Palindromic richness for languages invariant under more symmetries

Štěpán Starosta

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Outline

Reversal mapping and its fixed points

$\mathcal{A}$ - alphabet

reversal mapping $R : \mathcal{A} \to \mathcal{A}$

$$R(w_0 w_1 \dots w_n) = w_n \dots w_1 w_0 \qquad \text{where } w_i \in \mathcal{A}$$

$w \in \mathcal{A}^*$ is a **palindrome** if $w = R(w)$

examples: $\varepsilon, 0, 00, 010$

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Palindromic defect for finite words

$\mathrm{Pal}_R(w)$ = set of all palindromic factors of the word $w \in \mathcal{A}^*$

$\#\mathrm{Pal}_R(w) \leq |w| + 1$ [Droubay, Justin, Pirillo, 2001]

defect of a finite word w is defined as

$D(w) = |w| + 1 - \#\mathrm{Pal}_R(w)$ [Brlek, Hamel, Nivat, Reutenauer, 2004]

$D(w)$ = number of lacunas in w

lacuna of $w = w_1 \cdots w_n$ is an index i such that the longest palindromic suffix of $w_1 \cdots w_i$ has at least two occurrences in $w_1 \cdots w_i$ [Blondin Massé, Brlek, Garon, Labbé, 2008]

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defect of an infinite word $\mathbf{u}$ is defined as

$$D(\mathbf{u}) = \sup\{D(w) \mid w \text{ is factor of } \mathbf{u}\}$$

words with palindromic defect 0 are called **rich**

examples of rich words: episturmian words, words coding symmetric IET, codings of rotations on two intervals

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Generalized palindromes

Ψ - involutive antimorphism:

1. $\Psi^2 = \mathrm{Id}$ and
2. $\Psi(uv) = \Psi(v)\Psi(u)$ for every $u, v \in \mathcal{A}^*$

examples: $\mathcal{A} = \{0, 1\}$, $E : 0 \leftrightarrow 1$

E-palindromes: ε, 01, 10, 1010, ...

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Groups of morphisms and antimorphisms

Let G be a *finite* group consisting of morphisms and antimorphisms over $\mathcal{A}^*$ containing at least one antimorphism.

Example on $\{0,1\}$: $G = \{R, E, \mathrm{Id}, RE\}$

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Some definitions

A word $w \in \mathcal{A}^*$ is a G-**palindrome** if there exists an antimorphism $\Theta \in G$ such that $w = \Theta(w)$.

$[w] = \{v \in \mathcal{L} \colon w = \nu(v), \nu \in G\}$

Let $w, v \in \mathcal{A}^*$. A G-**occurrence** of w in v is an index i such that there exists $w' \in [w]$ having occurrence i in v.

We denote by $G\text{-}\mathrm{lps}(w)$ the longest G-palindromic suffix of the word w.

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G-defect

For $w = w_1 \cdots w_n \in \mathcal{A}^*$ we say that an integer i such that $1 \leq i \leq n$ is a G-**lacuna** in w if w_i and $G\text{-}\mathrm{lps}(w_1 \cdots w_i)$ have each at least two G-occurrences in $w_1 \cdots w_i$.

For $w \in \mathcal{A}^*$ we define its G-**defect**

$$D_G(w) = \text{number of } G\text{-lacunas in } w.$$

For an infinite word $\mathbf{u}$ we define

$$D_G(\mathbf{u}) = \sup_{w \text{ is a factor of } \mathbf{u}} \{D_G(w)\}.$$

A word with G-defect 0 is also said to be G-**rich** [Pelantová, S., 2013]

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The Thue–Morse word

The Thue–Morse word $\mathbf{u}_{TM}$ (fixed point of the morphism $0 \mapsto 01, 1 \mapsto 10$) satisfies the following:

