Regular Ideal Languages and Synchronizing Automata

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Synchronizing Automata

- A deterministic finite automaton (DFA) is a triple $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ (more appropriate to call semiautomaton). When it is a language recognizer $\mathscr{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$ it is also called DFA.
- The transition function $\delta : Q \times \Sigma \to Q$ naturally extends to the free monoid Σ^*, this extension is still denoted by δ; also for $S \subseteq Q$ and $w \in \Sigma^*$ we write $\delta(S, w) = \{\delta(q, w) \mid q \in S\} = S \cdot w$.
- A DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is called *synchronizing* if there is a word w whose action *resets* $\mathscr{A}$, that is, leaves the automaton in one particular state no matter which state in Q we start at: $\delta(q, w) = \delta(q', w)$ for all $q, q' \in Q$, equiv. $|Q \cdot w| = |\delta(Q, w)| = 1$.
- Any such w is called *synchronizing* or *reset* word for $\mathscr{A}$. The set of resets words is denoted by $\mathrm{Syn}(\mathscr{A})$.

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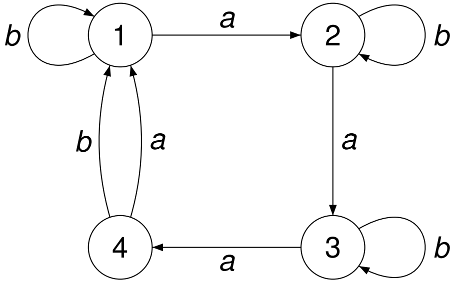
Synchronizing Automata and The Černý's conjecture



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- In 1964 Jan Černý found an infinite series of n-state reset automata whose shortest reset word has length $(n-1)^2$.
- This leads him to state the following:

Conjecture

Any synchronizing automaton with n states has a reset word of length at most $(n-1)^2$.

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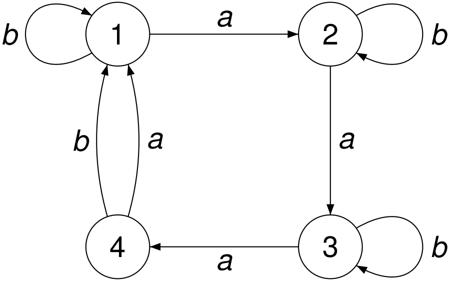

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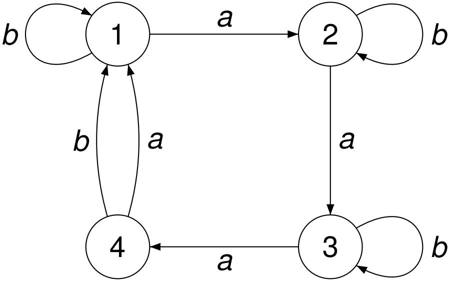

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The language of reset words

- $\mathrm{Syn}(\mathscr{A})$ is a regular language which is also a two-sided ideal (short ideal):

$$\Sigma^* \mathrm{Syn}(\mathscr{A})\Sigma^* \subseteq \mathrm{Syn}(\mathscr{A})$$

- Left (right) ideals: $\Sigma^* I \subseteq I$, ($I\Sigma^* \subseteq I$)
- Henceforth we consider just regular ideal languages.

Main Question

What is the relationship between ideals and synchronizing automata?

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The notion of reset complexity

Some simple remarks

- For each ideal I, the minimal DFA $\mathscr{A}_I$ recognizing I is synchronizing.
- $\mathscr{A}_I = \langle Q, \Sigma, \delta, q_0, \{s\} \rangle$ is synchronizing with a sink s, i.e. $s \cdot \Sigma = s$
- $\mathrm{Syn}(\mathscr{A}_I) = I = L[\mathscr{A}_I]$.

Therefore each ideal serves as the set of reset words for some synchronizing automaton:

Definition (Maslennikova)

The reset complexity $\mathrm{rc}(I)$ is the number of states of the smallest synchronizing automaton $\mathscr{A}$ with $\mathrm{Syn}(\mathscr{A}) = I$

Succinctness of the reset complexity

- Reset complexity is a "measure" of the complexity to describe an ideal.
- Cerny's conjecture holds iff $\mathrm{rc}(I) \geq \sqrt{\|I\|} + 1$, $\|I\| = \min\{|u| : u \in I\}$.
- It is similar to the state complexity of a regular language $\mathrm{sc}(L)$ as the number of the states of the minimal DFA recognizing L.
- Clearly $\mathrm{rc}(I) \leq \mathrm{sc}(I)$ since the minimal DFA $\mathscr{A}_I$ recognizing I is synchronizing. How much can we "compress" an ideal?

