When images of substitution shifts are trace shifts

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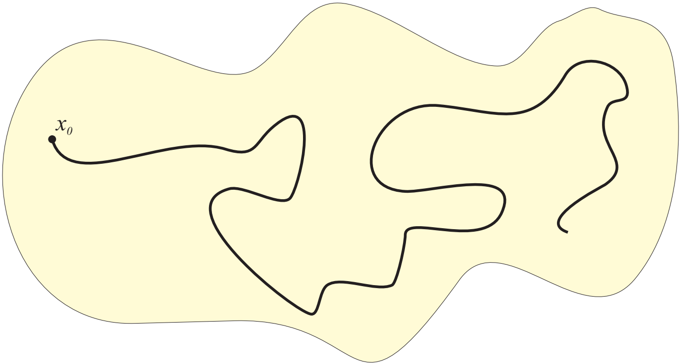
Dynamics of sequential processes

A dynamical system, in general, is a pair (X, T), where X is a compact metric space and T a continuous mapping.
Dynamical systems originally arose in the study of systems of differential equations used to model physical phenomena. The motions of the planets, or of mechanical systems, or of molecules in a gas can be modelled by such systems.



One simplification in this study is to discretize time, so that the state of the system (X, T) is observed only at discrete ticks of a clock, like a motion picture. This leads to the study of the iterates of a transformation T -

$$x_0, T(x_0), T^2(x_0), ..., T^n(x_0), ...$$

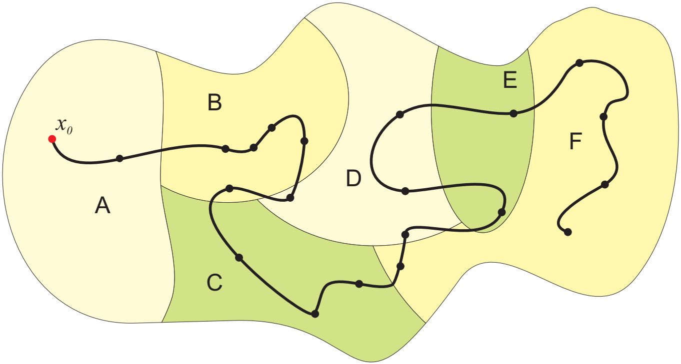
Dynamics of sequential processes

In this study one could be interested in:

- quantitative behavior, such as the average time spent in a certain region,
- qualitative behavior, such as whether a state eventually becomes periodic or tends to infinity.

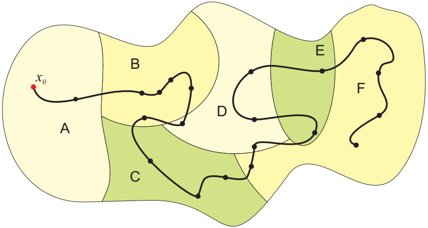
Symbolic dynamics of sequential processes

After discretizing time it is also possible to discretize space.

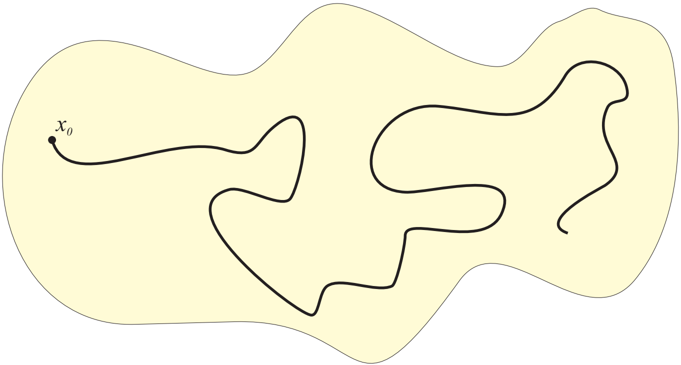


Symbolic dynamics of sequential processes

SYMBOLIC DYNAMICS arose as an attempt to study dynamics by means of **discretizing space as well as time**.