- its set of factors is closed under R and E,
- for $G = \{R, \mathrm{Id}\}$, we have $D_G(\mathbf{u}_{TM}) = D(\mathbf{u}_{TM}) = +\infty$,
- for $G = \{E, \mathrm{Id}\}$, we have $D_G(\mathbf{u}_{TM}) = +\infty$,
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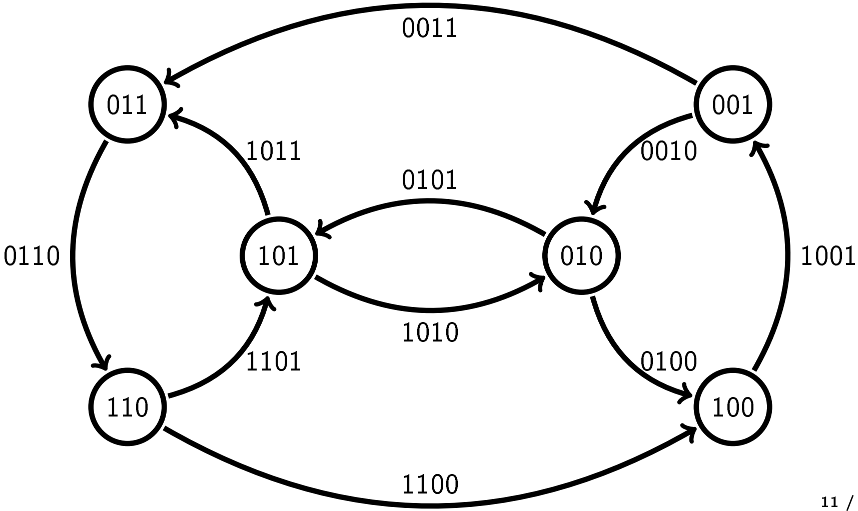
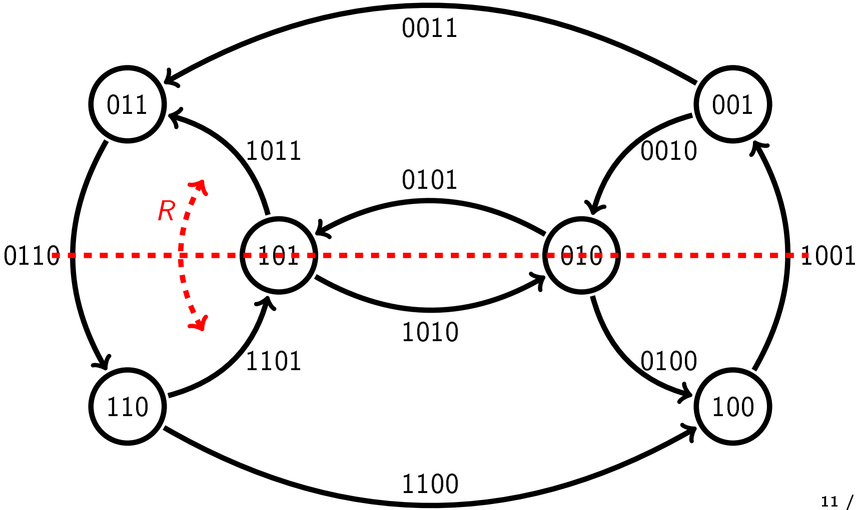
G-richness and Rauzy graphs - example

