linear part and Ψ_k is the holomorphic part of Ψ on $\mathbb{W}$. Consider any $b \in \mathbb{W}$. Then $a = h(b) \in \mathbb{V}$ and $\Gamma_k(b) = -\Phi_k(a)\Psi(b) \in L(F_2)$. Pick any $y_0 \in F_2$. Then $x_0 = \Psi(b_0)y_0 \in F$. There exist an open neighborhood $\mathbb{V}_1 \subset h(\mathbb{W})$ of a_0, an open neighborhood $\mathfrak{U}$ of x_0 and a bounded subset B of F_2 such that $\Phi_k(a)x \in B$ for all $(a, x) \in \mathbb{V}_1 \times \mathfrak{U}$. By continuity of the map $(b, y) \to \Psi(b)y$, there exist an open neighborhood $\mathbb{W}_1 \subset g(\mathbb{V}_1)$ of b_0 and an open neighborhood $\mathfrak{S}$ of y_0 such that $\Psi(b)y \in \mathfrak{U}$ for all (b, y) in $W_1 \times \mathfrak{S}$. Fix any $(b, y) \in W_1 \times \mathfrak{S}$. Then we get $a = h(b) \in \mathbb{V}_1$ and $x = \Psi(b)y \in \mathfrak{U}$. It is simple to verify that

$$\Gamma_k(b)y = -\Phi_k(a)x \in -B.$$

Therefore the map $(b, y) \to \Gamma_k(b)y$ is bounded on the open neighborhood $W_1 \times \mathfrak{S}$ of (b_0, y_0). Also from $y = \Phi(a)x = x + \Phi_k(a)x$, we have

$$\Gamma_k(b)y = -\Phi_k(a)x = x - y = \Psi(b)y - y = \Psi_j y - y + \Psi_k(b)y\,.$$

Hence the bounded map $(b, y) \to \Gamma_k(b)y$ is separately holomorphic on $W_1 \times \mathfrak{S}$ and it is jointly holomorphic by the Generalized Hartogs' Theorem. From $G^{-1}(b, y) = \big(h(b), \Gamma(b)y\big)$ where $\Gamma(b)y = y + \Gamma_k(b)y = x$, the tag G^{-1} is also special. Finally if G is special locally compact, replacement of B by a compact subset of F_2 completes the proof. □

3.5. **Theorem**. The composite of tags is a tag. If all factors are special, then so is the composite. If one of them is locally compact, then so is the composite.

Proof. Let E, E_2, F, F_2, F_3 be quasi-complete locally convex spaces and let X, X_2, X_3 be open subsets of E, E_2, E_3 respectively. Suppose that

$$X \times F \xrightarrow{G} X_2 \times F_2 \xrightarrow{H} X_3 \times F_3$$

are tags with the parameter parts g, h and the main parts Φ, Ψ respectively. For every (a, x) in $X \times F$, define $b = g(a)$, $q(a) = h(b)$, $\Gamma(a) = \Psi(b)\Phi(a)$ and also define $Q(a, x) = \big(q(a), \Gamma(a)x\big)$. Clearly $Q : X \times F \to X_3 \times F_3$ is a parameterized linear map and the parameter part $q = hg : X \to X_3$ is a morphism. Take any $a_0 \in X$. Let Φ_j, Ψ_j be the linear parts and let Φ_k, Ψ_k be the holomorphic parts of Φ, Ψ on some open neighborhoods $\mathbb{V} \subset X$, $\mathbb{W} \subset X_2$ of a_0, $b_0 = g(a_0)$ respectively. By continuity of the morphism g, we may assume $g(\mathbb{V}) \subset \mathbb{W}$. For all $(a, x) \in \mathbb{V} \times F$, we obtain $\Gamma(a) = \Gamma_j + \Gamma_k(a)$ where $\Gamma_j = \Psi_j \Phi_j \in L(F, F_3)$ and

$$\Gamma_k(a) = \Psi_k(b)\Phi_j + \Psi_j \Phi_k(a) + \Psi_k(b)\Phi_k(a) \in L(F, F_3).$$

For the last term as an example, the maps $h_1 : (a, x) \to \Phi_k(a)x$ and also $h_2 : (b, y) \to \Psi_k(b)y$ are holomorphic. Note that the continuous linear map $p : (a, x) \to a$ is a morphism. Thus $h_2(gp, h_1) : (a, x) \to \Psi_k(b)\Phi_k(a)x$ is holomorphic by [11, 2.9]. Hence the map $(a, x) \to \Gamma_k(a)x$ from $\mathbb{V} \times F$ into F_3 is holomorphic. Since $a_0 \in X$ is arbitrary, Q is a tag. If both G, H are special, then $\Gamma_j = \Psi_j = \Phi_j$ is the identity map on $F = F_2 = F_3$ and hence the composite Q is also special. Finally if

$$(a, x) \to \Phi_k(a)x : \mathbb{V} \times F \to F_2$$

or

$$(b, y) \to \Psi_k(b)y : \mathbb{W} \times F_2 \to F_3$$

is a locally compact map, then $(a, x) \to \Gamma(a)x : \mathbb{V} \times F \to F_3$ is also a locally compact map by [11, 2.9]. This completes the proof. □

3.6. Although it can be proved that products and direct sums of tags are tags, yet the notation does not fit in what we need in the constructions later. So they are embedded into the proofs of §§6.13, 8.4.

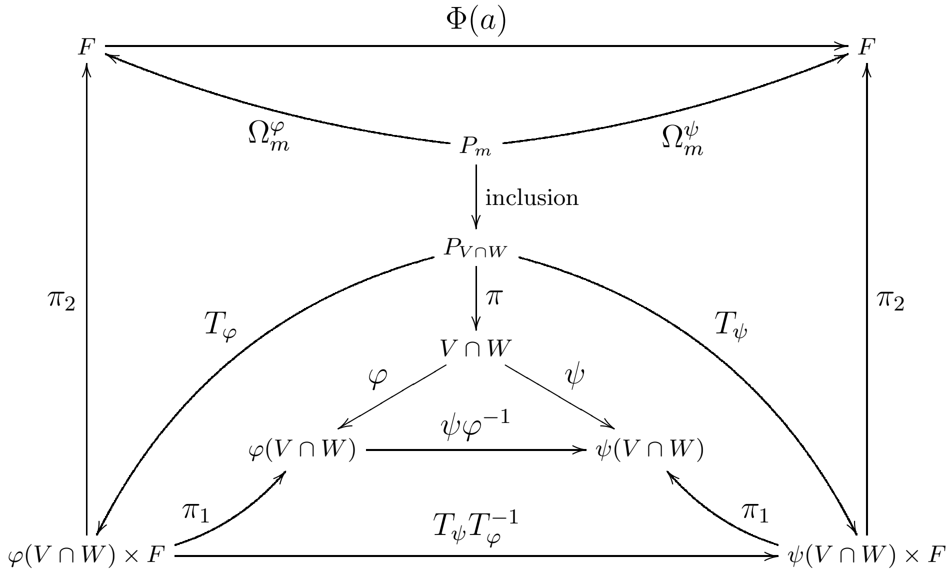
4. Vector Bundles

4.1. Let M be a holomorphic manifold modelled on E with an atlas $\mathscr{A}$. Suppose that π is a surjection from a set P onto M. The set $P_m = \pi^{-1}(m)$ is called the *fiber* over $m \in M$. For every subset V of M, we write

$$P_V = \bigcup_{m \in V} P_m = \pi^{-1}(V).$$

The projections of $E \times F$ onto E, F are denoted by π_1, π_2 respectively.

4.2. A triple (V, φ, T_φ) is called a *bundle patch* on M with *fiber space* F if (V, φ) is a chart on M and $T_\varphi : P_V \to \varphi(V) \times F$ is a bijection such that $\varphi\pi = \pi_1 T_\varphi$. For every $m \in V$, the bijection $\Omega_m^\varphi = \pi_2 T_\varphi \big| P_m$ from P_m onto F is called the *fiber representation* at m. It follows by definition that $T_\varphi(A) = \big(a, \Omega_m^\varphi(A)\big)$ for all $A \in P_m$ where $a = \varphi(m)$. We shall turn each P_m into a vector space which is topologically isomorphic to F.



4.3. Let (V, φ, T_φ), (W, ψ, T_ψ) be two bundle patches on M. Then (V, φ, T_φ) is *compatible with* (W, ψ, T_ψ) if the *bundle transformation*

$$T_\psi T_\varphi^{-1} : \varphi(V \cap W) \times F \to \psi(V \cap W) \times F$$

is a special locally compact isomorphic tag. In this case, let Φ be the main part of $T_\psi T_\varphi^{-1}$. Pick any $a_0 \in \varphi(V)$. Choose an open neighborhood $\mathbb{V}$ of a_0 such that $\Phi(a) = \Phi_j + \Phi_k(a)$ for every $a \in \mathbb{V} \subset \varphi(V \cap W)$ where Φ_j is the identity map on F and Φ_k is the holomorphic part of Φ. Replacing $\mathbb{V}$ by a smaller one, we may assume that $\psi\varphi^{-1} = I + K$ on $\mathbb{V}$ where I is the identity map on E and $K : \varphi(V \cap W) \to E$ is a locally compact holomorphic map. For every $(a, x) \in \varphi(\mathbb{V}) \times F$, write $A = T_\varphi^{-1}(a, x)$ and $(b, y) = T_\psi(A)$. Then we have

$$(b, y) = T_\psi T_\varphi^{-1}(a, x) = \big(\psi\varphi^{-1}(a), \Phi(a)x\big) = (a, x) + \big(Ka, \Phi_k(a)x\big). \qquad (a)$$

Hence $T_\psi T_\varphi^{-1}$ is a special locally compact morphism. For $m = \varphi^{-1}(a)$ in $V \cap W$, we obtain $\Omega_m^\psi A = y = \Phi(a)x = \Phi(a)\Omega_m^\varphi(A)$, that is

$$\Phi(a) = \Omega_m^\psi(\Omega_m^\varphi)^{-1} = \Phi_j + \Phi_k(a). \qquad (b)$$

If either V or W can be replaced by smaller ones, we may assume $\mathbb{V} = \varphi(V \cap W)$. To avoid too much repetition, we shall use the above notation involving m, a_0, $\mathbb{V}$, A, a, x, b, y, Φ, Φ_j, Φ_k, Ω_m^φ, Ω_m^ψ, I and K whenever §4.3 is quoted.

4.4. A bundle patch (V, φ, T_φ) *contains* $m \in M$ or *is at* m if $m \in V$. A family $\mathscr{B}$ of pairwise compatible bundle patches on M is called a *bundle atlas* for π if $\mathscr{B}$ covers M. In this case, the triple $(P, \pi, \mathscr{B})$ is called a *vector bundle* over M. Note that we frequently construct $\mathscr{B}$ from $\mathscr{A}$ as in tangent bundles.

4.5. Let $(P, \pi, \mathscr{B})$ be a vector bundle with fiber space F over a holomorphic manifold M modelled on E. Then P is called the *total space*, M the *base manifold*, E the *base space* and π the *projection*. A bundle patch (V, φ, T_φ) is called a *bundle chart* if it is compatible with every bundle patch in $\mathscr{B}$. Clearly any two bundle charts are compatible as a result of §§3.4,5. The family of all bundle charts is called the *bundle structure* of P. If the projection is not specified explicitly, the same symbol π is assumed for different vector bundles. Because we always work with bundle charts, the transitional role of $\mathscr{B}$ is rarely mentioned except during the initial construction of new vector bundles. We also say that the symbol P, or the pair (P, π), or the surjection $\pi : P \to M$ is a vector bundle.

4.6. **Theorem**. Every fiber P_m is a quasi-complete locally convex space such that for every bundle chart (V, φ, T_φ) at m, the fiber representation Ω_m^φ from P_m onto F is a topological isomorphism.