Representation of an ideal language by means of a synchronizing automaton can be exponentially smaller then the minimal DFA:

Theorem (Maslennikova)

There are ideals I_n for every $n \geq 3$ such that $\mathrm{rc}(I_n) = n$ and $\mathrm{sc}(I_n) = 2^n - n$.

Strongly connected synchronizing automata

- Cerny's conjecture holds iff it holds for strongly connected synchronizing automata.
- The minimal DFA recognizing an ideal I is always a synchronizing automaton whose set of reset words is I, but it is never strongly connected (unless it is trivial).

Question:

Given an ideal I, is there always a strongly connected synchronizing automaton $\mathscr{A}$ with $\mathrm{Syn}(\mathscr{A}) = I$?

Theorem (Gusev, Maslennikova, Pribavkina)

For a principal ideal $I = \Sigma^* w \Sigma^*$ there is an algorithm that produces a strongly connected synchronizing automaton $\mathscr{A}$ with $\mathrm{Syn}(\mathscr{A}) = I$ and with $|w| + 1$ states (as $\mathscr{A}_I$).

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Reset left (right) regular decompositions

Definition (Provisional)

An ideal I is called strongly connected if there is a strongly connected synchronizing automaton $\mathscr{A}$ with $\mathrm{Syn}(\mathscr{A}) = I$.

Definition (Reset left regular decomposition)

A reset left regular decomposition is a finite collection $\{I_i\}_{i\in F}$ of disjoint left ideals I_i of Σ^* such that:

i) For any $a \in \Sigma$ and $i \in F$, there is a $j \in F$ such that $I_i a \subseteq I_j$

ii) Let $I = \biguplus_{i\in F} I_i$, for any $u \in \Sigma^*$ if there is an $i \in F$ such that $Iu \subseteq I_i$, then $u \in I$.

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What is the relationship between strongly connected synchronizing automata and reset left regular decompositions?

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A categorical equivalence

- **SCSA**$_\Sigma$ category of strongly connected synchronizing automata on Σ, arrows are homomorphisms $\varphi : \mathscr{A} \to \mathscr{B}$.
- **RLD**$_\Sigma$ category of the reset left regular decompositions, arrows $f : \{I_i\}_{i\in F} \to \{J_i\}_{i\in H}$ is any function $f : F \to H$ such that for any $i \in F$ we have $I_i \subseteq J_{f(i)}$.

A categorical equivalence

Theorem (Réis, R.)

*I is strongly connected iff it has a reset left regular decomposition. Moreover **RLD**$_\Sigma$ and **SCSA**$_\Sigma$ are equivalent categories via the two functors $\mathcal{A}, \mathcal{I}$ defined by:*

- $\mathcal{A}$: **RLD**$_\Sigma \to$ **SCSA**$_\Sigma$ *which sends*

$$\mathcal{A} : \{I_i\}_{i\in F} \mapsto \mathcal{A}(\{I_i\}_{i\in F}) = \langle \{I_i\}_{i\in F}, \Sigma, \eta \rangle$$

  with $\eta(I_i, a) = I_j$ *iff* $I_i a \subseteq I_j$, *and if* $f : \{I_i\}_{i\in F} \to \{J_i\}_{i\in H}$ *then* $\mathcal{A}(f) : \mathcal{A}(\{I_i\}_{i\in F}) \to \mathcal{A}(\{J_i\}_{i\in H})$ *with* $I_i \mapsto J_m$ *if* $I_i \subseteq J_m$.

- $\mathcal{I}$: **SCSA**$_\Sigma \to$ **RLD**$_\Sigma$ *sending*

$$\mathcal{I} : \mathscr{A} = \langle Q, \Sigma, \delta \rangle \mapsto \mathcal{I}(\mathscr{A}) = \{\mathcal{I}(\mathscr{A})_q\}_{q\in Q}$$

  where $\mathcal{I}(\mathscr{A})_q = \{u \in \Sigma^* : \delta(Q, u) = q\}$, *and if* $\varphi : \langle Q, \Sigma, \delta \rangle \to \langle T, \Sigma, \xi \rangle$, *then* $\mathcal{I}(\varphi)$ *is the arrow* $f : Q \to T$ *with* $q \mapsto \varphi(q)$.

Ideals are actually strongly connected!

Corollary

Let I be an ideal over a unary alphabet Σ, then I is strongly connected iff $I = \Sigma^*$.

Theorem (Réis, R.)

Let I be an ideal on a non-unary alphabet, then I is strongly connected.