Rauzy graph of order 3 of the Thue-Morse word



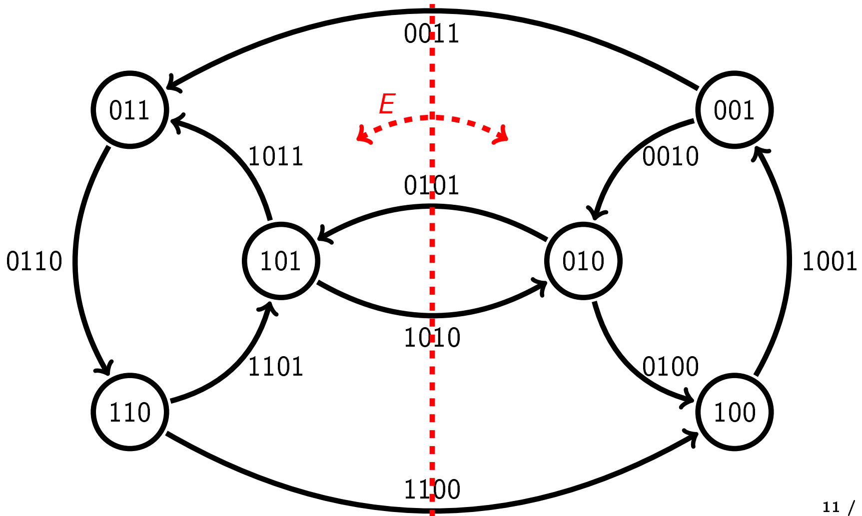
G-richness and Rauzy graphs - example

Rauzy graph of order 3 of the Thue-Morse word
its language is invariant under R



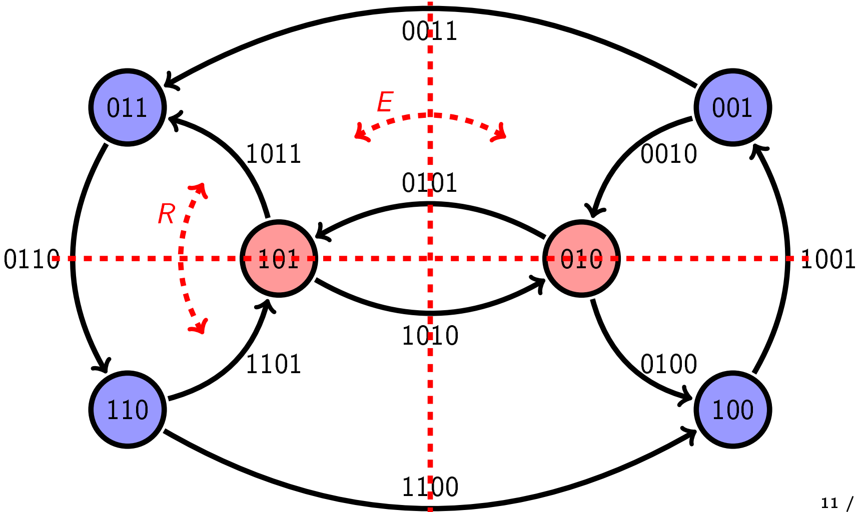
G-richness and Rauzy graphs - example

Rauzy graph of order 3 of the Thue-Morse word
its language is invariant under E



G-richness and Rauzy graphs - example

Rauzy graph of order 3 of the Thue-Morse word
its language is invariant under all elements of $G = \{R, E, \mathrm{Id}, RE\}$



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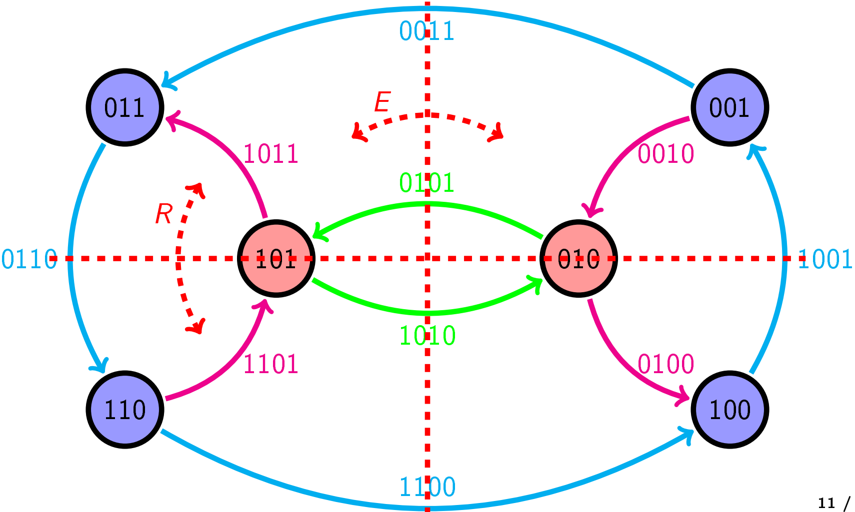

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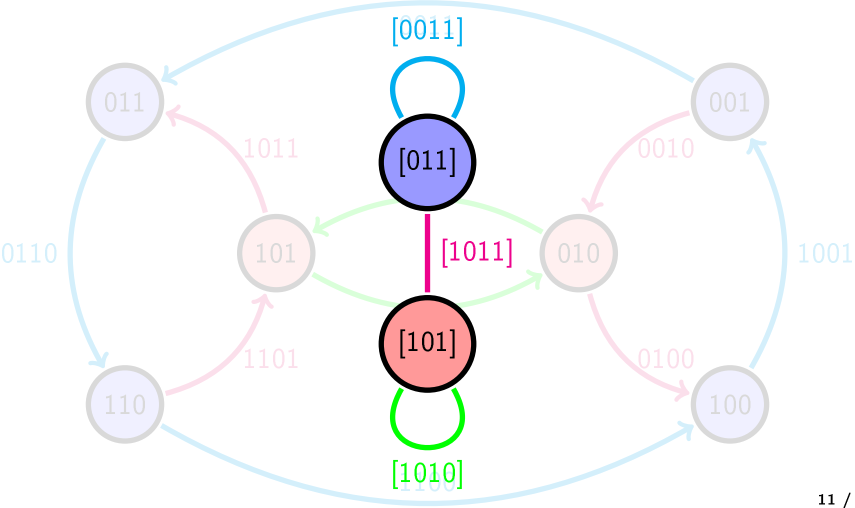
Rauzy graph of order 3 of the Thue-Morse word
undirected graph of symmetries of order 3