Proof. Since Ω_m^φ is a bijection, the linear combinations in P_m are defined by

$$\Omega_m^\varphi(\alpha A + \beta B) = \alpha \Omega_m^\varphi(A) + \beta \Omega_m^\varphi(B) \quad \text{for all } A, B \in P_m \text{ and } \alpha, \beta \in \mathbb{C}.$$

Suppose that the topology of F is given by a family of seminorms $x \to \|x\|_\theta$ for θ in an index set $\mathfrak{S}$. Then the seminorms $A \to \|\Omega_m^\varphi A\|_\theta$ for $\theta \in \mathfrak{S}$ also define a locally convex topology on P_m. By definition, P_m becomes a quasi-complete locally convex space such that Ω_m^φ is a topological isomorphism. For any bundle chart (W, ψ, T_ψ) at m, because $\Phi(a)$ in §4.3b is an algebraic automorphism on F, we get

$$\begin{aligned}
\Omega_m^\psi(\alpha A + \beta B) &= \Phi(a)\Omega_m^\varphi(\alpha A + \beta B) \\
&= \Phi(a)\left[\alpha \Omega_m^\varphi(A) + \beta \Omega_m^\varphi(B)\right] \\
&= \alpha \Phi(a)\Omega_m^\varphi(A) + \beta \Phi(a)\Omega_m^\varphi(B) \\
&= \alpha \Omega_m^\psi(A) + \beta \Omega_m^\psi(B).
\end{aligned}$$

Therefore the linear combinations in P_m are independent of the choice of (V, φ, T_φ). Similarly since $\Phi(a)$ is a topological automorphism on F, both bundle charts define the same locally convex topology on P_m. This completes the proof. □

4.7. **Theorem**. (a) The total space P is a holomorphic manifold modelled on $E \times F$ under the atlas $\mathscr{B}_P = \{(P_V, T_\varphi) : (V, \varphi, T_\varphi) \in \mathscr{B}\}$.

(b) If (V, φ, T_φ) is a bundle chart, then (P_V, T_φ) is a chart on P.

(c) The projection $\pi : P \to M$ is a morphism [14, 4.1].

<u>*Proof*</u>. Let (V, φ, T_φ) and (W, ψ, T_ψ) be bundle charts. It suffices to verify [14, 3.1, 2]. Clearly the map T_φ is injective from P_V into $E \times F$. The set

$$T_\varphi(P_V \cap P_W) = \varphi(V \cap W) \times F$$

is open in $E \times F$. By §4.3a, $T_\psi T_\varphi^{-1}$ is a special locally compact morphism. Therefore (P_V, T_φ), (P_W, T_ψ) are compatible patches of P by symmetry. Next, take any $A \in P$. Choose $m \in M$ with $A \in P_m$. Select $(V, \varphi, T_\varphi) \in \mathscr{B}$ with $m \in V$. Then $A \in P_m \subset P_V$. Thus $\mathscr{B}_P$ covers P. Therefore $\mathscr{B}_P$ is an atlas on P. Part (b) follows by definition. To show that the manifold topology is separated, let $A \neq B$ in P. If $m = \pi(A) \neq \pi(B) = n$, choose disjoint open subsets G, H of M with $m \in G$ and $n \in H$. Let (V, φ, T_φ) and (W, ψ, T_ψ) be bundle charts containing m, n respectively. Then $(V \cap G, \varphi, T_\varphi)$ and $(W \cap H, \psi, T_\psi)$ are bundle charts of P. So, $(P_{V\cap G}, T_\varphi)$ and $(P_{W\cap H}, T_\psi)$ are disjoint charts on P containing A, B respectively. On the other hand, if $\pi(A) = \pi(B) = m$, then for every bundle chart (V, φ, T_φ) at m, (P_V, T_φ) is a chart of P containing both A, B. Hence A, B can also be separated [14, 3.12] by open sets in P. Therefore the manifold topology of P is separated. Consequently P becomes a manifold modelled on $E \times F$. Finally, take any (a, x) in $\varphi(V) \times F$ and let $A = T_\varphi^{-1}(a, x)$. Then we have

$$\varphi\pi T_\varphi^{-1}(a, x) = \varphi\pi(A) = \varphi(m) = a.$$

Since $\varphi\pi T_\varphi^{-1}$ is the projection onto the first coordinate, it is also a morphism. As a result, π is also a morphism. □

4.8. Consider any point A in the manifold P and any bundle chart (V, φ, T_φ) at $m = \pi(A)$. For the chart (P_V, T_φ) at A on the manifold P, the fiber representation denoted by $T_{\varphi A} = (T_\varphi)_A$ is a topological isomorphism from the tangent space $T_A P$ onto the model space $E \times F$.

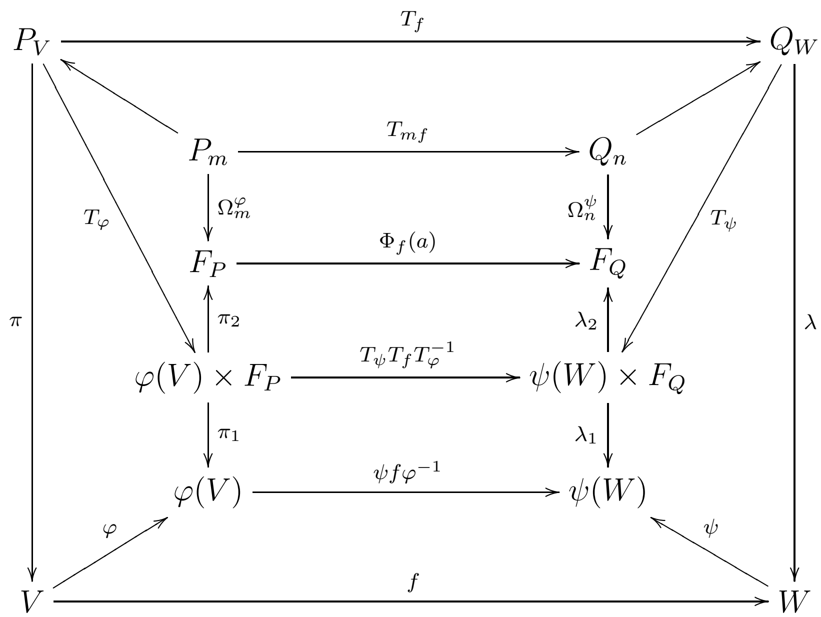
4.9. <u>**Theorem**</u>. The projection $\pi : P \to M$ is a submersion [15, 3.2]. More precisely for every $A \in P$, the differential $d\pi(A) : T_A P \to T_m M$ is a surjection and the kernel of $d\pi(A)$ splits in $T_A P$.

<u>*Proof*</u>. As the projection $\varphi\pi T_\varphi^{-1}$ from $E \times F$ onto E, it is surjective and its kernel $\{0\} \times F$ splits in $E \times F$. The result follows by translation through the fiber representations φ_m and $T_{\varphi A}$. □

4.10. Let $\mathscr{B}, \mathscr{C}$ be bundle atlases with fiber space F for the same surjection π from a set P onto an holomorphic manifold M modelled on E. The bundle structures of $\mathscr{B}, \mathscr{C}$ are denoted by $\mathbb{S}\mathscr{B}, \mathbb{S}\mathscr{C}$ respectively. Clearly every bundle patch is a bundle chart, that is $\mathscr{B} \subset \mathbb{S}\mathscr{B}$. Every bundle structure $\mathbb{S}\mathscr{B}$ is a bundle atlas. If $\mathscr{B} \subset \mathscr{C}$, then $\mathbb{S}\mathscr{C} \subset \mathbb{S}\mathscr{B}$. In particular, $\mathbb{S}\mathscr{B}$ is maximal. The bundle structure of $\mathbb{S}\mathscr{B}$ is $\mathbb{S}\mathscr{B}$, that is $\mathbb{S}\mathbb{S}\mathscr{B} = \mathbb{S}\mathscr{B}$.

5. Vector Bundle Maps

5.1. Let $\pi : P \to M$, $\lambda : Q \to N$ be vector bundles over holomorphic manifolds M, N modelled on quasi-complete locally convex spaces E_M, E_N with fiber spaces F_P, F_Q respectively. Consider a pair of maps $f : M \to N$ and $T_f : P \to Q$. Clearly $\lambda T_f = f\pi$ iff T_f is *fiber preserving*, that is $T_f(P_m) \subset Q_n$ for every $m \in M$ where $n = f(m)$. The restriction of T_f to P_m is denoted by T_{mf}.



5.2. A fiber preserving map $T_f : P \to Q$ over a morphism $f : M \to N$ is called a *vector bundle map* if for every $m \in M$, there exist a bundle chart (V, φ, T_φ) at m and a bundle chart (W, ψ, T_ψ) at $n = f(m)$ such that $f(V) \subset W$ and the *bundle representation*

$$T_\psi T_f T_\varphi^{-1} : \varphi(V) \times F_P \to \psi(W) \times F_Q$$

is a tag. Naturally a vector bundle map T_f is *locally compact* if every point $m \in M$ has a locally compact bundle representation $T_\psi T_f T_\varphi^{-1}$. Similarly special vector bundle maps and special locally compact vector bundle maps are defined in terms of their bundle representations. A vector bundle map T_f is *isomorphic* if f is a diffeomorphism [14, 4.1], T_f is bijective and T_f^{-1} is a vector bundle map over f^{-1}.

5.3. **Lemma**. Let $T_f : P \to Q$ be a vector bundle map over a morphism $f : M \to N$. For every $m \in M$ and every bundle chart (W, ψ, T_ψ) at $n = f(m)$, there exists a bundle chart (V, φ, T_φ) at m with $f(V) \subset W$ such that the map $T_\psi T_f T_\varphi^{-1}$ is a tag. Furthermore if T_f is a locally compact bundle map, then $T_\psi T_f T_\varphi^{-1}$ is also a locally compact tag.

Proof. Let (H, h, T_h) and (Q, q, T_q) be bundle charts of P, Q at m, n respectively with $f(H) \subset Q$ such that the bundle representation

$$T_q T_f T_h^{-1} : h(H) \times F_P \to q(Q) \times F_Q$$

is a tag. Since (Q, q, T_q) and (W, ψ, T_ψ) are compatible, the bundle transformation

$$T_\psi T_q^{-1} : q(Q \cap W) \times F_Q \to \psi(Q \cap W) \times F_Q$$

is also a tag. Then $V = H \cap f^{-1}(Q \cap W)$ is an open neighborhood of m. Let $\varphi = h|V$ and $T_\varphi = T_h|P_V$. Then (V, φ, T_φ) is a bundle chart of P with $f(V) \subset W$. Also the composite $T_\psi T_f T_\varphi^{-1} = (T_\psi T_q^{-1})\,(T_q T_f T_\varphi^{-1})$ of tags is a tag. The last statement follows immediately from §3.5. □

5.4. **Theorem**. Composites of vector bundle maps are vector bundle maps. Furthermore if all factors are special, then so is the composite. If one of them is locally compact, then so is the composite.

Proof. It follows immediately from the last lemma and §3.5. □

5.5. **Theorem**. Let $T_f : P \to Q$ be a vector bundle map over a morphism $f : M \to N$. Then:

(a) T_f is a morphism from the manifold P into the manifold Q.

(b) $T_{mf} : P_m \to Q_n$ is a continuous linear map where $n = f(m)$.

Proof. Take any $A \in P$. Let $m_0 = \pi(A)$, $n_0 = f(m_0)$ and $B = T_f(A)$. Let (V, φ, T_φ) be a bundle chart at m_0 and (W, ψ, T_ψ) a bundle chart at $n_0 = f(m_0)$ with $f(V) \subset W$ such that $T_\psi T_f T_\varphi^{-1}$ is a tag. Let Φ be the main part of $T_\psi T_f T_\varphi^{-1}$. Replacing V by a smaller one, we may assume that Φ_j is the linear part of Φ and Φ_k is the holomorphic part of Φ on V. Further replacement allows us to assume that $\psi f \varphi^{-1} = f_j + f_k$ is the standard representation [11, 2.7]. Then for every $m \in V$, the linear part $f_j = D(\psi f \varphi^{-1})(a) : E_M \to E_N$ is continuous linear where $a = \varphi(m)$ and the nonlinear part $f_k : \varphi(V) \to E_N$ is holomorphic.

(a) Observe that (P_V, T_φ) and (Q_W, T_ψ) are charts on the manifolds P, Q respectively. Clearly

$$(a, x) \to \big(f_j(a), \Phi_j(x)\big) : E_M \times F_P \to E_N \times F_Q$$

is a continuous linear map and $(a,x) \to \big(f_k(a), \Phi_k(a)x\big)$ is holomorphic on $\varphi(V)\times F_P$. Now

$$T_\psi T_f T_\varphi^{-1}(a,x) = (\psi f \varphi^{-1}(a), \Phi(a)x) = \big(f_j(a), \Phi_j(x)\big) + \big(f_k(a), \Phi_k(a)x\big)$$

shows that T_f is a morphism.