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The Sketch of the proof (I)

- It is enough to show that I^R has a reset right regular decomposition $\{I_k\}_{k\in F}$, then $\{I_k^R\}_{k\in F}$ is a reset left regular decomposition, then apply the previous Theorem.
- Consider the minimal DFA $\mathscr{A}_{I^R} = \langle Q, \Sigma, \delta, q_0, \{s\}\rangle$.

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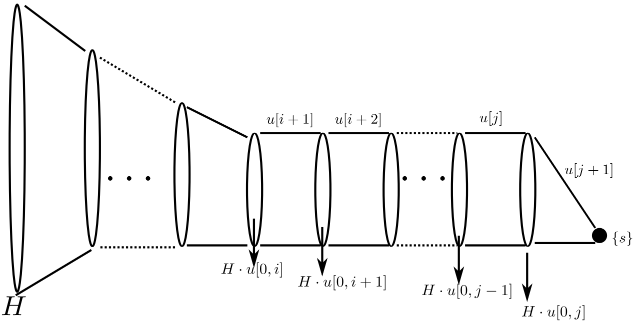
- Let i be the minimum index s.t. $|H \cdot u[0, i]| = |S|$, the tail of (H, u) is the element of $\mathbb{Z}_\ell([2^Q]_r \uplus \Sigma)$, where $\ell = \frac{n^2+n}{2} + 1$, r=|S|



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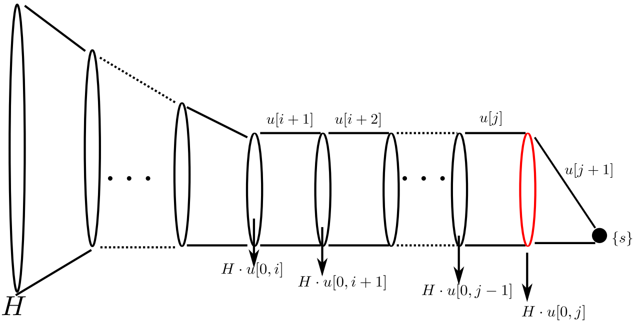

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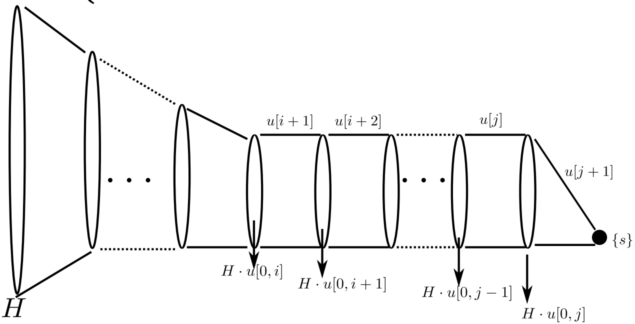
$$\mathcal{T}(H, u) = \begin{cases} \sum_{m=i}^{j-1} (H \cdot u[0, m] + u[m+1]), & \text{if } u[0, i] = u \\ \sum_{m=i}^{j} (H \cdot u[0, m] + u[m+1]), & \text{otherwise.} \end{cases}$$



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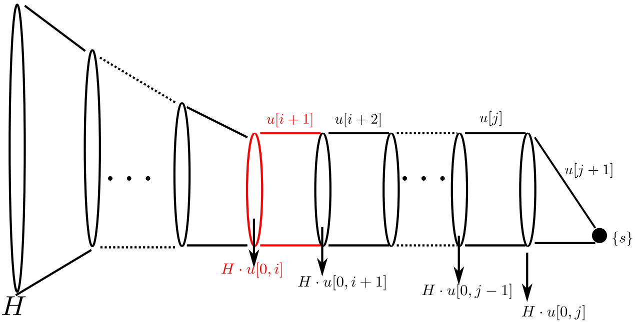

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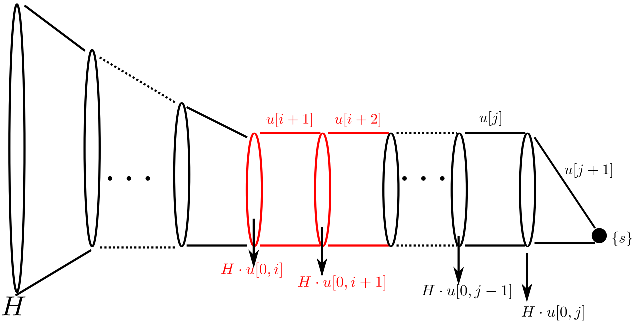