Examples: generalized Thue–Morse words

The generalized Thue–Morse word $\mathbf{t}_{b,m}$ is defined for $b > 1$ and $m > 1$ as

$$\mathbf{t}_{b,m} = (s_b(n) \mod m)_{n=0}^{+\infty},$$

where $s_b(n)$ denotes the sum of digits in the base-b representation of n.

$\mathbf{t}_{2,2} = \mathbf{u}_{TM}$

$\mathbf{t}_{b,m}$ is G-rich where G is isomorphic to the dihedral group of order $2m$.

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Some more examples

Computer experiments suggest more examples:

fixed point of

$$0 \mapsto 012, 1 \mapsto 02, 2 \mapsto 1$$

is G-rich where $G = \{R, \Psi, \mathrm{Id}, R\Psi\}$ and Ψ is determined by $0 \leftrightarrow 2$;

periodic point of

$$1 \mapsto 53, 2 \mapsto 64, 3 \mapsto 65, 4 \mapsto 12, 5 \mapsto 13, 6 \mapsto 24$$

is G-rich where $G = \{\Theta, \Omega, \mathrm{Id}, \Theta\Omega\}$, Θ is determined by $1 \leftrightarrow 4$ and $3 \leftrightarrow 6$, and Ω by $1 \leftrightarrow 3$, $2 \leftrightarrow 5$ and $4 \leftrightarrow 6$.

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Ψ-palindromic closure

Let $w \in \mathcal{A}^*$, Ψ be an involutive antimorphism.

The shortest Ψ-palindrome having w as a prefix is the Ψ-*palindromic closure* w.

Notation w^Ψ.

Examples on $\mathcal{A} = \{0, 1\}$:
$\varepsilon^E = \varepsilon$
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Generalized pseudostandards words

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Let $\Theta = \vartheta_1\vartheta_2\vartheta_3\ldots \in \mathcal{I}^{\mathbb{N}}$ and $\Delta = \delta_1\delta_2\delta_3\ldots \in \mathcal{A}^{\mathbb{N}}$.

Denote

$$w_0 = \varepsilon \quad \text{and} \quad w_n = \left(w_{n-1}\delta_n\right)^{\vartheta_n} \quad \text{for every positive integer } n.$$

The word

$$\mathbf{u}_\Theta(\Delta) = \lim_{n\to\infty} w_n$$

is a **generalized pseudostandard word** with the directive sequence of letters Δ and the directive sequence of antimorphisms Θ. [de Luca, De Luca, 2006]

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$\Delta = (01)^{\omega}$

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$w_2 = (w_1 1)^R = (01)^R = 010$
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$\mathbf{u}_{\Theta}(\Delta) = 01001010010\ldots$ - the Fibonacci word

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$w_1 = (w_0 0)^R = 0^R = 0$
$w_2 = (w_1 1)^R = (01)^R = 010$
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$\vdots$
$\mathbf{u}_{\Theta}(\Delta) = 01001010010\ldots$ - the Fibonacci word

Generalized pseudostandard word example 1

$\mathcal{A} = \{0, 1\}$

$\Theta = R^\omega$
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Generalized pseudostandard word example 2

$\mathcal{A} = \{0, 1\}$

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$w_1 = (w_0 0)^E = 0^E = 01$
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$\mathbf{u}_\Theta(\Delta) = 0110100110010110\ldots$ - the Thue-Morse word

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The case $\#\mathcal{I} = 1$

Let $\#\mathcal{I} = 1$, i.e., $\mathcal{I} = \{\Psi\}$.