(b) Take any $A \in P_m$. For $x = \Omega_m^\varphi(A)$, we have

$$\begin{aligned} \Omega_n^\psi T_{mf}(A) &= \lambda_2 T_\psi T_f T_\varphi^{-1}(a,x) \\ &= \lambda_2\big(\psi f \varphi^{-1}(a), \Phi(a)x\big) \\ &= \Phi(a)x = \Phi(a)\Omega_m^\varphi(A). \end{aligned}$$

Therefore $T_{mf} = (\Omega_n^\psi)^{-1}\Phi(a)\Omega_m^\varphi$ is a continuous linear map. □

6. Simple Constructions

6.1. In this section, we shall construct restrictions, subbundles, quotient bundles. We shall study kernels, ranges of vector bundle maps. Finally we construct (direct) products of vector bundles.

6.2. Let P be a vector bundle with fiber space F over a holomorphic manifold M modelled on E under the projection $\pi : P \to M$. Let N be a submanifold [14, 8.2] of M modelled on a splitting subspace $\mathbb{E}$ of E. For $Q = \pi^{-1}(N)$, the map $\tau = \pi|Q$ is a surjection from Q onto N. A bundle chart (V, φ, T_φ) of P is *adapted* for N if (V, φ) is an adapted chart on M for N, that is $\varphi(V \cap N) = \varphi(V) \cap \mathbb{E}$. The set Q is called the *restriction* of P to N if N is covered by a family $\mathscr{B}$ of adapted bundle charts. Write

$$Q_V = \tau^{-1}(V), V_N = V \cap N, \varphi_N = \varphi|V_N, S_\varphi = T_\varphi|Q_V$$

and

$$\mathscr{B}|N = \{\, (V_N, \varphi_N, S_\varphi) : (V, \varphi, T_\varphi) \in \mathscr{B}\}.$$

6.3. **Theorem**. The restriction Q of P is a vector bundle over N with fiber space F under the bundle atlas $\mathscr{B}|N$. Furthermore Q is a submanifold of P.

Proof. Firstly, $S_\varphi = T_\varphi|Q_V$ is a bijection from $Q_V = P_{V\cap N}$ onto

$$\varphi_N(V_N) \times F = \varphi(V \cap N) \times F$$

satisfying $\varphi_N \tau = \varphi\pi = \pi_1 T_\varphi = \tau_1 S_\varphi$ on Q_V. So, $(V_N, \varphi_N, S_\varphi)$ is a bundle patch on N. Next, take any (W, ψ, T_ψ) in $\mathscr{B}$ and use the notation of §4.3. For every $(a,x) \in \varphi_N(V \cap W) \times F$, we have

$$S_\psi S_\varphi^{-1}(a,x) = T_\psi T_\varphi^{-1}(a,x) = (\psi\varphi^{-1}(a), \Phi(a)x).$$

Clearly the restriction of $(a, x) \to \Phi_k(a)x$ to $\varphi_N(V \cap W) \times F$ is locally compact holomorphic. By symmetry, $(V_N, \varphi_N, S_\varphi)$ and (W_N, ψ_N, S_ψ) are compatible. Therefore $\mathscr{B}|N$ is a bundle atlas on Q. Next, let $\mathbb{H} = E \ominus \mathbb{E}$ be any topological complement. Then $\mathbb{E} \times F$ splits in $E \times F$ because of

$$E \times F \simeq (\mathbb{E} \oplus \mathbb{H}) \times F \simeq (\mathbb{E} \times F) \oplus (\mathbb{H} \times F).$$

Since

$$\begin{aligned} T_\varphi(P_V \cap Q) &= T_\varphi(P_{V \cap N}) = \varphi(V \cap N) \times F = [\varphi(V) \cap \mathbb{E}] \times F \\ &= [\varphi(V) \times F] \cap (\mathbb{E} \times F) = T_\varphi(P_V) \cap (\mathbb{E} \times F), \end{aligned}$$

Q is a $(\mathbb{E} \times F)$-submanifold of P. □

6.4. Restrictions reduce the size of the index set from M to N while subbundles reduce the size of the fiber spaces. Let $\mathbb{G}$ be a splitting subspace of F, R a subset of P and $\lambda = \pi|R$ the restriction. A bundle chart (V, φ, T_φ) of P is called a *subbundle chart for* R with the *fiber subspace* $\mathbb{G}$ if $T_\varphi(P_V \cap R) = \varphi(V) \times \mathbb{G}$. A family of subbundle charts for R is a *subbundle atlas* on M of P for R if it covers M. Both the set R and the map λ are called a *subbundle* of P if there is a subbundle atlas on M for R.

$$\begin{array}{ccc} P_V & \xrightarrow{T_\varphi} & \varphi(V) \times F \\ \downarrow \pi & & \downarrow \pi_1 \\ V & \xrightarrow{\varphi} & \varphi(V) \\ \uparrow \lambda & & \uparrow \lambda_1 \\ R_V & \xrightarrow{S_\varphi = T_\varphi | R_V} & \varphi(V) \times \mathbb{G} \end{array}$$

6.5. **Theorem**. Let R be a subbundle of P with a subbundle atlas $\mathscr{B}$ on M and with a fiber subspace $\mathbb{G}$. Write $R_V = \lambda^{-1}(V) = P_V \cap R$, $S_\varphi = T_\varphi|R_V$ and $\mathscr{B}_R = \{(V, \varphi, S_\varphi) : (V, \varphi, T_\varphi) \in \mathscr{B}\}$. Then:

(a) R is a vector bundle with fiber space $\mathbb{G}$ and bundle atlas $\mathscr{B}_R$.

(b) $\Omega^\varphi_m|R_m$ is a topological isomorphism from the vector subspace R_m of P_m onto $\mathbb{G}$.

(c) If (V, φ, T_φ) and (W, ψ, T_ψ) are subbundle charts for R in $\mathscr{B}$, then $\Phi(a)$ in §4.3b is a topological automorphism on $\mathbb{G}$. Furthermore we have $\Phi_k(a)(\mathbb{G}) \subset \mathbb{G}$ and the map

$$(a, y) \to \Phi_k(a)y : \varphi(V \cap W) \times \mathbb{G} \to \mathbb{G}$$

is a locally compact holomorphic map.

(d) R is a submanifold of P.

Proof. Since $\mathscr{B}$ covers M, the restriction λ is surjective. Let (V, φ, T_φ), (W, ψ, T_ψ) be subbundle charts in $\mathscr{B}$. We use the notation of §4.3. By definition, S_φ is a bijection from R_V onto $\varphi(V) \times \mathbb{G}$ satisfying $\lambda_1 S_\varphi = \varphi\lambda$. Thus (V, φ, S_φ) is a bundle patch for R. Next, we want to prove that the map

$$S_\psi S_\varphi^{-1} : S_\varphi(R_V \cap R_W) = \varphi(V \cap W) \times \mathbb{G} \to S_\psi(R_V \cap R_W)$$

is a special locally compact tag. Take any (a, x) in $S_\varphi(R_V \cap R_W)$ and write $(b, y) = S_\psi S_\varphi^{-1}(a, x)$. Then $y = \Phi(a)x \in \mathbb{G}$. Hence we have

$$\Phi_k(a)x = \Phi(a)x - x \in \mathbb{G}.$$

Since the map $(c, z) \to \Phi_k(c)z$ from $\varphi(V \cap W) \times F$ into F is locally compact, there exist an open neighborhood $Y \subset \varphi(V)$ of a, an open neighborhood X_0 of x and a compact subset C_0 of F such that $\Phi_k(Y)X_0 \subset C_0$. Now the set $C = C_0 \cap \mathbb{G}$ is compact in the close subspace $\mathbb{G}$ of F and $X = X_0 \cap \mathbb{G}$ is an open neighborhood of $x \in \mathbb{G}$. The map $(c, z) \to \Phi_k(c)z$ from $Y \times X$ into $C = C_0 \cap \mathbb{G}$ is locally compact holomorphic. This proves (c). By symmetry, (V, φ, S_φ), (W, ψ, S_ψ) are compatible bundle patches of R. Therefore R is a vector bundle with fiber space $\mathbb{G}$ and bundle atlas $\mathscr{B}_R$. Part (b) follows from $\Omega_m^\varphi R_m = \mathbb{G}$ and $\Phi(a) = \Omega_m^\psi(\Omega_m^\varphi)^{-1}$. As a result of

$$\begin{aligned} T_\varphi(P_V \cap R) &= \varphi(V) \times \mathbb{G} \\ &= [\varphi(V) \times F] \cap (E \times \mathbb{G}) \\ &= T_\varphi(P_V) \cap (E \times \mathbb{G}) \end{aligned}$$

R is an $(E \times \mathbb{G})$-submanifold of P since $E \times \mathbb{G}$ splits in $E \times F$. □

6.6. Let $\mathbb{G}$ be a splitting subspace of F. As a closed subspace of F, any topological complement $\mathbb{H} = F \ominus \mathbb{G}$ is a quasi-complete locally convex space. Every $x \in F$ has a unique decomposition $x = y + z$ for some $y \in \mathbb{G}$ and $z \in \mathbb{H}$. The projection $\tau : F \to \mathbb{H}$ is given by $\tau(x) = z$. For the quotient map $\delta : F \to F/\mathbb{G}$, the restriction $\delta|\,\mathbb{H}$ is a topological isomorphism. Hence the quotient space $F/\mathbb{G}$ is also a quasi-complete locally convex space. For $\beta = (\delta|\,\mathbb{H})^{-1} : F/\mathbb{G} \to \mathbb{H}$, we have $\beta\delta = \tau$. Identification of the equivalent class $\delta(x)$ in $F/\mathbb{G}$ with the vector $\tau(x)$ in $\mathbb{H}$ means $\delta(x) = \tau(x)$ without writing the symbol β.

6.7. Let R be a subbundle of P on M with fiber subspace $\mathbb{G}$. For each $m \in M$, R_m is a vector subspace of P_m. Let ξ_m be the quotient map from P_m on to the quotient space $Q_m = P_m/R_m$. Then $Q = \bigcup_{m \in M} Q_m$ is a disjoint union. Define the projection $\mu : Q \to M$ by $\mu(Q_m) = m$ and the quotient map $\xi : P \to Q$ by $\xi|P_m = \xi_m$ for all $m \in M$. Clearly we have $\pi = \mu\xi$.

6.8. Let $\mathscr{B}$ be a subbundle atlas of P for R on M. Take any (V, φ, T_φ) in $\mathscr{B}$ and $B \in Q_V = \mu^{-1}(V)$. Then $B \in Q_m$ for some $m \in V$. Choose $A \in P_m$ satisfying $\xi(A) = B$. Write

$$T_\varphi(A) = (a, x) \in \varphi(V) \times F.$$

Define $S_\varphi : Q_V \to \varphi(V) \times F/\mathbb{G}$ by

$$S_\varphi(B) = \big(a, \delta(x)\big).$$

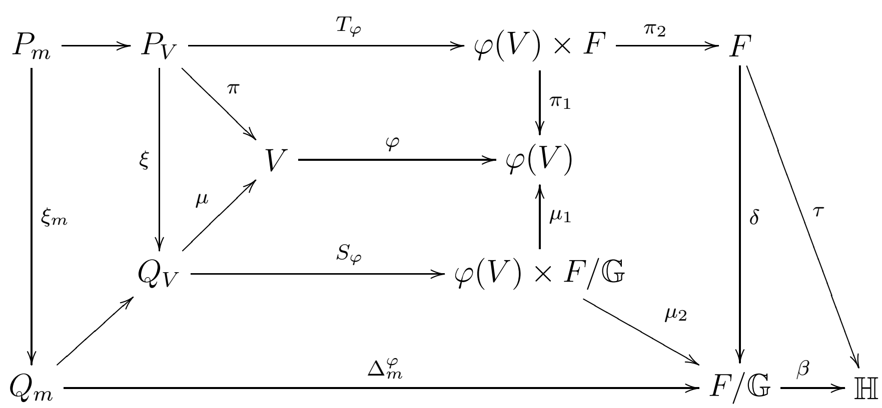
As usual, the projection of $\varphi(V) \times F/\mathbb{G}$ onto the first coordinate $\varphi(V)$ is also denoted by μ_1 and the projection to the second coordinate is denoted by μ_2. The quotient fiber representation is the map

$$\Delta_m^\varphi = \mu_2 S_\varphi | Q_m : Q_m \to F/\mathbb{G}.$$

Both μ and Q are called the *quotient bundle* of P over R. The family

$$\mathscr{B}/R = \{(V, \varphi, S_\varphi) : (V, \varphi, T_\varphi) \in \mathscr{B}\}$$

is called the *quotient bundle atlas*. We may write P/R instead of Q.



