Periodicity Forcing Words

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Question

For a word α and morphisms g, h, does the equality $g(\alpha) = h(\alpha)$ hold?

Question

Post Correspondence Problem:

Does there exist a word α such that, for two given morphisms, the equality $g(\alpha) = h(\alpha)$ holds?

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Word Equations:

For two words α, β, does there exist a morphism h such that $h(\alpha) = h(\beta)$?

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Word Equations:

For two words α, β, does there exist a morphism h such that $h(\alpha) = h(\beta)$?

If α is a (strict) renaming of β, this is equivalent to $g(\alpha) = h(\alpha)$ for morphisms g, h.

Question

Word Equations (A Special Case):

For a given word α, do there exist morphisms g, h, such that the equality $g(\alpha) = h(\alpha)$ holds?

Example: $x_1x_2x_2x_1 = x_3x_4x_4x_3$

Question

Ambiguity of Morphisms:

For a given word α and a given morphism g, does there exist a morphism h such that the equality $g(\alpha) = h(\alpha)$ holds?

Question

Morphism equivalence on languages:

For a set of words $\{\alpha_1, \alpha_2 \dots\}$ and morphisms g, h, do the equalities $g(\alpha_i) = h(\alpha_i)$ hold?

Periodic Morphisms

A morphism $g : A^* \to B^*$ is periodic if, for every $a \in A$, $g(a) \in \{w\}^*$ for some fixed word w.

Example: $g : \{x, y\}^* \to \{\mathtt{a}, \mathtt{b}\}^*$,

$g(x) = \mathtt{ab}$
$g(y) = \mathtt{abab}$

Periodicity Forcing Words

A word α is periodicity forcing if $g(\alpha) = h(\alpha)$ implies that at least one of g, h is periodic.

Periodicity Forcing Words

A word α is periodicity forcing if $g(\alpha) = h(\alpha)$ implies that at least one of g, h is periodic.

Question:

For a given word α, do there exist morphisms g, h, such that at least one is non-periodic, and the equality $g(\alpha) = h(\alpha)$ holds?

Periodicity Forcing Words

- Words which do not belong to a non-trivial equality set.
- Words for which the highest portion of morphisms are unambiguous.
- Also related to the topics of word equations, test sets etc.

Example

$$\alpha := 1 \cdot 2 \cdot 2 \cdot 1.$$

Example

$$g(1 \cdot 2 \cdot 2 \cdot 1) = h(1 \cdot 2 \cdot 2 \cdot 1)$$

Example

$$g(1)\, g(2)\, g(2)\, g(1) = h(1)\, h(2)\, h(2)\, h(1)$$

Example

$$\texttt{a}\; g(2)g(2)\; \texttt{a} = \texttt{aba}\; h(2)h(2)\; \texttt{aba}$$

Example

$$\mathtt{a}\ \mathtt{bab}\ \mathtt{bab}\ \mathtt{a} = \mathtt{aba}\ \mathtt{b}\ \mathtt{b}\ \mathtt{aba}$$

Example

$1 \cdot 2 \cdot 2 \cdot 1$ is not periodicity forcing.

$$g(1) = \mathtt{a},\ g(2) := \mathtt{bab},$$
$$h(1) = \mathtt{aba},\ h(2) := \mathtt{b}.$$

Example

$$\alpha' := 1 \cdot 2 \cdot 2 \cdot 1 \cdot 2.$$

α' is periodicity forcing.

The Results

- A generating 'prime' subset.
- Length bounds on the shortest periodicity forcing words.
- Periodicity forcing words can contain any given factor.

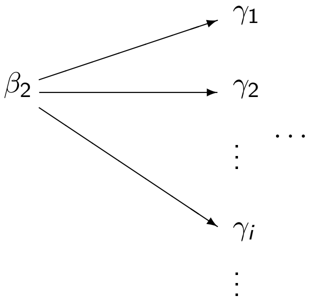
A Prime Subset

- From a previous paper, it is known that for any periodicity forcing word α, there exist non-trivial morphisms φ such that $\varphi(\alpha)$ is also periodicity forcing.
- Morphisms between periodicity forcing words are also characterised.
- Thus the set of all periodicity forcing words can be represented as sequences.