This case is known as Ψ-**standard** words.

Ψ-standard words have finite $\{\Psi, \mathrm{Id}\}$-defect. [Bucci, de Luca, De Luca, Zamboni, 2008]

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Generalized Thue–Morse words (1/2)

Recall that the generalized Thue-Morse word $\mathbf{t}_{b,m}$ is defined

$$\mathbf{t}_{b,m} = \left(s_b(n) \bmod m\right)_{n=0}^{\infty}$$

for $b > 1$ and $m > 1$.

Theorem (Jajcayová, Pelantová, S., 2013)

The generalized Thue–Morse word $\mathbf{t}_{b,m}$ is a generalized pseudostandard word if and only if $b \leq m$ or $b \equiv 1 \pmod{m}$.

Generalized Thue–Morse words (2/2)

The word $\mathbf{t}_{b,m}$ is defined over the alphabet $\mathbb{Z}_m = \{0, \ldots, m-1\}$.

Denote by $I_2(m)$ the group generated by the antimorphisms Ψ_x defined for every $x \in \mathbb{Z}_m$ by $\Psi_x(k) = x - k \quad$ for every $\; k \in \mathbb{Z}_m$.

Let $m, b \in \mathbb{N}$ such that $m > 1$ and $b > 1$.
Denote $\Delta = 0\Big(12\ldots(b-1)\Big)^{\omega} \in \mathbb{Z}_m^{\mathbb{N}} \quad$ and
$\Theta = \Big(\Psi_0\Psi_1\ldots\Psi_{m-1}\Big)^{\omega} \in I_2(m)^{\mathbb{N}}$.
If $b \leq m$ or $b \equiv 1 \pmod{m}$, then

$$\mathbf{u}_{\Theta}(\Delta) = \mathbf{t}_{b,m},$$

i.e., the generalized pseudostandard word $\mathbf{u}_{\Theta}(\Delta)$ with directive sequences Δ and Θ equals the generalized Thue–Morse word $\mathbf{t}_{b,m}$.

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Generalized pseudostandard word are not G-rich

$\mathcal{A} = \{0, 1, 2, 3\}$

$$\mathcal{I} = \left\{ \sigma_0 : \begin{pmatrix} 1 \leftrightarrow 3 \end{pmatrix}, \sigma_1 : \begin{pmatrix} 0 \leftrightarrow 1 \\ 2 \leftrightarrow 3 \end{pmatrix}, \sigma_2 : \begin{pmatrix} 0 \leftrightarrow 2 \end{pmatrix}, \sigma_3 : \begin{pmatrix} 0 \leftrightarrow 3 \\ 1 \leftrightarrow 2 \end{pmatrix} \right\}$$

$\Delta = 0102$

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$D_G(u_\Theta(\Delta)) = D_G(01021221121323) = 1$ for $G = \langle \mathcal{I} \rangle$

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Ambiguity for $\#\mathcal{I} > 1$

$\mathcal{A} = \{0, 1\}$

Example:
$\Theta_1 = E^\omega$ and $\Delta_1 = 0^\omega$
$0^E = 01$
$010^E = 0101$

$\Theta_2 = (RE)^\omega$ and $\Delta_2 = (01)^\omega$
$0^R = 0$
$01^E = 01$
$010^R = 010$

$\mathbf{u}_{\Theta_1}(\Delta_1) = \mathbf{u}_{\Theta_2}(\Delta_2)$

Normalized directive sequences

To deal with this, [Blondin Massé, Paquin, Tremblay, Vuillon, 2013] introduced normalization.

Directive sequences (Θ, Δ) are **normalized** if the following is verified: if p is a G-palindromic prefix of $\mathbf{u}_\Theta(\Delta)$, then there exists an integer n such that $w_n = p$.

In other words, we do not miss any G-palindromic prefix while constructing the generalized pseudostandard word.

On binary alphabet, a normalization process is described.

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Open questions

1. How to normalize the directive sequences of every generalized pseudostandard word?
2. How to recognize those generalized pseudostandard words that have finite G-defect?
3. Given a group G generated by involutive antimorphisms, is there a G-rich word?
4. Conjecture: if an infinite word is a fixed point of a primitive injective morphism, then its G-defect is 0 or $+\infty$. This conjecture is still open for the classical notion of palindromic defect.
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Thank you