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Close-to-convexity of quasihyperbolic balls

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Abstract. **We will consider close-to-convexity of metric balls defined by the quasihyperbolic metric and the** j**-metric.**

Definitions

Let $G \subsetneq \mathbb{R}^n$**,** $n \geq 2$**, be a domain and define**

- **The** ***quasihyperbolic distance*** **for** $x, y \in G$ **by**

  $$k_G(x,y) = \inf_{\alpha \in \Gamma_{xy}} \int_\alpha \frac{|dz|}{d(z)},$$

  where $d(z) = d(z, \partial G)$ **and** Γ_{xy} **is the collection of all rectifiable curves in** G **joining** x **and** y**.**

- **[MO] For** $x, y \in \mathbb{R}^n \setminus \{0\}$ **and** $n \geq 2$

  $$k_{\mathbb{R}^n \setminus \{0\}}(x,y) = \sqrt{\alpha^2 + \log^2 \frac{|x|}{|y|}},$$

  where $\alpha = \measuredangle(x, 0, y) \in [0, \pi]$**.**

- **The** j***-distance*** **for** $x, y \in G$ **by**

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x), d(y)\}}\right).$$

Note that $j_G(x,y) \le k_G(x,y)$ **for all** $x, y \in G$**.**

- **For** $m \in \{k_G, j_G\}$ **we define the** ***metric ball*** **(*****metric disk*** **in the case** $n = 2$**) for** $r > 0$ **and** $x \in G$ **by**

$$B_m(x,r) = \{y \in G : m(x,y) < r\}.$$

- G **is *starlike with respect to*** $x \in G$ **if for all** $y \in G$ **the line segment** $[x, y]$ **is contained in** G **and** G **is *strictly starlike with respect to*** x **if each line from the point** x **meets** ∂G **at exactly one point.**

- **If** G **is starlike for all** $x \in G$ **then it is *convex*.**

- **A domain** $G \subset \mathbb{R}^n$ **is *close-to-convex* if** $\mathbb{R}^n \setminus G$ **can be covered with non-intersecting half-lines (**$\{z \in \mathbb{R}^n : z = x + ty,\, x, y \in \mathbb{R}^n, y \neq 0, t > 0\}$ **or** $\{z \in \mathbb{R}^n : z = x + ty,\, x, y \in \mathbb{R}^n, y \neq 0, t \geq 0\}$**).**

Clearly convex domains are starlike and starlike domains are close-to-convex.

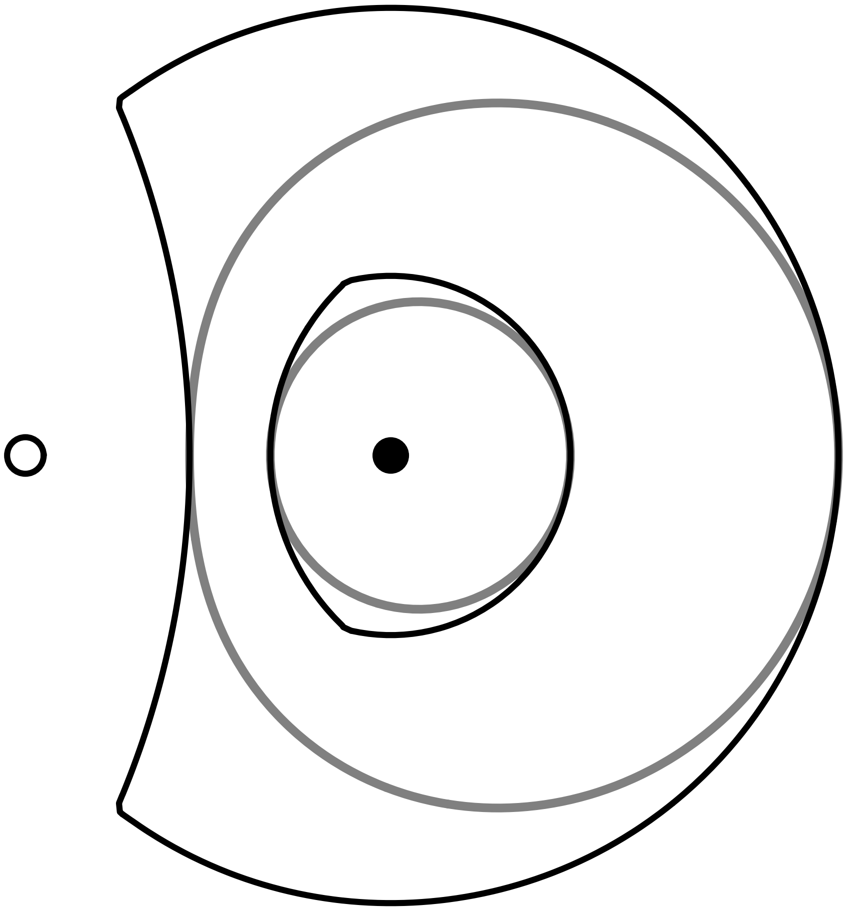

Figure 1: An example of quasihyperbolic and *j*-metric disks in a punctured plane.

Close-to-convexity

Let D **be the unit disk and** $f : D \to D' = f(D)$ **be a univalent function with** $f(0) = 0$**. Then**

- D' **is convex** $\Longleftrightarrow \operatorname{Re} \frac{zf'(z)}{f(z)} > \frac{1}{2}$ **[S, Corollary 2.15].**

- D' **is starlike w.r.t. 0** $\Longleftrightarrow \operatorname{Re} \frac{zf'(z)}{f(z)} > 0$ **[S, Theorem 2.2].**

Close-to-convex functions were introduced by W. Kaplan in 1952 [K].

- $f(z)$ **is close-to-convex iff there exists a convex** g **such that** $\operatorname{Re} \frac{f'(z)}{g'(z)} > 0$**.**

Lewandowski has shown that [S] that f **is close-to-convex iff** D' **is close-to-convex.**

Geometry of the quasihyperbolic balls

The work is based on the following open problem [Vu, 8.1]:

Does there exist $r_0 > 0$ such that $B_m(x,r)$ is convex (in Euclidean geometry) for all $r \in (0, r_0)$? ($G \subset \mathbb{R}^n$ domain, m metric)

Let $G \subsetneq \mathbb{R}^n$ be a domain and $x \in G$. Then $B_k(x,r)$ is

- **convex for $n = 2$ and $r \in (0,1]$ [V],**
- **convex for $r > 0$, if G is convex [MV],**
- **starlike w.r.t x for $r > 0$, if G is starlike w.r.t. x [K].**

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