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Quasisymmetric Structure and Quasisymmetry on 1-manifolds

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Abstract. The purpose of this paper is to study the group QS_2 of quasisymmetric homeomorphisms of a one-dimensional manifold J with respect to a given quasisymmetric structure on J. Under a natural neighborhood system of the identity, the group QS_2 is a partial topological group. Its characteristic topological subgroup is identified as the collection of all elements in QS_2 with vanishing ratio distortion. When J is a Jordan curve in the plane, denote the group of quasisymmetric homeomorphisms of J with respect to the Euclidean metric by QS_1. Sufficient conditions and necessary conditions are established for the two groups QS_1 and QS_2 to coincide with each other.

Keywords. quasisymmetric map, quasisymmetric structure, topological group, ratio distortion.

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1. Introduction

A homeomorphism $h : I_1 \to I_2$ between intervals on the real axis $\mathbb{R}$ is said to be *quasisymmetric* if there is constant M such that

$$\frac{1}{M} \le \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \le M \tag{1.1}$$

for all x and t for which $x - t, x, x + t \in I_1$. The condition (1.1) is the well known so called M-condition introduced by Beurling and Ahlfors [BA] in their study of boundary values of quasiconformal mappings. The class of all such quasisymmetric maps, denoted by QS, form a pseudogroup under composition. A *quasisymmetric structure* (or QS structure), like a differential structure, on any topological one-manifold J is a maximal atlas of coordinate charts h_α such that the transition maps $h_\alpha \circ h_\beta^{-1}$ are elements of QS whenever they are defined. Then, for a given QS structure on J, we say that a homeomorphism $f : J \to J$ is quasisymmetric if for each pair of charts h_α and h_β, the composition $h_\alpha \circ f \circ h_\beta^{-1}$, denoted by f_α^β, is in the pseudogroup QS whenever it is defined. It is easy to see that the set QS_2 of all QS homeomorphisms of a manifold J with respect to a given QS structure form a group.

In Section 2, we study the topological structure of the group QS_2 under a neighborhood system. In particular, we identify the characteristic topological subgroup of QS_2 for any compact one-dimensional manifold, extending a result of Gardiner and Sullivan [GS] on the unit circle.

Next, let J be a Jordan curve in the plane. Then one can define another group, denoted by $QS_1(J)$, of quasisymmetric homeomorphisms of J with respect to the Euclidean metric in the plane as follows. We say that a homeomorphism $f : J \to J$ is in the group QS_1 if there is a constant M such that

$$\frac{|a-x|}{|b-x|} \leq 1 \Longrightarrow \frac{|f(a)-f(x)|}{|f(b)-f(x)|} \leq M \tag{1.2}$$

for all $a, b, x \in J$. Therefore, there are two notions of quasisymmetry on a Jordan curve J. In Section 3, we will explore the relations between the two quasisymmetric groups QS_1 and QS_2 and find necessary conditions and sufficient conditions for them to coincide with each other.

2. The topology of QS_2

In this section we introduce a neighborhood system in the group QS_2 to make it a partial topological group and identify its characteristic topological subgroup.

Let J be a compact one-dimensional manifold and Q be a given QS structure on J. Following [GS] and [St], in order to define a system of neighborhoods of the identity in the group QS_2, we introduce a finer structure subordinate to the given QS structure. Suppose that we are given a finite $PSL(2,\mathbb{R})$-structure H subordinate to a QS structure Q. By this we mean that there is a finite collection of charts h_α for the given QS structure such that the composition $h_\alpha \circ h_\beta^{-1}$ is in $PSL(2,\mathbb{R})$ whenever it is defined.

In what follows we assume that J is a compact one-dimensional manifold and H a finite $PSL(2,\mathbb{R})$-structure subordinate to a given QS structure Q on J. As shown in [St, 2.1.3], the quasisymmetric groups $QS_2(H)$ with respect to H and $QS_2(Q)$ with respect to Q are the same. Thus the restriction to a finite $PSL(2,\mathbb{R})$-structure H will not change the group structure QS_2; it will only make the topology of QS_2 better.