6.9. **Theorem**. The map $\mu : Q \to M$ is a vector bundle under the quotient bundle atlas. Furthermore the quotient map $\xi : P \to Q$ is a vector bundle map over the identity map on M. Actually ξ is a submersion.

Proof. To show that $S_\varphi(B)$ is well-defined, suppose $B = \xi(A_2)$ for some $A_2 \in P$ and $T_\varphi(A_2) = (b, y)$. From $\pi(A_2) = \mu\xi(A_2) = \mu(B) = m$, we have $a = \varphi(m) = b$. Next, since

$$\xi(A - A_2) = \xi(A) - \xi(A_2) = B - B = 0\,,$$

we get $A - A_2 \in R_m = (\Omega_m^\varphi)^{-1}(\mathbb{G})$. Therefore we obtain

$$x - y = \Omega_m^\varphi(A) - \Omega_m^\varphi(A_2) = \Omega_m^\varphi(A - A_2) \in \mathbb{G}\,,$$

or $\delta(x) = \delta(y)$, that is $(a, \delta(x)) = (b, \delta(y))$. Consequently $S_\varphi(B)$ is independent of the choice of A and it is well-defined. Clearly, S_φ is surjective onto $\varphi(V) \times F/\mathbb{G}$. To prove that S_φ is injective, assume $S_\varphi(B) = S_\varphi(B_2)$, that is, $(a, \delta(x)) = (a_2, \delta(x_2))$. Choose $m \in V$ so that $\varphi(m) = a = a_2$. Now for $\delta(x) = \delta(x_2)$, we have

$$\Omega_m^\varphi(A - A_2) = \Omega_m^\varphi(A) - \Omega_m^\varphi(A_2) = x - x_2 \in \mathbb{G}\,,$$

or

$$T_\varphi(A - A_2) \in \varphi(V) \times \mathbb{G} = T_\varphi(P_V \cap R),$$

that is $A - A_2 \in P_m \cap R = R_m$. Hence $B = \xi(A) = \xi(A_2) = B_2$. Therefore S_φ is bijective. Next, pick any $m \in V$ and $B \in Q_m$. Write $B = \xi(A)$ for $A \in P_m$ and $T_\varphi(A) = (a, x)$. Then

$$\mu_1 S_\varphi(B) = \mu_1(a, \delta(x)) = a = \varphi[\mu(B)]\,.$$

Hence $\mu_1 S_\varphi = \varphi\mu$. Therefore (V, φ, S_φ) is a bundle patch of Q. Next take any $(V, \varphi, T_\varphi), (W, \psi, T_\psi) \in \mathscr{B}$. With the notation of §4.3, we have

$$T_\psi T_\varphi^{-1}(a, x) = \big(\psi\varphi^{-1}(a), \Phi(a)x\big)$$

and $\Phi(a) = \Phi_j + \Phi_k(a)$. Take any $(a, \underline{x}) \in \varphi(V \cap W) \times F/\mathbb{G}$. Write $\underline{x} = \delta(x)$ for some $x \in F$. Define $\theta_k(a)(\underline{x}) = \delta[\Phi_k(a)x]$. Suppose $\underline{x} = \delta(y)$ for some other $y \in F$. Then we obtain

$$\delta(x - y) = \delta(x) - \delta(y) = \underline{x} - \underline{x} = 0,$$

that is $x - y \in \mathbb{G}$. Hence $\Phi_k(a)(x - y) \in \mathbb{G}$ and

$$\delta[\Phi_k(a)x] - \delta[\Phi_k(a)y] = \delta\Phi_k(a)(x - y) = 0\,.$$

Thus $\theta_k(a)\underline{x}$ is independent of the choice of $x \in \delta^{-1}(\underline{x})$. Since $\delta : F \to F/\mathbb{G}$ is continuous linear, $(a, x) \to \delta[\Phi_k(a)x]$ is also a holomorphic locally compact map. In particular, the map $a \to \theta_k(a)(\underline{x}) = \delta[\Phi_k(a)x]$ is differentiable. The continuous linear map $\underline{x} \to \theta_k(a)(\underline{x})$ is also differentiable. Because $(a, \underline{x}) \to \theta_k(a)(\underline{x})$ is a locally compact map, it is holomorphic by the Generalized Hartogs' Theorem. Clearly, $\theta(a)(\underline{x}) = \underline{x} + \theta_k(a)(\underline{x})$. Hence the bundle transformation

$$S_\psi S_\varphi^{-1} : S_\varphi(Q_V \cap Q_W) \to S_\psi(Q_V \cap Q_W)$$

is a special locally compact tag. The bundle patches (V, φ, S_φ) and (W, ψ, S_ψ) are compatible by symmetry. Therefore $\mu : Q \to M$ is a vector bundle. From $S_\varphi \xi T_\varphi^{-1}(a, x) = \big(a, \delta(x)\big)$, the quotient map $\xi : P \to Q$ is a vector bundle map over the identity on M. Finally from

$$\Delta_m^\varphi \xi (\Omega_m^\varphi)^{-1}(x) = \Delta_m^\varphi \xi(A) = \Delta_m^\varphi(B) = \delta(x),$$

the quotient map

$$\Delta_m^\varphi[d\xi(A)](\Omega_m^\varphi)^{-1} = \Delta_m^\varphi \xi(\Omega_m^\varphi)^{-1} = \delta$$

is submersive and so is the differential $d\xi(A)$. Therefore the quotient map ξ is a submersion. □

6.10. Let $\pi : P \to M$, $\lambda : Q \to M$ be vector bundles over the same base space M with fiber spaces F, G respectively. Suppose that $S : P \to Q$ is a vector bundle map over the identity map on M. For each $m \in M$, the restriction $S_m : P_m \to Q_m$ is a continuous linear map. The *kernel* of S is defined by the disjoint union

$$ker(S) = \bigcup_{m\in M} \; ker(S_m) \subset P$$

and the *range* by

$$ran(S) = \bigcup_{m\in M} \; ran(S_m) \subset Q.$$

We identify $F_1 \oplus F_2 \simeq F_1 \times F_2$ in the following theorem as in [11, 9.1].

6.11. **Theorem**. Both $ker(S)$, $ran(S)$ are subbundles of P, Q respectively iff for every $m \in M$ there exist a bundle chart (V, φ, T_φ) of P at m, a bundle chart (W, ψ, T_ψ) of Q at m, split subspaces $F_1 \oplus F_2 = F$, $G_1 \oplus G_2 = G$ and topological isomorphisms $\Lambda(a) : F_1 \to G_1$ for each $a \in \varphi(V)$ such that $V \subset W$, $\Lambda(a) = \Lambda_j + \Lambda_k(a)$ and

$$T_\psi S T_\varphi^{-1}(a, x_1, x_2) = (\psi\varphi^{-1}(a), \Lambda(a)x_1, 0) \qquad (a)$$

for every $x_1 \in F_1$ and $x_2 \in F_2$ where Λ_j and $\Lambda_k(a)$ belong to $L(F_1, G_1)$ and the map $(a, x_1) \to \Lambda_k(a)x_1$ is holomorphic on $\varphi(V) \times F_1$. Furthermore if S is holomorphic, then we may assume that the map $(a, x_1) \to \Lambda(a)x_1$ is holomorphic on $\varphi(V) \times F_1$. A similar result holds for locally compact holomorphic map S.

Proof. Suppose that $ker(S)$, $ran(S)$ are subbundles of P, Q with fiber subspaces F_2, G_1 of F, G respectively. Take any $m \in M$. There are subbundle charts (V, φ, T_φ), (W, ψ, T_ψ) of P, Q at m respectively so that

$$T_\varphi[P_V \cap ker(S)] = \varphi(V) \times F_2$$

and

$$T_\psi[Q_W \cap ran(S)] = \psi(W) \times G_1\,.$$

After replacing V by a smaller one, we may assume that $V \subset W$ and that the bundle representation

$$T_\psi S T_\varphi^{-1} : \varphi(V) \times F \to \psi(W) \times G$$

is given by $T_\psi S T_\varphi^{-1}(a, x) = \big(\psi\varphi^{-1}(a), \Phi(a)x\big)$ where $\Phi(a) = \Phi_j + \Phi_k(a)$, $\Phi_j, \Phi_k(a)$ belong to $L(F, G)$ and the map $(a, x) \to \Phi_k(a)x$ is holomorphic. Suppose that

$F_1 = F \ominus F_2$ and $G_2 = G \ominus G_1$ are topological complements. Let $\sigma : F_1 \to F$ denote the natural injection and $\delta : G \to G_1$ denote the projection. Then all $\Lambda_j = \delta\Phi_j\sigma$, $\Lambda_k(a) = \delta\Phi_k(a)\sigma$, $\Lambda(a) = \Lambda_j + \Lambda_k(a)$ belong to $L(F_1, G_1)$. Since $(a, x_1) \to \Lambda_k(a)x_1$ is locally bounded and differentiable separately in $a \in \varphi(V)$ and $x_1 \in F_1$, it is holomorphic jointly in (a, x_1). For every $m \in V \cap W$, we have

$$\Omega_m^\varphi[ker(S_m)] = \pi_2 T_\varphi[P_m \cap ker(S)] = F_2$$

and

$$\Omega_m^\psi[ran(S_m)] = \Lambda_2 T_\psi[Q_m \cap ran(S)] = G_1 \,.$$

For every $A \in P_m$,

$$A \in ker(S_m) \;\text{ iff }\; \Omega_m^\psi S_m(A) = 0 \;\text{ iff }\; \Phi(a)\Omega_m^\varphi(A) = 0.$$

Hence $ker[\Phi(a)] = \Omega_m^\varphi[ker(S_m)] = F_2$. Similarly we have

$$ran[\Phi(a)] = \Omega_m^\psi[ran(S_m)] = G_1.$$

Therefore

$$\Lambda(a) : F_1 \simeq F/ker[\Phi(a)] \to ran[\Phi(a)] = G_1$$

is a topological isomorphism and

$$\Phi(a)(x_1, x_2) = \Lambda(a)x_1$$

for all (x_1, x_2) in $F = F_1 \times F_2$. Consequently, we have obtained the required equation (a). Furthermore if S is (locally compact) holomorphic, then we may choose $\Phi_j = 0$ and then $(a, x_1) \to \Phi(a)x_1 = \Phi_k(a)x_1$ is also (locally compact) holomorphic. Obviously the given condition is also sufficient for $ker(S)$, $ran(S)$ to be subbundles. □

6.12. Let P, Q be vector bundles under bundle atlases $\mathscr{A}, \mathscr{B}$ with fiber spaces F_P, F_Q over holomorphic manifolds M, N modelled on E_M, E_N respectively where E_M, E_N, F_P, F_Q are quasi-complete locally convex spaces. To construct the product bundle over the product manifold $M \times N$ [14, §7], consider the disjoint union

$$P \times Q = \bigcup_{(m,n)\in M\times N} P_m \times Q_n \,.$$

The projections from P, Q and $P \times Q$ onto M, N and $M \times N$ are denoted by π, λ, τ respectively. Let (V, φ, T_φ) in $\mathscr{A}$ and (W, ψ, T_ψ) in $\mathscr{B}$ be bundle charts of P, Q respectively. Define $\Omega_{mn}^{\varphi\psi} = \Omega_m^\varphi \times \Omega_n^\psi$ and

$$T_{\varphi\psi} : (P \times Q)_{V\times W} \to (\varphi \times \psi)(V \times W) \times (F_P \times F_Q)$$

by

$$T_{\varphi\psi}(A) = \big(\,(\varphi \times \psi)(m, n), \Omega_{mn}^{\varphi\psi}(A)\,\big)$$

for each $A \in (P \times Q)_{V\times W}$ where $(m, n) = \tau(A) \in V \times W$.

6.13. **Theorem**. $P \times Q$ is a vector bundle with fiber space $F_P \times F_Q$ over the product manifold $M \times N$ under the *product bundle atlas*

$$\mathscr{A} \times \mathscr{B} = \{(V \times W, \varphi \times \psi, T_{\varphi\psi}) : (V, \varphi, T_\varphi) \in \mathscr{A},\ (W, \psi, T_\psi) \in \mathscr{B}\}\,.$$

Naturally the projection $\tau : P \times Q \to M \times N$ is called the *product vector bundle* of P, Q.