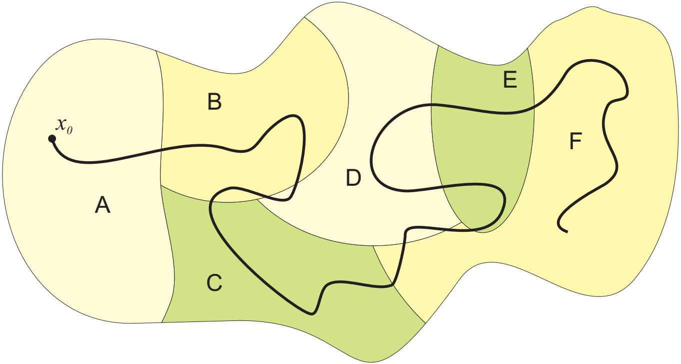


Dynamics of sequential processes



1. *AAB B BBDBCCCFDEDDEFFFF*
2. *AABBBBDBCCCFDEDDEFFFF*

Dynamics of sequential processes



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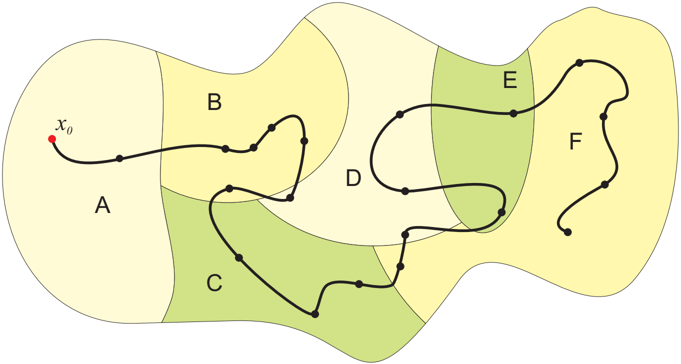
Dynamics of sequential processes



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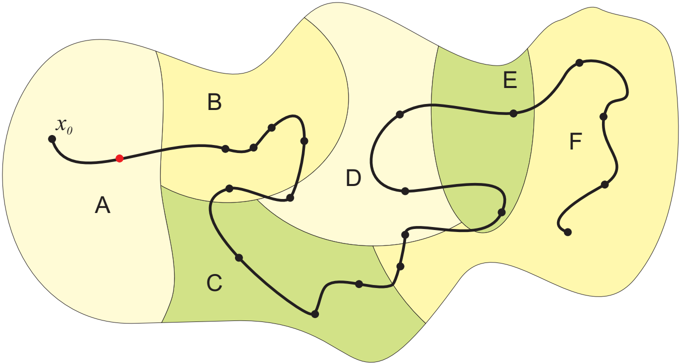
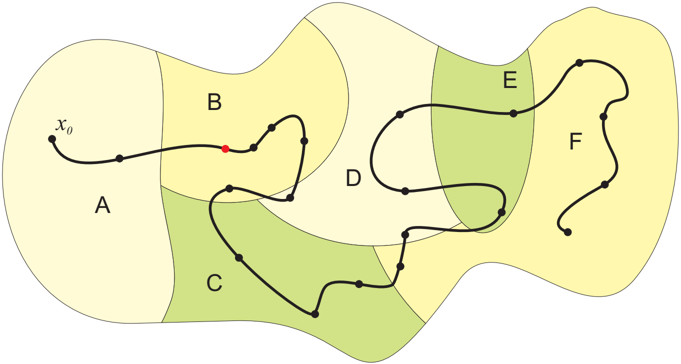
2. *AABBBBDBCCCFDEDDEFFFF*