2.1. Characteristic topological subgroup of a partial topological group.

Recall that a topological group is a group G with a topological structure such that the group operations are continuous under the topological structure. A *partial topological group* is a group with a neighborhood system at the identity with the property that given a neighborhood V of the identity, there is a neighborhood U such that $U \circ U \subset V$ and $U^{-1} \subset V$, that is, the neighborhood system is complete under group operations. In general, a partial topological group may not be a topological group.

Lemma 2.1. [GS] *Let G be a partial topological group. Then G is a topological group with the given neighborhood system of the identity if and only if the conjugate map: $h \to g \circ h \circ g^{-1}$ is continuous at the identity for every $g \in G$.*

For a given partial topological group G, let

$$G_1 = \{g \in G : \text{the map } h \to g \circ h \circ g^{-1} \text{ is continuous at the identity}\}.$$

Then one can show that G_1 is a topological subgroup of G with the given neighborhood system. This subgroup is called the *characteristic topological subgroup* of G. For more details on this, we refer the reader to [GS] and [St].

2.2. Vanishing ratio distortion. To define a topology in QS_2, we need the following concept. Let $f : I_1 \to I_2$ be a QS map of intervals. The *ratio distortion* of f is the function ψ_f defined by

$$\psi_f(x,t) = \frac{f(x+t) - f(x)}{f(x) - f(x-t)} - 1$$

for all $t > 0$ and x such that the interval $[x-t, x+t]$ is contained in I_1. Furthermore, f is said to have *vanishing ratio distortion* if there is a function $\epsilon_f(t)$ converging to zero as $t \to 0$ such that $|\psi_f(x,t)| \le \epsilon_f(t)$ whenever the ratio distortion $\psi_f(x,t)$ is defined. Similarly, an element $f \in QS_2$ is said to have *vanishing ratio distortion* if for each pair of charts h_α and h_β in the given $PSL(2,\mathbb{R})$ structure, the composition map $f_\alpha^\beta = h_\alpha \circ f \circ h_\beta^{-1}$ has vanishing ratio distortion in the above sense.

Note that the class of homeomorphisms of S^1 with vanishing ratio distortion is precisely the class of symmetric homeomorphisms [BY], which was used to characterize symmetric quasicircles.

2.3. QS_2 as a partial topological group. Now we can define a neighborhood system of the identity in QS_2 that will make QS_2 a partial topological group. For any ϵ: $0 < \epsilon < 1$, we define neighborhood N_ϵ to be the collection of all $f \in QS_2$ such that for each chart h_α in the given finite $PSL(2,\mathbb{R})$-structure H on J we have

$$|f_\alpha^\alpha(x) - x| < \epsilon \text{ and } |(f_\alpha^\alpha)^{-1}(x) - x| < \epsilon \tag{2.1}$$

whenever $f_\alpha^\alpha(x)$ and $(f_\alpha^\alpha)^{-1}(x)$ are defined, and

$$-\epsilon < \psi_{f_\alpha^\alpha}(x,t) < \epsilon \text{ and } -\epsilon < \psi_{(f_\alpha^\alpha)^{-1}} < \epsilon \tag{2.2}$$

whenever the functions are defined. Recall that $f_\alpha^\alpha = h_\alpha \circ f \circ h_\alpha^{-1}$ is the conjugate map on an interval in $\mathbb{R}$ with respect to a given chart h_α. In words, a neighborhood of identity is defined by small movement of points x (condition (2.1)) and small distortion of symmetric triples $x - t, x$ and $x + t$ (condition (2.2)), when viewed on each coordinate chart. Using quasiconformal extension properties of quasisymmetric maps on the real line (see[Ah, GS]), one can show that QS_2 is a partial topological group under the neighborhood system $\{N_\epsilon\}$.