Proof. Clearly $\mathscr{A} \times \mathscr{B}$ covers $M \times N$. Observe that the linear map

$$\xi : E_M \times F_P \times E_N \times F_Q \to E_M \times E_N \times F_P \times F_Q$$

given by $\xi(a, x, b, y) = (a, b, x, y)$ is a topological isomorphism. For each $A \in (P \times Q)_{V \times W}$, we have $A = (A^P, A^Q)$ for some $A^P \in P_m$ and $A^Q \in Q_n$ where $m \in V$ and $n \in W$. Write $a = \varphi(m)$, $b = \psi(n)$, $x = \Omega^\varphi_m(A^P)$ and $y = \Omega^\psi_n(A^Q)$. From

$$\begin{aligned} T_{\varphi\psi}(A) &= (a, b, x, y) = \xi(a, x, b, y) \\ &= \xi\big(T_\varphi(A^P), T_\psi(A^Q)\big) = \xi(T_\varphi \times T_\psi)(A)\,, \end{aligned}$$

the map

$$T_{\varphi\psi} = \xi(T_\varphi \times T_\psi) : (P \times Q)_{V \times W} \to (\varphi \times \psi)(V \times W) \times (F_P \times F_Q)$$

is bijective. Next, let π_1, λ_1, τ_1 be the projections of $E_M \times F_P, E_N \times F_Q$ and $(E_M \times E_N) \times (F_P \times F_Q)$ onto the first coordinates E_M, E_N and $E_M \times E_N$ respectively. By

$$\tau_1 T_{\varphi\psi}(A) = (a, b) = (\varphi \times \psi)(\pi A^P, \lambda A^Q) = (\varphi \times \psi)\tau(A)\,,$$

we get $\tau_1 T_{\varphi\psi} = (\varphi \times \psi)\tau$. Therefore $T_{\varphi\psi}$ is a bundle patch on $M \times N$. To study the bundle transformations, let $(V_2, \varphi_2, T_{\varphi_2}), (W_2, \psi_2, T_{\psi_2})$ be bundle charts of P, Q in $\mathscr{A}, \mathscr{B}$ respectively. Then both

$$T_{\varphi_2} T_\varphi^{-1} : \varphi(V \cap V_2) \times F_P \to \varphi_2(V \cap V_2) \times F_P$$

and
$$T_{\psi_2} T_\psi^{-1} : \psi(W \cap W_2) \times F_Q \to \psi_2(W \cap W_2) \times F_Q$$

are special locally compact tags. Let Φ, Ψ be the holomorphic parts of $T_{\varphi_2} T_\varphi^{-1}$, $T_{\psi_2} T_\psi^{-1}$ respectively. Pick any $a \in \varphi(V \cap V_2)$, $b \in \psi(W \cap W_2)$, $x \in F_P$ and $y \in F_Q$. Define

$$\Lambda(a, b)(x, y) = \big(\Phi(a)(x), \Psi(b)(y)\big).$$

Choose $m \in V \cap V_2$ with $a = \varphi(m)$ and $n \in W \cap W_2$ with $b = \psi(n)$. Since

$$\Omega^{\varphi\psi}_{mn} = \Omega^\varphi_m \times \Omega^\psi_n : P_m \times Q_n \to F_P \times F_Q$$

is an isomorphism, there exists $A = (A^P, A^Q) \in P_m \times Q_n$ such that $\Omega^{\varphi\psi}_{mn}(A) = (x, y)$, or equivalently $x = \Omega^{\varphi}_m(A^P)$ and $y = \Omega^{\psi}_n(A^Q)$. Clearly $(\varphi_2 \times \psi_2)(\varphi \times \psi)^{-1}$ is a special morphism from the following calculation:

$$\begin{aligned} & T_{\varphi_2\psi_2}T^{-1}_{\varphi\psi}(a,b,x,y) \\ = \; & T_{\varphi_2\psi_2}A \\ = \; & \big((\varphi_2 \times \psi_2)(m,n), \Omega^{\varphi_2\psi_2}_{mn}(A)\ \big) \\ = \; & \big((\varphi_2 \times \psi_2)(\varphi \times \psi)^{-1}(a,b), (\Omega^{\varphi_2}_m \times \Omega^{\psi_2}_n)(\Omega^{\varphi}_m \times \Omega^{\psi}_n)^{-1}(a,b)(x,y)\big) \\ = \; & \big((\varphi_2\varphi^{-1}(a), \psi_2\psi^{-1}(b), \Omega^{\varphi_2}_m(\Omega^{\varphi}_m)^{-1}(a)x, \Omega^{\psi_2}_n(\Omega^{\psi}_n)^{-1}(b)y\big) \\ = \; & \big((\varphi_2\varphi^{-1}(a), \psi_2\psi^{-1}(b), \Phi(a)(x), \Psi(b)(y)\big) \\ = \; & \big((\varphi_2 \times \psi_2)(\varphi \times \psi)^{-1}(a,b), \Lambda(a,b)(x,y)\big)\,. \end{aligned}$$

Next, take any $a_0 \in \varphi(V \cap V_2)$ and $b_0 \in \psi(W \cap W_2)$. Select open neighborhoods $\mathbb{V} \subset \varphi(V \cap V_2)$, $\mathbb{W} \subset \psi(W \cap W_2)$ of a_0, b_0 respectively such that $\Phi(a) = \Phi_j + \Phi_k(a)$ and $\Psi(b) = \Psi_j + \Psi_k(b)$ for all $a \in \mathbb{V}$, $b \in \mathbb{W}$ where Φ_j, Ψ_j are the identity maps on F_P, F_Q and Φ_k, Ψ_k are the holomorphic parts of Φ, Ψ on $\mathbb{V}, \mathbb{W}$ respectively. Define

$$\Lambda_j(x,y) = \Phi_j(x) + \Psi_j(y) \quad \text{and} \quad \Lambda_k(a,b)(x,y) = \Phi_k(a)(x) + \Psi_k(b)(y)$$

for all $a \in \mathbb{V}$, $b \in \mathbb{W}$, $x \in F_P$ and $y \in F_Q$. Clearly Λ_j is the identity map on $F_P \times F_Q$. The map

$$(a,b,x,y) \to \Phi_k(a)(x) + \Psi_k(b)(y)$$

is locally compact and separately differentiable. By the Generalized Hartogs' Theorem, this map is holomorphic. Since $\Lambda(a,b) = \Lambda_j + \Lambda(a,b)$, the bundle transformation $T_{\varphi_2\psi_2}T^{-1}_{\varphi\psi}$ is a special locally compact tag on

$$(\varphi \times \psi)[(V \times W) \times (V_2 \times W_2)] \times (F_P \times F_Q)\,.$$

By symmetry, $T_{\varphi_2\psi_2}$, $T_{\varphi\psi}$ are compatible. Consequently, $P \times Q$ is a vector bundle over $M \times N$ with the bundle atlas $\mathscr{A} \times \mathscr{B}$. □

6.14. It is easy to show that the projections from $P \times Q$ onto P, Q are vector bundle maps.

7. Common Basic Atlas for Several Vector Bundles

7.1. Let P be a vector bundle with fiber space F over a holomorphic manifold M modelled on E. For every bundle chart (V, φ, T_φ) of P, the pair (V, φ) is called the *basic chart* of (V, φ, T_φ). An atlas $\mathscr{A}$ on M is *basic* for P if every chart in $\mathscr{A}$ is a basic chart of some bundle chart of P.

7.2. **Lemma**. For every chart (U,ξ) at $m \in M$, there is a bundle chart (V,φ,T_φ) of P at m such that $V \subset U$ and $\varphi = \xi|V$.

Proof. Take any bundle chart (R,θ,T_θ) at m. Let $V = U \cap R$ and $\varphi = \xi|V$. Define $T_\varphi : P_V \to \varphi(V) \times F$ by $T_\varphi(A) = \big(\varphi(v), \Omega_v^\theta A\big)$ where $v = \pi(A) \in V$. Since $\Omega_v^\theta : P_v \to F$ is an isomorphism, T_φ is a bijection. By

$$\pi_1 T_\varphi(A) = \varphi(v) = \varphi\pi(A),$$

the map T_φ is a bundle patch at m. Next let (W,ψ,T_ψ) be a bundle chart of P. Since (R,θ,T_θ) and (W,ψ,T_ψ) are compatible, the bundle transformation $T_\psi T_\theta^{-1}$ is a special locally compact tag. Let Φ be the main part of $T_\psi T_\theta^{-1}$, that is

$$T_\psi T_\theta^{-1}(c,z) = \big(\psi\theta^{-1}(c), \Phi(c)z\big)\,, \quad \forall\ (c,z) \in \theta(U \cap W) \times F\,.$$

To study $T_\psi T_\varphi^{-1}$, take any $(a,x) \in \varphi(V \cap W) \times F$. Let $(b,y) = T_\psi T_\varphi^{-1}(a,x)$. Then we have $v = \varphi^{-1}(a) \in V \cap W$ and $A = T_\varphi^{-1}(a,x) \in P_{V\cap W}$. Thus $a = \varphi(v) = \xi(v)$ and $b = \psi\pi(A) = \psi(v)$, that is $b = \psi\xi^{-1}(a)$. Next from $y = \pi_2 T_\psi(A) = \Omega_v^\psi(A)$ and $x = \pi_2 T_\varphi(A) = \Omega_v^\theta(A)$, we obtain

$$y = \Omega_v^\psi(\Omega_v^\theta)^{-1}x = \Phi(\theta v)x = \Phi(\theta\varphi^{-1})(a)x$$

by §4.3. Hence $T_\psi T_\varphi^{-1}(a,x) = \big(\psi\xi^{-1}(a), \Phi(\theta\varphi^{-1})(a)x\big)$. Consequently $T_\psi T_\varphi^{-1}$ is a special locally compact tag by §3.5. Similarly, $T_\varphi T_\psi^{-1}$ is also a special locally compact tag. Therefore (V,φ,T_f) and (W,ψ,T_ψ) are compatible. As a result, (V,φ,T_f) is a bundle chart of P at m. □

7.3. **Theorem**. If $P^1, P^2, \cdots, P^r$ are vector bundles over the same holomorphic manifold M, then there is an atlas on M that is basic for all $P^1, P^2, \ldots, P^r$.

Proof. It suffices to prove the case when $r = 2$. Let E, F_P, F_Q be quasi-complete locally convex spaces and let P, Q be vector bundles over M modelled on E with fiber spaces F_P, F_Q respectively. The projections of P, Q onto M are denoted by the same symbol π. Let $\mathscr{A}$ be the family of charts (V,φ) on M such that there are bundle charts (V,φ,T_φ^P), (V,φ,T_φ^Q) of P, Q respectively. Take any $m \in M$. There is a bundle chart (W,ψ,T_ψ^Q) of Q at m. There is a bundle chart (V,φ,T_φ^P) of P at m such that $V \subset W$ and $\varphi = \psi|V$. Clearly for the restriction $T_\varphi^Q = T_\psi^Q|Q_V$, the triple (V,φ,T_φ^Q) is also a bundle chart of Q at m. Therefore $\mathscr{A}$ covers M. Consequently it is an atlas on M. □

7.4. Let E, F, G be quasi-complete locally convex spaces. Suppose that $\pi : P \to M$ is a vector bundle with fiber space F over a holomorphic manifold M modelled on E. Assume that $\lambda : Q \to P$ is a vector bundle with fiber space G over the holomorphic manifold P modelled on $E \times F$ according to §4.7.

It would be nice to know the conditions for the composite $\pi\lambda$ to be a vector bundle on M with fiber space $F \times G$.

8. Direct Sums and Spaces of Compact Linear Maps

8.1. Let P, Q be vector bundles over the same manifold M modelled on E with fiber spaces F_P, F_Q respectively where E, F_P, F_Q are quasi-complete locally convex spaces. The projections of P, Q onto M are denoted by the same symbol π. Let $\mathscr{A}$ be an atlas on M that is basic to both P, Q. By definition, for each $(V, \varphi) \in \mathscr{A}$, there are bundle charts $(V, \varphi, T_\varphi^P)$, $(V, \varphi, T_\varphi^Q)$ of P, Q respectively. Write

$$\mathscr{A}_P = \{(V, \varphi, T_\varphi^P) : (V, \varphi) \in \mathscr{A}\} \quad \text{and} \quad \mathscr{A}_Q = \{(V, \varphi, T_\varphi^Q) : (V, \varphi) \in \mathscr{A}\}.$$

Then for every $m \in V$, the fiber representations

$$\Omega_m^{P\varphi} = \pi_2 T_\varphi^P | P_m : P_m \to F_P \ \text{ and } \ \Omega_m^{Q\varphi} : Q_m \to F_Q$$

are topological isomorphisms.