Dynamics of sequential processes



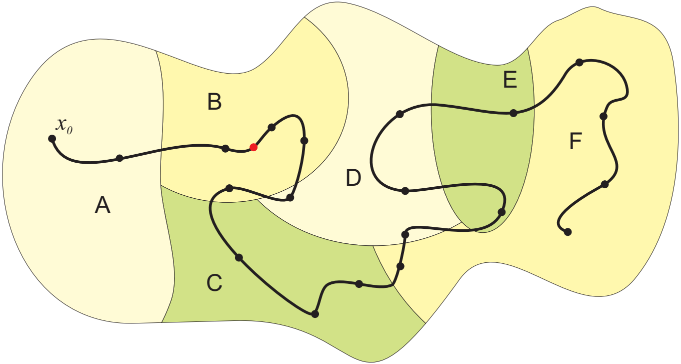
1. $AABBBBDBCCCFDEDDEFFFF$
2. $ABBBBDBCCCFDEDDEFFFF$

Dynamics of sequential processes



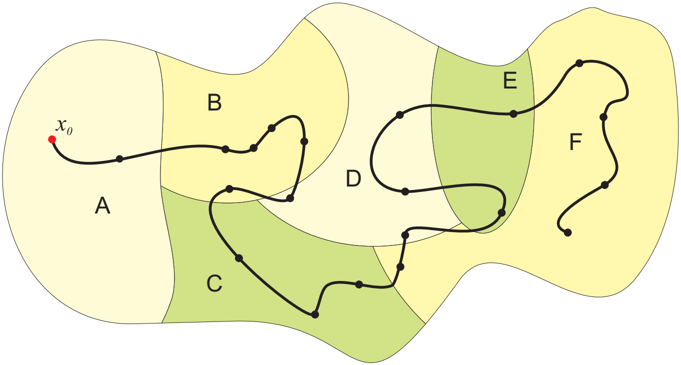
1. $AA\textcolor{red}{B}BBBDBCCCFDEDDEFFFF$
2. $BBBBDBCCCFDEDDEFFFF$

Dynamics of sequential processes



1. *AAB**B**BBDBCCCFDEDDEFFFF*
2. *BBBDBCCCFDEDDEFFFF*

Dynamics of sequential processes



1. $AABBBBDBCCCFDEDDEFFFF\ldots$

SHIFTS - Discrete sequential dynamics - a bit more formal view

(X, σ) such that:

1. $X \subset \mathcal{A}^{\mathbb{N}}$ and closed
2. X invariant, that is $\sigma(X) \subset X$ where
   $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$, $\sigma(x)_i = x_{i+1}$ - shift mapping

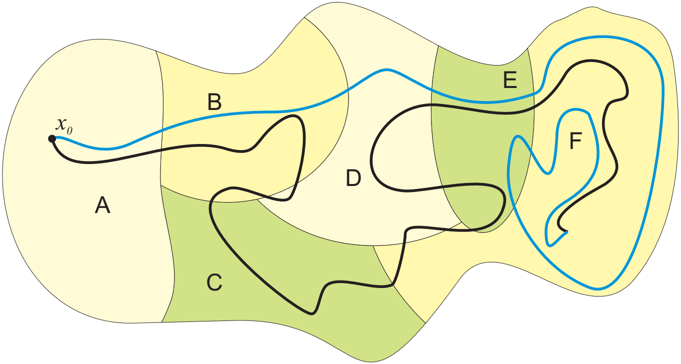
SHIFT – symbolic description of a sequential dynamics
σ cuts off the first letter of an infinite sequence

Topology in $\mathcal{A}^{\mathbb{N}}$ given by the metric

$$d(x,y) = \begin{cases} 0 & ,x = y \\ 2^{-k} & ,x \neq y \end{cases}$$

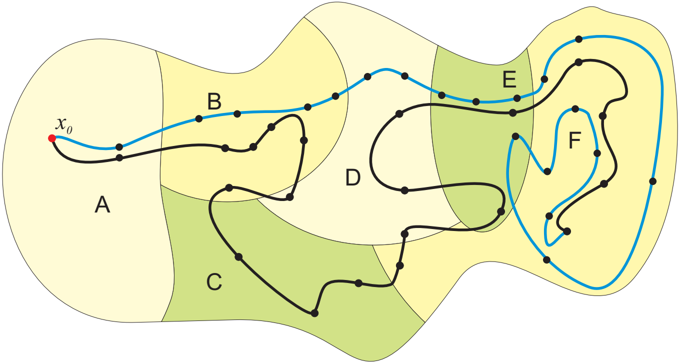
where $k = \max\{i \in \mathbb{N} \,:\, x_{[0,i)} = y_{[0,i)}\}$.