2.4. The characteristic topological subgroup of QS_2. Now we can identify the characteristic topological subgroup of QS_2.

Theorem 2.2. *Let J be a compact one-dimensional manifold with a finite $PSL(2,\mathbb{R})$-structure H. Let QS_2 be the partial topological group of quasisymmetric homeomorphisms of J with respect to H. Then the characteristic topological subgroup of QS_2 is the class of all $f \in QS_2$ with vanishing ratio distortion.*

Proof. According to Lemma 2.2 and the definition of characteristic topological subgroup given above, it suffices to show that for any $f \in QS_2$ the conjugate map: $f \circ h \circ f^{-1}$ is continuous at the identity in QS_2 if and only if f has vanishing ratio distortion.

Assume $f \in QS_2$ has vanishing ratio distortion. We need to show that the map $f \circ h \circ f^{-1}$ is near the identity when $h \in QS_2$ is near enough to the identity in the sense of (2.1) and (2.2). Note that the notion of quasisymmetry, vanishing ratio distortion and neighborhoods of identity in QS_2 are all defined in terms of charts in the given finite $PSL(2,\mathbb{R})$ structure of J. Therefore, in proving the above statement, one can assume that all maps are defined on finite intervals in $\mathbb{R}$. Then condition (2.1) for $f\circ h\circ f^{-1}$ follows easily from the uniform continuity of quasisymmetric maps and the fact that h is near the identity. To verify condition (2.2) for $f \circ h \circ f^{-1}$, we need to show that its distortion function

$$\psi_{f\circ h\circ f^{-1}}(x,t) = \frac{f \circ h \circ f^{-1}(x+t) - f \circ h \circ f^{-1}(x)}{f \circ h \circ f^{-1}(x) - f \circ h \circ f^{-1}(x-t)} - 1$$

has small absolute value for all $t > 0$ and x. This can be achieved by using similar ideas as in [GS, Lemma 2.1]. For small values of t, one uses the fact that f and f^{-1}, having vanishing ratio distortion, distort small symmetric intervals $[x-t,x]$ and $[x,x+t]$ by no more than $1+\epsilon$. For large values of t, one uses the Hölder continuity of f and f^{-1}. The details, which can be found in [St, Lemma 2.13], are omitted here.

Conversely, assume $f \in QS_2$ does not have vanishing ratio distortion. We need to show that conjugation by f is not continuous at the identity in QS_2. Again, as noted above, by composing with coordinate charts we may assume that all maps are defined on finite intervals in $\mathbb{R}$.

Since f satisfies an M-condition and does not have vanishing ratio distortion, there exists a sequence $x_n \to x$ and a decreasing sequence of positive numbers $t_n \to 0$ such that

$$\lim_{n\to\infty} \frac{f(x_n+t_n)-f(x_n)}{f(x_n)-f(x_n-t_n)} = L \neq 1.$$

By choosing appropriate coordinate charts, we may further assume that all x_n, x_n+t_n, and x are contained in an open interval I_1 where f satisfies an M-condition. Without loss of generality, we can also assume that $L>1$. Thus there is an integer N such that

$$\frac{f(x_n+t_n)-f(x_n)}{f(x_n)-f(x_n-t_n)} \geq \frac{N+1}{N} \tag{2.3}$$

for all (large) n.

To show that the conjugation map $h \to f\circ h\circ f^{-1}$ is not continuous at the identity, we consider the sequence $< s_n >$ of translation maps given by $s_n(x) = x+t_n$. Since $t_n \to 0$, the sequence $< s_n >$ converges to the identity. It remains to show that the conjugate sequence $f\circ s_n \circ f^{-1}$ does not converge to the identity. Suppose otherwise. We will derive a contradiction with the M-condition satisfied by f.