8.2. In addition, consider bundle charts (W, ψ, T_ψ^P), (W, ψ, T_ψ^Q) of P, Q respectively. Let Λ, Δ, Γ be the main parts of the locally compact tags $T_\psi^P (T_\varphi^P)^{-1}$, $T_\psi^Q (T_\varphi^Q)^{-1}$, $T_\varphi^P (T_\psi^P)^{-1}$ respectively. The first two will be used in the construction of the direct sum $P \oplus Q$ and the last two in the construction of the bundle $L_k(P, Q)$ of compact linear maps. Then for all $a \in \varphi(V \cap W)$, $y \in F_P$, $z \in F_Q$, we have

$$T_\psi^P (T_\varphi^P)^{-1}(a, y) = \big(\psi\varphi^{-1}(a), \Lambda(a)(y)\big)$$

$$T_\psi^Q (T_\varphi^Q)^{-1}(a, z) = \big(\psi\varphi^{-1}(a), \Delta(a)(z)\big),$$

and

$$T_\varphi^P (T_\psi^P)^{-1}(a, y) = \big(\varphi\psi^{-1}(a), \Gamma(a)(y)\big).$$

Suppose that Λ_j, Δ_j, Γ_j denote the identity maps on F_P, F_Q, F_P respectively. For each $a_0 \in \varphi(V \cap W)$, let $\Lambda_k, \Delta_k, \Gamma_k$ be the holomorphic parts of Λ, Δ, Γ respectively on some open neighborhood $\mathbb{V} \subset \varphi(V \cap W)$ of a_0. Then for each $a \in \mathbb{V}$, we obtain

$$\begin{aligned}
\Lambda(a) &= \Lambda_j + \Lambda_k(a) = \Omega_m^{P\psi} (\Omega_m^{P\varphi})^{-1} \\
\Delta(a) &= \Delta_j + \Delta_k(a) = \Omega_m^{Q\psi} (\Omega_m^{Q\varphi})^{-1} \\
\Gamma(a) &= \Gamma_j + \Gamma_k(a) = \Omega_m^{P\varphi} (\Omega_m^{P\psi})^{-1}
\end{aligned}$$

where $m = \varphi^{-1}(a) \in V \cap W$. Furthermore all maps

$$\begin{aligned} (a,y) \to \Lambda_k(a)(y) &\;:\; \mathbb{V} \times F_P \to F_P \\ (a,z) \to \Delta_k(a)(z) &\;:\; \mathbb{V} \times F_Q \to F_Q \\ (a,y) \to \Gamma_k(a)(y) &\;:\; \mathbb{V} \times F_P \to F_P \end{aligned}$$

are locally compact holomorphic.

8.3. Since the union $P \oplus Q = \bigcup_{m \in M} P_m \oplus Q_m$ is disjoint, the projection π from the $P \oplus Q$ onto M is uniquely defined by $\pi(A) = m$ for every $A \in P_m \oplus Q_m$. Take any $(V, \varphi, T_\varphi^P) \in \mathscr{A}_P$ and $(V, \varphi, T_\varphi^Q) \in \mathscr{A}_Q$. The map

$$\Omega_m^\varphi = \Omega_m^{P\varphi} \oplus \Omega_m^{Q\varphi} : P_m \oplus Q_m \to F_P \oplus F_Q$$

is an isomorphism. Define

$$T_\varphi : (P \oplus Q)_V \to \varphi(V) \times (F_P \oplus F_Q)$$

by $T_\varphi(A) = \big(a, \Omega_m^\varphi(A)\big)$ where $m = \pi(A)$ and $a = \varphi(m)$.

8.4. **Theorem**. The *direct sum* $P \oplus Q$ is a vector bundle over M with the bundle atlas $\mathscr{A}_P \oplus \mathscr{A}_Q = \{(V, \varphi, T_\varphi) : (V, \varphi) \in \mathscr{A}\}$.

Proof. Clearly each T_φ is a bijection satisfying $\pi_1 T_\varphi = \varphi\pi$. Hence (V, φ, T_φ) is a bundle patch on $P \oplus Q$. Next, for each $a \in \mathbb{V}$ define

$$\Phi(a) = \Lambda(a) \oplus \Delta(a), \;\; \Phi_k(a) = \Lambda_k(a) \oplus \Delta_k(a) \;\; \text{and} \;\; \Phi_j = \Lambda_j \oplus \Delta_j.$$

Take any $x \in P_m \oplus Q_m$. Write $(a, x) = T_\varphi(A)$ for some $A \in (P \oplus Q)_{V \cap W}$. Then $m = \varphi^{-1}(a) = \pi(A) \in V \cap W$. Therefore we have

$$\begin{aligned} T_\psi T_\varphi^{-1}(a,x) &= T_\psi(A) \\ &= \big(\psi\varphi^{-1}(a), \Omega_m^\varphi(A)\;\big) \\ &= \big(\psi\varphi^{-1}(a), (\Omega_m^{P\psi} \oplus \Omega_m^{Q\psi})(A)\;\big) \\ &= \big(\psi\varphi^{-1}(a), (\Omega_m^{P\psi} \oplus \Omega_m^{Q\psi})(\Omega_m^{P\varphi} \oplus \Omega_m^{Q\varphi})^{-1}(x)\;\big) \\ &= \big(\psi\varphi^{-1}(a), \big[\Omega_m^{P\psi}(\Omega_m^{P\varphi})^{-1} \oplus \Omega_m^{Q\psi}(\Omega_m^{Q\varphi})^{-1}\big](x)\;\big) \\ &= \big(\psi\varphi^{-1}(a), \big[\Lambda(a) \oplus \Delta(a)\big](x)\big) \\ &= \big(\psi\varphi^{-1}(a), \Phi(a)(x)\big). \end{aligned}$$

The map

$$(a,x) = (a,y,z) \to \Phi_k(a)(x) = \big(\Lambda_k(a)(y), \Delta_k(a)(z)\big)$$

is locally compact and separately differentiable in $a \in \mathbb{V}$, $y \in F_P$ and $z \in F_Q$. Hence it is locally compact holomorphic on $\mathbb{V} \times (F_P \oplus F_Q)$. Clearly $\Phi_j = \Lambda_j \oplus \Delta_j$ is the identity map on $F_P \oplus F_Q$. Because $\Phi(a) = \Phi_j + \Phi_k(a)$, the bundle transformation $T_\psi T_\varphi^{-1}$ is a special locally compact tag. By symmetry, (V, φ, T_φ) and (W, ψ, T_ψ) are compatible. Therefore $\mathscr{A}_P \oplus \mathscr{A}_Q$ is a bundle atlas of $P \oplus Q$. □

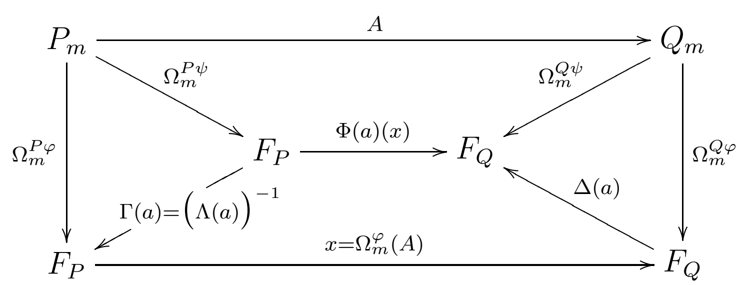
8.5. Because the proof of §7.3 works only for a finite number of vector bundles, it is difficult for the time being to construct the direct sums of arbitrary families of vector bundles. We do not know how to handle the tensor products of two vector bundles either.

8.6. Take any bundle charts $(V, \varphi, T_\varphi^P)$ in $\mathscr{A}_P$ and $(V, \varphi, T_\varphi^Q)$ in $\mathscr{A}_Q$. For each $m \in V$, since $\Omega_m^{P\varphi}, \Omega_m^{Q\varphi}$ are topological isomorphisms, the map

$$\Omega_m^\varphi : L_k(P_m, Q_m) \to L_k(F_P, F_Q)$$

defined by $\Omega_m^\varphi(A) = \Omega_m^{Q\varphi} A (\Omega_m^{P\varphi})^{-1}$ is a topological isomorphism. Now for every $A \in L_k(P_m, Q_m)$, let $T_\varphi(A) = \big(a, \Omega_m^\varphi(A)\big)$ where $a = \varphi(m)$. The projection from the disjoint union $L_k(P, Q) = \bigcup_{m \in M} L_k(P_m, Q_m)$ onto M is also denoted by π for convenience. Since $\varphi\pi = \pi_1 T_\varphi$, we have a bundle patch (V, φ, T_φ) of $L_k(P, Q)$. Interested people may consider quasi-completions as alternative assumptions in the following theorem that $L_k(F_P, F_Q)$ is quasi-complete.

8.7. **Theorem**. If $L_k(F_P, F_Q)$ is quasi-complete, then $L_k(P, Q)$ is a vector bundle over M with fiber space $L_k(F_P, F_Q)$ equipped with the compact-open topology under the bundle atlas $\mathscr{A}_L = \{(V, \varphi, T_\varphi) : (V, A) \in \mathscr{A}\}$.



Proof. We use the notation of §8.2. For every m in $V \cap W$ and x in $L_k(F_P, F_Q)$, let $\Phi(a)(x) = \Delta(a)\, x\, \Gamma(a)$ where $a = \varphi(m)$. Because both $\Delta(a)$ and $\Gamma(a)$ are topological isomorphisms by §3.4b, each $\Phi(a)$ is a continuous linear operator on $L_k(F_P, F_Q)$. Next, take any a in $\varphi(V \cap W)$ and x in $L_k(F_P, F_Q)$. Then $A = T_\varphi^{-1}(a, x) \in L_k(P_m, Q_m)$ where $m = \varphi^{-1}(a) \in V \cap W$. By

$$x = \Omega_m^\varphi(A) = \Omega_m^{Q\varphi} A (\Omega_m^{P\varphi})^{-1},$$

we get

$$\begin{aligned} T_\psi T_\varphi^{-1}(a,x) &= T_\psi(A) \\ &= \big(\psi\varphi^{-1}(a), \Omega_m^\psi(A)\big) \\ &= \big(\psi\varphi^{-1}(a), \Omega_m^{Q\psi} A (\Omega_m^{P\psi})^{-1}\big) \\ &= \big(\psi\varphi^{-1}(a), \Omega_m^{Q\psi} (\Omega_m^{Q\varphi})^{-1} x \Omega_m^{P\varphi} (\Omega_m^{P\psi})^{-1}\big) \\ &= \big(\psi\varphi^{-1}(a), \Delta(a) x \Gamma(a)\big) \\ &= \big(\psi\varphi^{-1}(a), \Phi(a)(x)\big) \ . \end{aligned}$$

Next, take any $a_0 \in \varphi(V \cap W)$. Choose $\mathbb{V}$ according to §8.2. For every $a \in \mathbb{V}$ and $x \in L_k(F_P, F_Q)$, define

$$\begin{aligned} \xi_Q(a)(x) &= \Delta_k(a)\, x,\ \xi_P(a)(x) = x\,\Gamma_k(a), \\ \xi(a)(x) &= \Delta_k(a)\, x\, \Gamma_k(a), \quad \text{and} \quad \Phi_k(a) = \xi_Q(a) + \xi_P(a) + \xi(a). \end{aligned}$$

From

$$\Phi(a)(x) = [\Delta_j + \Delta_k(a)]\, x\, [\Gamma_j + \Gamma_k(a)] = [\Phi_j + \Phi_k(a)](x),$$

we obtain $\Phi(a) = \Phi_j + \Phi_k(a)$ on $L_k(F_P, F_Q)$ where Φ_j is the identity map on $L_k(F_P, F_Q)$. We claim that the maps $(a,x) \to \xi_Q(a)x$, $(a,x) \to \xi_P(a)x$ and $(a,x) \to \xi(a)x$ are locally compact and holomorphic. In this case, the map $(a,x) \to \Phi_k(a)x$ is also locally compact and holomorphic. Hence T_φ, T_ψ are compatible by symmetry. Consequently, $L_k(P,Q)$ is a vector bundle over M with the bundle atlas $\mathscr{A}_L$. This would complete the proof. Actually we only prove that $(a,x) \to \xi(a)x$ is locally compact and holomorphic because the other two ξ_P, ξ_Q would follow in a similar but easier way.