Parallel processes



1. AABBBBDBCCCFDEDDEFFFF
2. AABBBBDDEEEFFFFFEFFFFF
3. AABBBBD $\begin{matrix}\text{BCCC}\\ \text{DEEE}\end{matrix}$ F $\begin{matrix}\text{DEDDE}\\ \text{FFFEF}\end{matrix}$ FFFF

Parallel processes



1. *AABBBBDBCCCFDEDDEFFFF*
2. *AABBBBDDEEEFFFFEFFFFF*
3. $AABBBBD \begin{matrix} BCCC \\ DEEE \end{matrix} F \begin{matrix} DEDDE \\ FFFEF \end{matrix} FFFF$

Traces (finite)

1. $I \subset \mathcal{A} \times \mathcal{A}$ symmetric, irreflexive relation
2. the relation I induces a congruence $\sim_I$ on $\mathcal{A}^*$
3. Let $u, v \in \mathcal{A}^*$. We write $u \sim_I w$, if:
   1. there exists a sequence $v_1, \dots, v_k$ of words such that $u = v_1$, $w = v_k$ and
   2. for any $i = 1, \dots, k-1$ there exist words v_i' v_i'' and letters $a_i, b_i \in \mathcal{A}$ such that

$$v_i = v_i' a_i b_i v_i'', \quad v_{i+1} = v_i' b_i a_i v_i'' \quad and \quad (a_i, b_i) \in I$$

Example: $I = \{(a, b), (b, a)\}$, $aaabcbbbba \sim_I abaacabbbb$.

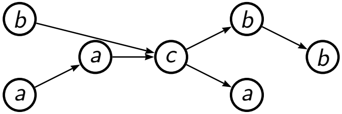
I – INDEPENDENCE (COMMUTATION) RELATION
$D = (\mathcal{A} \times \mathcal{A}) \setminus I$ – DEPENDENCE RELATION
TRACE (finite) – equivalence class of $\sim_I$ on $\mathcal{A}^*$ – $t = [w]_{\sim_I}$

Trace representation - Foata normal form

1. A word $w \in \mathcal{A}^*$ is in Foata normal form, if it is the empty word or if there exist an integer $n > 0$ and nonempty words $v_1, ..., v_n \in \mathcal{A}^+$ such that:
    1. $w = v_1.....v_n$,
    2. for any $i = 1, ..., n$ the word v_i is a catenation of pairwise independent letters and v_i is minimal with respect to the lexicographic ordering,
    3. for any $i = 1, ..., n-1$ and for any letter $a \in alph(v_{i+1})$ there exists a letter $b \in alph(v_i)$ such that $(a, b) \in D$.
2. Foata normal form is unique for the class $[u]_{\sim_I}$. Example: $I = \{(a, b), (b, a)\}$ and $u = aabcbba$, FNF: $[ab][a][c][ab][b]$

Trace representation - dependence graph

Traces can be represented also as dependence graphs (Hasse diagrams) - $u = aabcbba$, FNF: $[ab][a][c][ab][b]$ and



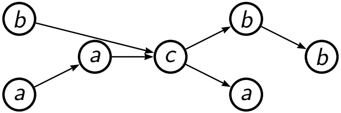
Hence traces can be considered as:

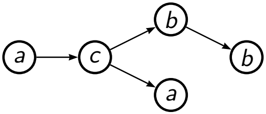
- equivalent classes of $\sim_I$ defined on $\mathcal{A}^*$ — $t = [u]_{\sim_I}$
- labelled, directed and acyclic graphs
- words in Foata Normal Form

Infinite traces

$\mathcal{A}, I$ fixed.

1. For an infinite word $x = a_1 a_2 \ldots\ldots \in \mathcal{A}^{\mathbb{N}}$ we define the dependence graph $\varphi_{\mathbb{G}}(x) = [V, E, \lambda]$ where:
   1. $V = \mathbb{N}$ and
   2. $\lambda(i) = a_i$ for any $i \in \mathbb{N}$.
   3. There exists an arrow $(i, j) \in E$, if and only if $i < j$ and $(a_i, a_j) \in D$.
2. We extend the shift map $\sigma(x)_i = x_{i+1}$ to infinite traces to obtain a map denoted Φ which removes the first Foata step:





Shifts on infinite Traces - a more formal view

define a metric d :

$$d_{\mathbb{R}}(s,\ t) = \begin{cases} 2^{-l_{\mathbb{R}}(s,\,t)} & \textit{if} \quad x \neq y \\ 0 & \textit{otherwise} \end{cases}$$

where

$l_{\mathbb{R}}(s,\ t) = \sup\{n \in \mathbb{N}\ :\ \forall p \in \mathbb{M}(\Sigma, I),\ |\ p\ | \leq n,\ \ p \leq s \Leftrightarrow p \leq t\}$

p is a prefix in a trace sense.