For simplicity of notation, let $y_n = f(x_n)$, $y_n^- = f(x_n-t_n)$ and $y_n^+ = f(x_n+t_n)$. By (2.3), one can choose $\tilde{y}_n^+$ such that $y_n < \tilde{y}_n^+ \leq y_n^+$ and

$$\frac{\tilde{y}_n^+ - y_n}{y_n - y_n^-} = 1 + \frac{1}{N}.$$

This allows one to partition the interval $[y_n^-, \tilde{y}_n^+]$ into exactly $2N+1$ intervals of equal length by

$$y_n^- = p_n^{(0)} < p_n^{(1)} < \cdots < p_n^{(N)} = y_n < p_n^{(N+1)} < \cdots < p_n^{(2N+1)} = \tilde{y}_n^+.$$

By examining the distortions on consecutive symmetric intervals $[p_n^{(i-1)}, p_n^{(i)}]$ and $[p_n^{(i)}, p_n^{(i+1)}]$, one can deduce that there exists $\delta > 0$ such that if a map g is in the δ-neighborhood N_δ of the identity, then

$$\frac{g(y_n^+)-g(y_n)}{g(y_n)-g(y_n^-)} \geq \frac{g(\tilde{y}_n^+)-g(y_n)}{g(y_n)-g(y_n^-)} \geq 1 + \frac{1}{2N}. \tag{2.4}$$

Next, fix an integer k so that $(1+\frac{1}{2N})^k > M$, where M is the constant in the M-condition satisfied by f. By our contrapositive hypothesis that the sequence $g_n = f\circ s_n \circ f^{-1}$ converges to the identity, we can fix a sufficiently large n such that for each $m : 1 \leq m \leq k$, the m-th iteration $(g_n)^m = f\circ (s_n)^m \circ f^{-1}$ is in N_δ.

Finally, applying (2.4) to $g = (g_n)^m$ and its inverse, respectively, we obtain that

$$\frac{f(x_n + (m+1)t_n) - f(x_n + mt_n)}{f(x_n + mt_n) - f(x_n + (m-1)t_n)} = \frac{(g_n)^m(y_n^+) - (g_n)^m(y_n)}{(g_n)^m(y_n) - (g_n)^m(y_n^-)} \geq 1 + \frac{1}{2N}$$

and

$$\frac{f(x_n - (m-1)t_n) - f(x_n - mt_n)}{f(x_n - mt_n) - f(x_n - (m+1)t_n)} = \frac{(g_n)^{-m}(y_n^+) - (g_n)^{-m}(y_n)}{(g_n)^{-m}(y_n) - (g_n)^{-m}(y_n^-)} \geq 1 + \frac{1}{2N}.$$

This yields that

$$\begin{aligned}\frac{f(x_n + kt_n) - f(x_n)}{f(x_n) - f(x_n - kt_n)} &= \frac{\sum_{m=0}^{k-1}[f(x_n + (m+1)t_n) - f(x_n + mt_n)]}{\sum_{m=1} k[f(x_n - (m-1)t_n) - f(x_n - mt_n)]} \\ &\geq \frac{\sum_{m=0}^{k-1}(1 + \frac{1}{2N})^m}{\sum_{m=1} k(1 + \frac{1}{2N})^{-m}} = (1 + \frac{1}{2N})^k > M,\end{aligned}$$

which contradicts the M-condition satisfied by f. Therefore, conjugation by f is not continuous at the identity. This completes the proof of Theorem 2.2. ■

3. Relation between QS_1 and QS_2

Let J be a Jordan curve in the plane. In this section we investigate the relation between QS_1, the group of QS maps of J with respect to the Euclidean metric, and QS_2, the group of QS maps of J with respect to a given $PSL(2,\mathbb{R})$-structure H. We begin with two examples.