Pick any $x_0 \in L_k(F_P, F_Q)$. Since x_0 is a compact linear map, there is a 0-neighborhood $\mathfrak{U}_0$ of F_P and a compact subset C_0^Q of F_Q such that

$$x_0(\mathfrak{U}_0) \subset C_0^Q.$$

Because the map

$$(a,z) \to \Delta_k(a)(z) : \mathbb{V} \times F_Q \to F_Q$$

is locally compact, for every $z \in C_0^Q$ there is an open neighborhood $\mathbb{V}_z \subset \mathbb{V}$ of a_0, an open convex balanced 0-neighborhood $\mathfrak{W}_z$ of F_Q and a compact subset C_z^Q of F_Q such that

$$\Delta_k(\mathbb{V}_z)(z + 2\,\mathfrak{W}_z) \subset C_z^Q.$$

There is a finite subset H of C_0^Q such that $C_0^Q \subset \bigcup_{z\in H}(z + \mathfrak{W}_z)$. Clearly the closed convex balanced hull C^Q of $\bigcup_{z\in H} C_z^Q$ is compact in F_Q, $\mathbb{V}_1 = \bigcap_{z\in H} \mathbb{V}_z$ is a neighborhood of a_0 and $\mathfrak{W} = \bigcap_{z\in H} \mathfrak{W}_z$ is a balanced 0-neighborhood of F_Q.

Take any $a \in \mathbb{V}_1$, any $c_0^Q \in C_0^Q$ and any $w \in \mathfrak{W}$. There is $z \in H$ such that $c_0^Q \in z + \mathfrak{W}_z$. Since $\mathfrak{W}_z$ is convex, we have

$$\begin{aligned} & \Delta_k(a)(c_0^Q + w) \in \Delta_k(a)(z + \mathfrak{W}_z + \mathfrak{W}) \\ \subset \ & \Delta_k(a)(z + \mathfrak{W}_z + \mathfrak{W}_z) \subset \Delta_k(a)(z + 2\,\mathfrak{W}_z) \subset C^Q. \end{aligned}$$

Therefore we conclude

$$\Delta_k(\mathbb{V}_1)(C_0^Q + \mathfrak{W}) \subset C^Q.$$

On the other hand, because the map

$$(a, y) \to \Gamma_k(a)(y) : \mathbb{V} \times F_P \to F_P$$

is locally compact at $(a_0, 0) \in \mathbb{V} \times F_P$, there is an open neighborhood $\mathbb{V}_2 \subset \mathbb{V}_1$ of a_0, a 0-neighborhood $\mathfrak{U}_1$ of F_P and a compact subset C^P of F_P such that $\Gamma_k(\mathbb{V}_2)(\mathfrak{U}_1) \subset C^P$. Choose $\lambda > 1$ satisfying $C^P \subset \lambda\,\mathfrak{U}_0$. Now

$$\mathbb{N} = \{\ell \in L_k(F_P, F_Q) : \ell(C^P) \subset \mathfrak{W}\}$$

is a 0-neighborhood of $L_k(F_P, F_Q)$. We claim that

$$\xi(\mathbb{V}_2)(x_0 + \mathbb{N})(\mathfrak{U}_1) \subset \lambda C^Q.$$

Indeed, take any $a \in \mathbb{V}_2$, $x \in x_0 + \mathbb{N}$ and $y \in \mathfrak{U}_1$. Then we have

$$\xi(a)(x)(y) = \Delta_k(a)\, x\, \Gamma_k(a)(y) \in \Delta_k(a)\, x(C^P).$$

Suppose $\xi(a)(x)(y) = \Delta_k(a)\, x(c^P)$ for some $c^P \in C^P$. Write $c^P = \lambda u_0$ for some $u_0 \in \mathfrak{U}_0$. Then $w = (x - x_0)(c^P) \in \mathfrak{W}$, that is

$$x(c^P) = x_0(c^P) + w = \lambda x_0(u_0) + w = \lambda \left[x_0(u_0) + \frac{w}{\lambda} \right] \in \lambda(C_0^Q + \mathfrak{W})$$

because $\lambda > 1$ and $\mathfrak{W}$ is balanced. Hence we get

$$\xi(a)(x)(y) = \Delta_k(a)\, x(c^P) \in \lambda \Delta_k(a)(C_0^Q + \mathfrak{W}) \subset \lambda C^Q.$$

Now, let $\mathfrak{W}^Q$ denote any open convex balanced 0-neighborhood of F_Q. Choose $\tau > 1$ such that $C^Q \subset \tau\,\mathfrak{W}^Q$. As a result of

$$\xi(\mathbb{V}_2)(x_0 + \mathbb{N})(\mathfrak{U}_1) \subset \lambda C^Q \subset \lambda\tau\,\mathfrak{W}^Q,$$

we obtain

$$\xi(\mathbb{V}_2)(x_0 + \mathbb{N})(\mathfrak{U}_2) \subset \mathfrak{W}^Q$$

where $\mathfrak{U}_2 = \mathfrak{U}_1/(\lambda\tau) \subset \mathfrak{U}_1$ is also a 0-neighborhood of F_P. Therefore the set $\xi(\mathbb{V}_2)(x_0 + \mathbb{N})$ is equicontinuous.

Next, take any $y \in F_P$. Choose $\theta > 0$ with $y \in \theta\,\mathfrak{U}_1$. As a subset of the compact set $\theta\lambda C^Q$, the set $\xi(\mathbb{V}_2)(x_0 + \mathbb{N})(y)$ is relatively compact in F_Q.

By Ascoli's Theorem, the set $\xi(\mathbb{V}_2)(x_0+\mathbb{N})$ is relatively compact in $L_k(F_P,F_Q)$ equipped with the compact-open topology.

It remains to show that the map $(a,x)\to\xi(a)(x)$ is differentiable on $\mathbb{V}_2\times L_k(F_P,F_Q)$. Since $x\to\xi(a)(x)$ is linear, it is differentiable. To study $a\to\xi(a)(x)$, without loss of generality we may assume $x=x_0$ so that we can use the symbols such as $\mathbb{V}_2$, $\mathfrak{U}_1$, C^P, λ, C_0^Q, c^Q, $\mathfrak{W}^Q$ and τ again. Take any $a\in\mathbb{V}_2$ and $e\in E$. Select an open convex balanced 0-neighborhood $\mathbb{V}_0$ of E such that $a+3\mathbb{V}_0\subset\mathbb{V}_2$. Choose $\delta>0$ such that $\delta e\in\mathbb{V}_0$. Then for all $\beta_j\in\mathbb{C}$ with $|\beta_j|\le\delta$, we have

$$a+\beta_1e+\beta_2e+\beta_3e\in\mathbb{V}_2$$

so that all the terms below are well-defined. Write

$$\begin{aligned}
\xi_{ae}(t) &= \frac{\xi(a+te)(x_0)-\xi(a)(x_0)}{t}\,,\\
\xi_{ae}^{\Delta}(t) &= \frac{\Delta_k(a+te)x_0\,\Gamma_k(a+te)-\Delta_k(a)x_0\,\Gamma_k(a+te)}{t}\,,\\
\xi_{ae}^{\Gamma}(t) &= \frac{\Delta_k(a)\,x_0\Gamma_k(a+te)-\Delta_k(a)\,x_0\Gamma_k(a)}{t}
\end{aligned}$$

for all $t\in\mathbb{C}$ with $0<|t|\le\delta$. From $\Gamma_k(\mathbb{V}_2)(\mathfrak{U}_1)\subset C^P$, each $\Gamma_k(a+te)$ is a compact linear operator on F_P. Also from $\Delta_k(\mathbb{V}_1)x_0(\mathfrak{U}_0)\subset C^Q$, each $\Delta_k(a+te)x_0$ is a compact linear map from F_P into F_Q. Thus all $\xi_{ae}(t),\xi_{ae}^{\Delta}(t),\xi_{ae}^{\Gamma}(t)$ belong to $L_k(F_P,F_Q)$. It is routine to verify $\xi_{ae}(t)=\xi_{ae}^{\Delta}(t)+\xi_{ae}^{\Gamma}(t)$. We claim that the limits of $\xi_{ae}^{\Delta}(t)$, $\xi_{ae}^{\Gamma}(t)$ and hence also the limit of $\xi_{ae}(t)$ exist as $t\to0$. Note that the partial derivative $\partial_a[\Gamma_k(a)(y)]$ of the holomorphic map $(a,y)\to\Gamma_k(a)(y)$ is a continuous linear map from E into F_P. Observe that for every $y\in F_P$, we get

$$\begin{aligned}
\xi_{ae}^{\Gamma}(t)(y) &= \Delta_k(a)\,x_0\left\{\frac{\Gamma_k(a+te)(y)-\Gamma_k(a)(y)}{t}\right\}\\
&= \Delta_k(a)\,x_0\left\{\int_0^1\partial_a[\Gamma_k(a+\theta_1te)(y)](e)\,d\theta_1\right\}\\
&= \Delta_k(a)\,x_0\left\{\int_0^1\frac{\partial}{\partial\beta_2}\,[\Gamma_k(a+\theta_1te+\beta_2e)(y)]\,d\theta_1\right\}\qquad\text{at }\beta_2=0\\
&= \Delta_k(a)\,x_0\left\{\int_0^1\frac{1}{2\pi i}\int_{|\beta_2|=\delta}\frac{\Gamma_k(a+\theta_1te+\beta_2e)(y)}{\beta_2^2}\,d\beta_2\,d\theta_1\right\}\\
&= \frac{1}{2\pi i}\int_0^1\int_{|\beta_2|=\delta}\frac{\Delta_k(a)\,x_0\Gamma_k(a+\theta_1te+\beta_2e)(y)}{\beta_2^2}\,d\beta_2\,d\theta_1\,.
\end{aligned}$$

On the other hand, for every $t \in \mathbb{C}$ with $|t| \le \delta$, the map $\Gamma_{ae} : F_P \to F_P$ defined by

$$\begin{aligned}\Gamma_{ae}(y) &= \partial_a[\Gamma_k(a)(y)](e) \\ &= \frac{d}{d\beta_2}\,[\Gamma_k(a+\beta_2 e)(y)]\Bigg|_{\beta_2=0} \\ &= \frac{1}{2\pi i}\int_{|\beta_2|=\delta}\frac{\Gamma_k(a+\beta_2 e)(y)}{\beta_2^2}\,d\beta_2\end{aligned}$$

is linear. Since $\Delta_k(a)x_0$ is a compact linear map, it is continuous. For any $y \in \mathfrak{U}_1$, since C^Q is closed convex balanced we have

$$\Delta_k(a)x_0\Gamma_{ae}(y) = \frac{1}{2\pi i}\int_{|\beta_2|=\delta}\frac{\Delta_k(a)x_0\Gamma_k(a+\beta_2 e)(y)}{\beta_2^2}\,d\beta_2 \in \frac{C^Q}{\delta^2}\,.$$