Trace Shifts

(X, Φ), where X a set of infinite traces and

1. $X \subset \mathbb{R}^{\omega}(\Sigma, I)$ closed
2. invariant - $\Phi(X) \subset X$ (Φ cuts off the first Foata step in an infinite trace)

t-SHIFT – Trace shift - symbolic description of a parallel dynamics

Shifts on Traces - general research problem

Find relations between the dynamics of sequential computations and their parallel (trace) counterparts

Remark
Notice that problems which we consider are of the dual nature – combinatorial (combinatorics on words) and topological.

Basic observations - Problems

1. The map $\varphi_{\mathbb{G}}$ which transfers infinite sequences to labelled, directed and acyclic graphs is not continuous,
2. converting a shift (X, σ) by $\varphi_{\mathbb{G}}$ results in a set of infinite traces (dependence graphs). It may happened that the resulted set $\varphi_{\mathbb{G}}(X)$ is not a t-shift,
3. if (Y, Φ) is a t-shift then it may happened that $\varphi_{\mathbb{G}}(X) \neq Y$ for any shift (X, σ)

Minimal shifts, minimal t-shifts

Definition 1

(X, σ) *is a minimal shift if it is closed, nonempty, invariant and contains no proper subset with these three properties.*

- If X is a minimal shift, then $\varphi_{\mathbb{G}}(X)$ is closed.
- It may happened that a corresponding set of traces is not invariant.

Minimal substitution shifts and their trace counterparts

$h : \Sigma^* \longrightarrow \Sigma^*$ satisfies conditions UR - (uniform recurrence property):

1. For any letter $a \in \Sigma$ $\lim_{n\to\infty} \mid h^n(a) \mid = +\infty$,
2. there exists a letter $a_0 \in \Sigma$ such that $h(a_0)$ begins with a_0,
3. for any letter $a \in \Sigma$ there exists an integer $k \geq 0$ such that in $h^k(a)$ occurs the letter a_0.

$h - -substitution$

- $u = h^\omega(a_0) = lim_{n\to\infty} h^n(a_0)$ - an infinite uniformly recurrent word,
- $X = \mathrm{cl}(\mathrm{Orb}^+(u))$ where $\mathrm{Orb}^+(u) = \{u, \sigma(u), \sigma^2(u), \dots\}$ and *cl* stays for a topological closure - minimal shift

$u \in \Sigma^\omega$ is uniformly recurrent if every subword of u occurs infinitely often in u with bounded gaps.

Now consider a set of infinite traces $\varphi_{\mathbb{G}}(X)$ where $X = \text{cl}(\text{Orb}^+(u))$ is a minimal substitution shift. Basing on the theorem (WF, P.Oprocha):

Theorem 2

Let $X \subset \Sigma^\omega$ be a shift. If for every integer $n > 0$ there exists j such that for all $x \in X$ and $a \in alph(X)$ it holds $|x_{[0,j]}|_a \geq n$, then $\varphi_{\mathbb{G}}(X)$ is closed.

we obtain:

Corollary 3

If $X = \text{cl}(Orb^+(u))$ is a minimal substitution shift, then $\varphi_{\mathbb{G}}(X)$ is closed.

but not necessary invariant ...

Results

... but not necessary invariant.

Example 4

$\Sigma = \{a, b, c, d\}$, $I = \{(b, d), (d, b), (d, c), (c, d)\}$ *and*
$h(a) = ab, h(b) = da, h(c) = cb, h(d) = aa$
h fulfils conditions (UR) and thus $u = h^{\omega}(c)$ is uniformly recurrent
$X = \mathrm{cl}(Orb^{+}(u))$ *minimal substitution shift.*

Consider $x \in X$, $x = cbdaaaabababda.........$
$\varphi_{\mathbb{G}}(x) = [cd][b][a][a][a][a][b][a][b][a].... \in \varphi_{\mathbb{G}}(X)$.