Example 3.1. Let H be the $PSL(2,\mathbb{R})$ structure on the unit circle S^1 determined by the charts:

$$U_\alpha = \left\{ e^{2i\pi t} : -\frac{3}{8} < t < \frac{3}{8} \right\}; \quad h_\alpha(e^{2i\pi t}) = \begin{cases} t, & -\frac{3}{8} < t \leq 0 \text{ or } \frac{1}{8} \leq t < \frac{3}{8}, \\ 8t^2, & 0 \leq t \leq \frac{1}{8}; \end{cases}$$

$$U_\beta = \left\{ e^{2i\pi t} : \frac{1}{4} < t < \frac{3}{4} \right\}; \quad h_\beta(e^{2i\pi t}) = t.$$

Then any rotation given by $r_\theta(z) = e^{i\theta}z$ for $\theta \neq 2k\pi$ is in the group QS_1. But it is not in QS_2 because the transition map $h_\alpha \circ r_\theta \circ h_\alpha^{-1}(t)$ is not quasisymmetric. Recall that QS_2 is the group of homeomorphisms of S^1 that are quasisymmetric with respect to the given $PSL(2\mathbb{R})$ structure.

Example 3.2. Let G be another $PSL(2\mathbb{R})$ structure on S^1 determined by the charts:

$$U_1 = \left\{ e^{2i\pi t} : -\frac{3}{8} < t < \frac{3}{8} \right\}; \quad h_1(e^{2i\pi t}) = \begin{cases} t, & -\frac{3}{8} < t \leq 0 \text{ or } \frac{1}{8} \leq t < \frac{3}{8}, \\ 8t^2, & 0 \leq t \leq \frac{1}{8}; \end{cases}$$

$$U_2 = \left\{ e^{2i\pi t} : \frac{1}{8} < t < \frac{7}{8} \right\}; \;\; h_2(e^{2i\pi t}) = \begin{cases} t - \frac{1}{2}, & \frac{1}{8} < t \le \frac{3}{8} \text{ or } \frac{1}{2} \le t < \frac{7}{8}, \\ -8(t-\frac{1}{2})^2, & \frac{3}{8} \le t \le \frac{1}{2}. \end{cases}$$

Let $f : S^1 \to S^1$ be defined by

$$f(e^{2i\pi t}) = \begin{cases} t + \frac{1}{2}, & -\frac{3}{8} \le t \le -\frac{1}{8} \text{ or } \frac{1}{8} \le t \le \frac{3}{8}; \\ -\frac{1}{2\sqrt{2}}(-t)^{\frac{1}{2}} + \frac{1}{2}, & -\frac{1}{8} \le t \le 0; \\ 8t^2 + \frac{1}{2}, & 0 \le t \le \frac{1}{8}; \\ -8(t-\frac{1}{2})^2, & \frac{3}{8} \le t \le \frac{1}{2}; \\ \frac{1}{2\sqrt{2}}(t-\frac{1}{2})^{\frac{1}{2}}, & \frac{1}{2} \le t \le \frac{5}{8}. \end{cases}$$

Then f is in QS_2, but not in QS_1.

The above two examples show that in general the groups QS_1 and QS_2 do not have any containment relation with each other. However, we can still establish the following relations.

Theorem 3.3. *Let J be a Jordan curve in the plane with a finite $PSL(2,\mathbb{R})$ structure H. If each of the charts in H is quasisymmetric with respect to the Euclidean metric, then $QS_2 = QS_1$.*

Proof. Assume $f \in QS_1$. Then for each pair of charts h_α and h_β in H, the map $h_\alpha \circ f \circ h_\beta^{-1}$ is quasisymmetric whenever it is defined since it is a composition of quasisymmetric maps. Hence f is in QS_2.