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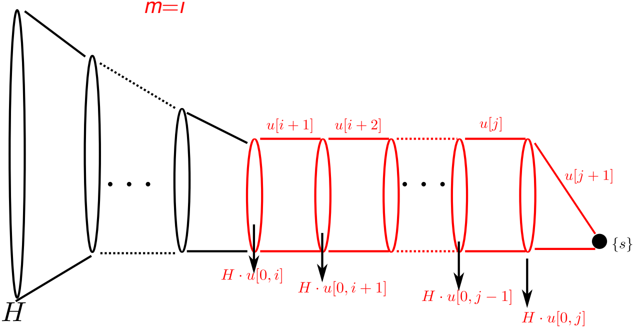

The Sketch of the proof (II)

- Let $\mathbb{T} = \uplus_{r=2}^{n} \mathbb{Z}_\ell([2^Q]_r \uplus \Sigma)$ we give a structure of graded semigroup, for $\mathcal{T}_1 \in \mathbb{Z}_\ell([2^Q]_i \uplus \Sigma)$, $\mathcal{T}_2 \in \mathbb{Z}_\ell([2^Q]_j \uplus \Sigma)$ we define:

$$\mathcal{T}_1 \diamond \mathcal{T}_2 = \begin{cases} \mathcal{T}_{\min\{i,j\}} & \text{if } i \neq j \\ \mathcal{T}_1 + \mathcal{T}_2 & \text{otherwise} \end{cases}$$

Lemma

Consider the map $\mu : \Sigma^* \to Hom(2^Q, \mathbb{T})$ defined by $\mu(u) = \tau_u$ with

$$\tau_u(H) = \begin{cases} \mathcal{T}(H, u) & \text{if } |H| > 1 \\ 0_n & \text{otherwise} \end{cases}$$

Then $Ker(\mu)$ is a left-congruence on Σ^*.

The Sketch of the proof (III)

- Proof is based on showing that the equivalence classes $\{J_i\}_{i\in F}$ of $Ker(\mu)\cap(I^R\times I^R)$ is a reset right regular decomposition.
- The fact that the J_i's are right ideals and satisfy condition i) is a consequence of the previous Lemma.
- More complicated to prove the reset condition ii)... (fundamental $|\Sigma|\geq 2$)

Corollary

Suppose that I^R has state complexity n, then there is a strongly connected synchronizing automata $\mathscr{B}$ with $\mathrm{Syn}(\mathscr{B})=I$ with at most

$$m^{k2^n}\left(\sum_{t=2}^{n} m^{\binom{n}{t}}\right)^{2^n}$$

states, where $k=|\Sigma|$ and $m=\left(\frac{n^2+n}{2}+1\right)$.

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Reset decomposition complexity

Another formulation of Cerny's conjecture via a new "descriptional complexity" measure

Definition

Given an non-unary ideal I, $rdc(I)$ is the smallest reset left regular decomposition of I (Equivalently the smallest strongly connected synchronizing automaton $\mathscr{A}$ with $\mathrm{Syn}(\mathscr{A}) = I$).

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- The number of states of $\mathscr{A}$ is a double exponential with respect to $\mathrm{sc}(I^R)$, is it tight? The general bound can not be better than an exponential...see the next talk!
- In general upper and lower bounds on $rdc(I)$ are interesting, especially lower bounds (by the previous Corollary). Fundamental is the issue of understanding these decompositions from a purely language theoretic point of view.

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- In general upper and lower bounds on $rdc(I)$ are interesting, especially lower bounds (by the previous Corollary). Fundamental is the issue of understanding these decompositions from a purely language theoretic point of view.

Conclusions

- We have proved that for a non-unary ideal I there is at least a strongly connected synchronizing automaton $\mathscr{A}$ with $\mathrm{Syn}(\mathscr{A}) = I$.
- The number of states of $\mathscr{A}$ is a double exponential with respect to $\mathrm{sc}(I^R)$, is it tight? The general bound can not be better than an exponential...see the next talk!
- In general upper and lower bounds on $rdc(I)$ are interesting, especially lower bounds (by the previous Corollary). Fundamental is the issue of understanding these decompositions from a purely language theoretic point of view.

Definition

A reset left regular decomposition is a finite collection $\{I_i\}_{i\in F}$ of disjoint left ideals I_i of Σ^* such that:

i) For any $a \in \Sigma$ and $i \in F$, there is a $j \in F$ such that $I_i a \subseteq I_j$.

ii) Let $I = \biguplus_{i\in F} I_i$, for any $u \in \Sigma^*$ if there is an $i \in F$ such that $Iu \subseteq I_i$, then $u \in I$.


The End