Hence in $\varphi_{\mathbb{G}}(X)$ should be also $[b][a][a][a][a][b][a][b][a]....$ but there is no any infinite word in X with a subword baaaa.
Thus $\varphi_{\mathbb{G}}(X)$ is not invariant.

Results

Theorem 5

$h : \Sigma^* \longrightarrow \Sigma^*$ a substitution + conditions (UR) and $X = \mathrm{cl}(Orb^+(u))$ minimal substitution shift. Assume that for any $a_i, a_j \in \Sigma$,
$\varphi_{\mathbb{G}}(h(a_i)) = v_1^i v_2^i \dots v_{r_i}^i$ $\varphi_{\mathbb{G}}(h(a_j)) = v_1^j v_2^j \dots v_{r_j}^j$
v_k^l - Foata step in the Foata normal form of $\varphi_{\mathbb{G}}(h(a_i))$, $r_i, r_j > 2$. If one of the following conditions hold:

(a) *for any $y \in alph(v_1^j)$ there exists $x \in alph(v_{r_i}^i)$ such that xDy*

(b) $v_{r_i}^i \sim_I v_1^j$ *and the condition (a) holds for $v_1^j, v_{r_{i-1}}^i$.*

Then $\widehat{w} = (v_1^0)^{-1}\varphi_{\mathbb{G}}(u)$ is a minimal point (almost periodic) where $\varphi_{\mathbb{G}}(h(a_0)) = v_1^0 \dots v_{r_0}^0$. That is $\mathrm{cl}(Orb^+(\widehat{w}))$ is a minimal t-shift.

Observation 1

1. $\Sigma = \{a_1, ..., a_k\}$,
2. I independence relation given by a cycle C_k,
3. morphism h is identity on letters $a_1, ..., a_{k-1}$ and $h(a_k) = a_1.....a_k$.

$u = h^{\omega}(a_k) = (a_1.....a_{k-1})^{\infty}$
$X = \mathrm{cl}(\mathrm{Orb}^{+}(u))$ - minimal substitution shift
$\varphi_{\mathbb{G}}(X)$ t-shift but not minimal – it splits into several minimal shifts.

For k even:
$\mathrm{cl}(\mathrm{Orb}^{+}((F_1.....F_{k/2})^{\infty}))$, where $F_i = a_i a_{i+1}$ and
$\mathrm{cl}(\mathrm{Orb}^{+}(F_1.....F_{k/2})^{\infty})$, where $F_i = a_{i+1} a_{i+2}$ and $F_{k/2} = a_k a_1$.

Observation 2

1. $\Sigma = \{a_1, \ldots.., a_k\}$,
2. I independence relation given by a path P_k,
3. morphism h is identity on letters $a_1, \ldots.., a_{k-1}$ and $h(a_k) = a_1 \ldots.. a_k$.

$u = h^\omega(a_k) = (a_1 \ldots.. a_{k-1})^\infty$
$X = \mathrm{cl}(\mathrm{Orb}^+(u))$ - minimal substitution shift

$\varphi_{\mathbb{G}}(X)$ - t-shift
It contains a minimal t-shift - (k even):
$\mathrm{cl}(\mathrm{Orb}^+((F_1 \ldots.. F_{k/2})^\infty)$, where $F_i = a_i a_{i+1}$

THANK YOU FOR YOUR ATTENTION

MOTIVATION

The problem is well known in the theory of computing. Namely, a one-tape Turing machine is equivalent, in the sense of a computational power, to a Turing machine with multiple tapes. However, if we analyze moves of the tape then a "simple" computation of a multi-tape machine may force quite "complex" behavior of a one-tape machine during a simulation process. There are also other models of computation considered in the literature. For example, CRCW P-RAM - Concurrent Read Concurrent Write Parallel RAM that realizes the situation in which more than one processor can concurrently read from or write into the same memory location. These models could be described as dynamical systems and then one could analyze some of their dynamical properties. Basing our research on trace theory we try to combine, in some sense, these two approaches - parallel or concurrent execution of computational processes and its dynamics description using symbolic dynamics tools.