Next assume $f \in QS_2$. Then for each pair of charts h_α and h_β in H, the map $h_\alpha \circ f \circ h_\beta^{-1}$ is quasisymmetric. Since h_α and h_β are quasisymmetric, it follows that $f = h_\alpha^{-1} \circ h_\alpha \circ f \circ h_\beta^{-1} \circ h_\beta$ satisfies an M-condition on the open arc where the right hand side is defined. Covering J by a finite number of such open arcs, we see that f satisfies an M-condition on each of these open arcs. Therefore, by [TV, Theorem 2.23], f satisfies a global M-condition on J. Hence f is in QS_1. ■

As seen in the above examples, it is not always the case that each chart is quasisymmetric. But, nevertheless, we have the following necessary condition for $QS_1 = QS_2$.

Theorem 3.4. *Let J be a Jordan curve in the plane with a finite $PSL(2,\mathbb{R})$ structure H. If $QS_1(J) = QS_2(J,H)$, then either each chart in H is quasisymmetric, or every chart is nowhere quasisymmetric.*

We say that a map h from an open arc on J into the plane is *nowhere quasisymmetric* if the restriction to any subarc fails to be quasisymmetric.

Proof of Theorem 3.4. As shown in [St, 3.12], piecing together a minimal collection of charts in H covering J and using the exponential covering map

$\exp(2\pi i\theta) : \mathbb{R} \to S^1$, one can construct a homeomorphism $f : J \to S^1$ such that locally f is the composition of a chart in H and the above exponential map. This map f, called a complementary homeomorphism of H, enables one to lift quasisymmetric homeomorphisms of S^1 to maps in the class QS_2.

Assume that $QS_1 = QS_2$. By the construction of the complementary map f, it is sufficient to show that f (or its inverse, denoted by g) is either quasisymmetric or nowhere quasisymmetric. For each $s \in \mathbb{R}$ define a rotation $R_s : S^1 \to S^1$ by $R_s(e^{2i\pi t}) = e^{2i\pi(t+s)}$. Since the charts in H are quasisymmetrically compatible with each other, one can verify that the map $g_s = g \circ R_s \circ g^{-1} : J \to J$ is in the class QS_2. Thus, by the assumption that $QS_1 = QS_2$, g_s is quasisymmetric in the Euclidean metric for each s. Suppose that $g : S^1 \to J$ is not quasisymmetric. Then there exists a sequence of real numbers $x_n \in [0, 1]$ with $x_n \to x_0 \in [0, 1]$ and a sequence of positive numbers $t_n \to 0$ such that

$$\lim_{n\to\infty} \frac{|g(e^{2i\pi(x_n+t_n)}) - g(e^{2i\pi x_n})|}{|g(e^{2i\pi x_n}) - g(e^{2i\pi(x_n-t_n)})|} = \infty \text{ or } 0.$$

Since g_s is quasisymmetric and

$$g(e^{2i\pi(x+s)}) = g \circ R_s(e^{2i\pi x}) = g_s \circ g(e^{2i\pi x}),$$

for any x, it follows that

$$\lim_{n\to\infty} \frac{|g(e^{2i\pi((x_n+s)+t_n)}) - g(e^{2i\pi(x_n+s)})|}{|g(e^{2i\pi(x_n+s)}) - g(e^{2i\pi((x_n+s)-t_n)})|} = \frac{|g_s \circ g(e^{2i\pi(x_n+t_n)}) - g_s \circ g(e^{2i\pi x_n})|}{|g_s \circ g(e^{2i\pi x_n}) - g_s \circ g(e^{2i\pi(x_n-t_n)})|}$$
$$= \infty \text{ or } 0.$$

This shows that g is not quasisymmetric in any neighborhood of $e^{2i\pi(x_0+s)}$ for all s. Thus g, and hence f, is nowhere quasisymmetric. This completes the proof of Theorem 3.4.

Remarks. It is not known whether the condition in Theorem 3.3 is necessary. However, it is true that if each chart in a finite $PSL(2, \mathbb{R})$ structure J is quasisymmetric, then J is a quasicircle. It is also open whether there is a finite $PSL(2, \mathbb{R})$ structure such that each chart is nowhere quasisymmetric.

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