Although we do not know whether Γ_{ae} is continuous, yet the map

$$\Gamma^\bullet_{ae} = \Delta_k(a)x_0\Gamma_{ae} : F_P \to F_Q$$

is compact linear and consequently it is continuous. We want to prove $\xi^\Gamma_{ae}(t) \to \Gamma^\bullet_{ae}$ in $L_k(F_P, F_Q)$ under the compact-open topology. Let C_2^P be a compact subset of F_P and we use the arbitrary 0-neighborhood $\mathfrak{W}^Q$ of F_Q again. Now

$$\mathbb{N}_2 = \{\ell \in L_k(F_P, F_Q) : \ell(C_2^P) \subset \mathfrak{W}^Q\}$$

is a 0-neighborhood of $L_k(F_P, F_Q)$. Choose $\mu > 1$ with $C_2^P \subset \mu\,\mathfrak{U}_1$. Pick any $y \in C_2^Q$. It is a routine calculation to get

$$\xi^\Gamma_{ae}(t)(y) - \Gamma^\bullet_{ae}(y) = \frac{1}{2\pi i}\int_0^1\int_{|\beta_2|=\delta}\frac{\Gamma_\xi(t,\theta_1,\beta_2,y)}{\beta_2^2}\,d\beta_2 d\theta_1$$

where

$$\begin{aligned}&\Gamma_\xi(t,\theta_1,\beta_2,y) \\ =\ & \Delta_k(a)\,x_0\Gamma_k(a+\theta_1 te+\beta_2 e)(y) - \Delta_k(a)\,x_0\Gamma_k(a+\beta_2 e)(y) \\ =\ & \Delta_k(a)\,x_0[\,\Gamma_k(a+\theta_1 te+\beta_2 e)(y) - \Gamma_k(a+\beta_2 e)(y)\,] \\ =\ & \Delta_k(a)\,x_0\int_0^1 \partial_a[\Gamma_k(a+\theta_1\theta_3 te+\beta_2 e)(y)\,]\,(\theta_1 te)\ d\theta_3 \\ =\ & \theta_1 t\Delta_k(a)\,x_0\int_0^1 \partial_a[\Gamma_k(a+\theta_1\theta_3 te+\beta_2 e)(y)\,]\,(e)\ d\theta_3 \\ =\ & \theta_1 t\Delta_k(a)\,x_0\int_0^1 \frac{\partial}{\partial\beta_4}\Gamma_k(a+\theta_1\theta_3 te+\beta_2 e+\beta_4 e)(y)\,d\theta_3 \quad \text{at } \beta_4 = 0 \\ =\ & \theta_1 t\Delta_k(a)\,x_0\int_0^1 \frac{1}{2\pi i}\int_{|\beta_4|=\delta}\frac{\Gamma_k(a+\theta_1\theta_3 te+\beta_2 e+\beta_4 e)(y)}{\beta_4^2}\,d\beta_4\,d\theta_3\,.\end{aligned}$$

Hence we obtain

$$\xi^{\Gamma}_{ae}(t)(y) - \Gamma^{\bullet}_{ae}(y) = \frac{t}{(2\pi i)^2}\int_0^1 \int_{|\beta_2|=\delta}\int_0^1\int_{|\beta_4|=\delta} \frac{\theta_1 \mathbb{F}(y)}{\beta_2^2\,\beta_4^2}\; d\beta_4 d\theta_3 d\beta_2 d\theta_1$$

where $\mathbb{F} = \Delta_k(a)\, x_0\Gamma_k(a+\theta_1\theta_3 te + \beta_2 e + \beta_4 e)$. Therefore we have

$$\begin{aligned} & \mathbb{F}(C_2^P) \subset \Delta_k(\mathbb{V}_1)x_0\Gamma_k(\mathbb{V}_2)(C_2^P) \\ \subset\ & \mu\, \Delta_k(\mathbb{V}_1)x_0\Gamma_k(\mathbb{V}_2)(\mathfrak{U}_1) \subset \mu\, \Delta_k(\mathbb{V}_1)x_0(C^P) \\ \subset\ & \mu\lambda\Delta_k(\mathbb{V}_1)x_0(\mathfrak{U}_0) \subset \mu\lambda\Delta_k(\mathbb{V}_1)(C_0^Q) \subset \mu\lambda C^Q \subset \mu\lambda\tau\mathfrak{W}^Q. \end{aligned}$$

Let $\delta_1 = \delta/\mu\lambda\tau$. Then for all $|t| < \delta_1$, we deduce

$$\left[\,\xi^{\Gamma}_{ae}(t) - \Gamma^{\bullet}_{ae}\right](C_2^P) \subset t\mu\lambda\tau\mathfrak{W}^Q \subset \mathfrak{W}^Q,$$

that is $\xi^{\Gamma}_{ae}(t) - \Gamma^{\bullet}_{ae} \in \mathbb{N}_2$. We have proved $\xi^{\Gamma}_{ae}(t) \to \Gamma^{\bullet}_{ae}$ in $L_k(F_P, F_Q)$ as $t \to 0$.

Similarly for every $z \in F_Q$ and $t \in \mathbb{C}$ with $0 < |t| \le \delta_1$, we get

$$\begin{aligned} \frac{\Delta_k(a+te)(z) - \Delta_k(a)(z)}{t} &= \int_0^1 \partial_a[\Delta_k(a+\theta_1 te)(z)](e)\, d\theta_1 \\ &= \int_0^1 \frac{\partial}{\partial\beta_2}\,[\Delta_k(a+\theta_1 te + \beta_2 e)(z)]\, d\theta_1 \quad \text{at } \beta_2 = 0 \\ &= \int_0^1 \frac{1}{2\pi i}\int_{|\beta_2|=\delta} \frac{\Delta_k(a+\theta_1 te+\beta_2 e)(z)}{\beta_2^2}\, d\beta_2\, d\theta_1\,. \end{aligned}$$

Replacing $z = x_0\Gamma_k(a+te)(y)$ where $y \in F_P$, we have

$$\begin{aligned} \xi^{\Delta}_{ae}(t)(y) &= \frac{\Delta_k(a+te)x_0\Gamma_k(a+te)(y) - \Delta(a)x_0\Gamma_k(a+te)(y)}{t} \\ &= \frac{1}{2\pi i}\int_0^1\int_{|\beta_2|=\delta} \frac{\Delta_k(a+\theta_1 te+\beta_2 e)x_0\Gamma_k(a+te)(y)}{\beta_2^2}\, d\beta_2\, d\theta_1\,. \end{aligned}$$

On the other hand, for every $t \in \mathbb{C}$ with $|t| \le \delta$, the map $\Delta_{ae} : F_P \to F_P$ defined by

$$\begin{aligned} \Delta_{ae}(z) &= \partial_a[\Delta_k(a)(z)](e) \\ &= \frac{d}{d\beta_2}\,[\Delta_k(a+\beta_2 e)(z)]\Big|_{\beta_2=0} \\ &= \frac{1}{2\pi i}\int_{|\beta_2|=\delta} \frac{\Delta_k(a+\beta_2 e)(z)}{\beta_2^2}\; d\beta_2 \end{aligned}$$

is linear. Replacing $z = x_0\Gamma_k(a+te)(y)$ where $y \in F_P$, we obtain

$$\Delta_{ae}[x_0\Gamma_k(a+te)(y)] = \frac{1}{2\pi i}\int_{|\beta_2|=\delta} \frac{\Delta_k(a+\beta_2 e)x_0\Gamma_k(a+te)(y)}{\beta_2^2}\, d\beta_2\,.$$

Now the map $\Delta_{ae}^{\bullet} : F_P \to F_Q$ given by

$$\Delta_{ae}^{\bullet}(y) = \Delta_{ae}[x_0\Gamma_k(a+te)(y)]$$

is compact linear because $\Delta_{ae}^{\bullet}(\mathfrak{U}_1) \subset C^Q/\delta^2$. It is a routine calculation to get

$$\xi_{ae}^{\Delta}(t)(y) - \Delta_{ae}^{\bullet}(y) = \frac{1}{2\pi i}\int_0^1 \int_{|\beta_2|=\delta} \frac{\Delta_\xi(t,\theta_1,\beta_2,y)}{\beta_2^2}\, d\beta_2 d\theta_1$$

where

$$\begin{aligned}
&\Delta_\xi(t,\theta_1,\beta_2,y)\\
=\;& \Delta_k(a+\theta_1 te+\beta_2 e)x_0\Gamma_k(a+te)(y) - \Delta_k(a+\beta_2 e)x_0\Gamma_k(a+te)(y)\\
=\;& \Delta_k(a+\theta_1 te+\beta_2 e)(z) - \Delta_k(a+\beta_2 e)(z) \quad \text{for } z = x_0\Gamma_k(a+te)(y)\\
=\;& \int_0^1 \partial_a[\Delta_k(a+\theta_1\theta_3 te+\beta_2 e)(z)\,]\,(\theta_1 te)\; d\theta_3\\
=\;& \theta_1 t\int_0^1 \partial_a[\Delta_k(a+\theta_1\theta_3 te+\beta_2 e)(z)\,]\,(e)\; d\theta_3\\
=\;& \theta_1 t\int_0^1 \frac{\partial}{\partial\beta_4}\Delta_k(a+\theta_1\theta_3 te+\beta_2 e+\beta_4 e)(z)\, d\theta_3 \quad \text{at } \beta_4 = 0\\
=\;& \theta_1 t\int_0^1 \frac{1}{2\pi i}\int_{|\beta_4|=\delta} \frac{\Delta_k(a+\theta_1\theta_3 te+\beta_2 e+\beta_4 e)(z)}{\beta_4^2}\, d\beta_4 d\theta_3\,.
\end{aligned}$$

Hence we obtain

$$\xi_{ae}^{\Delta}(t)(y) - \Delta_{ae}^{\bullet}(y) = \frac{t}{(2\pi i)^2}\int_0^1 \int_{|\beta_2|=\delta}\int_0^1 \int_{|\beta_4|=\delta} \frac{\theta_1\mathbb{G}(y)}{\beta_2^2\,\beta_4^2}\; d\beta_4 d\theta_3 d\beta_2 d\theta_1$$

where

$$\mathbb{G}(y) = \Delta_k(a+\theta_1\theta_3 te+\beta_2 e+\beta_4 e)x_0\Gamma_k(a+te)(y).$$

Therefore we have

$$\mathbb{G}(C_2^P) \subset \Delta_k(\mathbb{V}_1)x_0\Gamma_k(\mathbb{V}_2)(C_2^P) \subset \mu\lambda\tau\mathfrak{W}^Q.$$

Then for all $|t| < \delta_1$, we deduce

$$\left[\,\xi_{ae}^{\Delta}(t) - \Delta_{ae}^{\bullet}\right](C_2^P) \subset t\mu\lambda\tau\mathfrak{W}^Q \subset \mathfrak{W}^Q,$$

that is $\xi_{ae}^{\Delta}(t) - \Delta_{ae}^{\bullet} \in \mathbb{N}_2$. We have proved

$$\xi_{ae}^{\Delta}(t) \to \Delta_{ae}^{\bullet} \;\text{ in } L_k(F_P, F_Q) \text{ as } t \to 0.$$

As a result, we conclude

$$\lim_{t\to 0} \frac{\xi(a+te)(x_0)-\xi(a)(x_0)}{t} = \Gamma^{\bullet}_{ae} + \Delta^{\bullet}_{ae}$$

in $L_k(F_P, F_Q)$. Therefore the map $a \to \xi(a)(x)$ is differentiable. By the Generalized Hartogs' theorem, $(a,x) \to \xi(a)(x)$ is also a holomorphic map from $\mathbb{V}_2 \times L_k(F_P, F_Q)$ into $L_k(F_P, F_Q)$. This completes the proof. □

8.8. Let E, F be quasi-complete locally convex spaces and let M be a holomorphic manifold modelled on E with an atlas $\mathscr{A}$. Suppose that π is the projection of the product space $Q = M \times F$ onto its first coordinate M. For all charts (V, φ) and (W, ψ) on M, let

$$T_\varphi(A) = T_\psi(A) = x \;\text{ for every } A = (m,x) \text{ in } Q_{V\cap W}.$$

Clearly (V, φ, T_φ) and (W, ψ, T_ψ) are compatible bundle patches with

$$T_\psi T_\varphi^{-1}(a,x) = \big(\psi\varphi^{-1}(a), \Phi(a)(x)\big) \;\text{ for all } (a,x) \in \varphi(V\cap W)\times F,$$

where $\Phi(a)$ is the identity map on F. Therefore Q is a vector bundle under the bundle atlas $\mathscr{A}_F = \{(V, \varphi, T_\varphi) : (V,\varphi) \in \mathscr{A}\}$. It is called the *trivial bundle* over M with fiber space F. When $F = \mathbb{C}$, it is also called the *trivial line bundle* over M.

8.9. Let P be a vector bundle with fiber space F over a holomorphic manifold M modelled on E and Q the trivial line bundle over M. Every continuous linear form on F is a compact linear map. Suppose that the topological dual space F^* of all continuous linear forms is quasi-complete under the compact-open topology and this is the case when F is a barrelled space. Then $P^* = L_k(P,Q)$ is called the *dual vector bundle* of P.

8.10. Let M be a holomorphic manifold modelled on a barelled space E. For each $m \in M$, the cotangent space $T^*_m M$ is defined in terms of local functions at $m \in M$. The natural isomorphism [14, 6.3] from $T^*_m M$ onto the dual space $(T_m M)^*$ of the tangent space $T_m M$ is trivially extended to a bijection from the *cotangent bundle* $T^*M = \bigcup_{m\in M} T^*_m M$ onto the dual vector bundle $(TM)^*$. The topological properties of T^*M are derived from $(TM)^*$ accordingly.

Acknowledgment. *The author would like to thank all those who helped the fabulous phoenix to rise from the ash, new and fresh and young.*

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