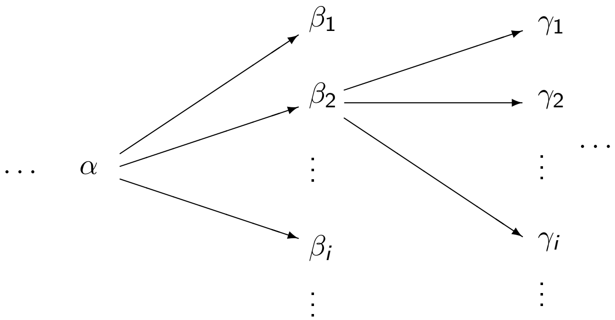
A Prime Subset

β_2

A Prime Subset



A Prime Subset



A Prime Subset

Theorem:

There exist periodicity forcing words which are not morphic images of any other periodicity forcing word.

Example: $1 \cdot 2 \cdot 1 \cdot 1 \cdot 2$.

A Prime Subset

Theorem:

There exist periodicity forcing words which are not morphic images of any other periodicity forcing word.

Consequence:

There exists a proper subset of periodicity forcing words from which the whole set can be generated using morphisms.

Shortest Periodicity Forcing Words

- It is immediately clear that the shortest periodicity forcing words must grow with respect to alphabet size.
- The rate of growth provides an indication of the restrictiveness of periodicity forcing words.
- Also suggests how many periodicity forcing words there might be (how general the set is).
- (Lower) bounds on length are useful when looking for prime words.

Shortest Periodicity Forcing Words

Question:

For any n, what is the length k of the shortest periodicity forcing word with exactly n different letters?

Shortest Periodicity Forcing Words

- Trivially, if $n = 1$, $k = 1$.
- $k = 5$ for $n = 2$ (Culik II, Karhumäki 1980).
- What about for $n > 2$?

Shortest Periodicity Forcing Words

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- What about for $n > 2$?

Theorem:

$$n^2 \leq k \leq 5n^2 + 2n.$$

Lower Bound (n^2)

- $|\alpha| < n^2$ implies that some y occurs less than n times.
- Write $\alpha := \beta_1 \cdot y \cdot \beta_2 \cdot y \cdots y \cdot \beta_m$, where y does not occur in any β_i and $m \leq n$.
- Consider the morphisms of the form:

$$\sigma(x) := \begin{cases} \mathtt{a}^p\, \mathtt{b}\mathtt{a}^q & \text{if } x = y \text{, and} \\ \mathtt{a}^{r_x} & \text{otherwise.} \end{cases}$$

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Example:

$$\alpha := \overbrace{1}^{\beta_1} \cdot \overbrace{2}^{y} \cdot \overbrace{3 \cdot 1 \cdot 1}^{\beta_2} \cdot \overbrace{2}^{y} \cdot \overbrace{3 \cdot 3}^{\beta_3}$$

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Example:

$$\alpha := \overbrace{\mathtt{a}^{r_1}}^{\beta_1} \cdot \overbrace{\mathtt{a}^p\mathtt{b}\mathtt{a}^q}^{y} \cdot \overbrace{\mathtt{a}^{r_3}\mathtt{a}^{r_1}\mathtt{a}^{r_1}}^{\beta_2} \cdot \overbrace{\mathtt{a}^p\mathtt{b}\mathtt{a}^q}^{y} \cdot \overbrace{\mathtt{a}^{r_3}\mathtt{a}^{r_3}}^{\beta_3}$$

Lower Bound (n^2)

- Two (different) σs agree on α if and only if the exponents of a coincide between each occurrence of y.
- This produces a system of linear Diophantine equations which can be solved to show that two different choices for the exponents may produce the same result.

$$r_1 + p = r_1' + p'$$
$$2r_1 + q + r_3 + p = 2r_1' + q' + r_3' + p'$$
$$q + 2r_3 = q' + 2r_3'$$

Lower Bound (n^2)

- Two (different) σs agree on α if and only if the exponents of a coincide between each occurrence of y.
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$$3 + 1 = 2 + 2$$

$$6 + 2 + 2 + 1 = 4 + 4 + 1 + 2$$

$$2 + 4 = 4 + 2$$

Lower Bound (n^2)

α is not periodicity forcing.

Upper Bound ($5n^2 + 2n$)

- Consider the set S_n of all permutations of the word $1 \cdot 2 \cdots n$.
- E.g.: $S_3 := \{1 \cdot 2 \cdot 3,\ 1 \cdot 3 \cdot 2,\ 2 \cdot 1 \cdot 3,\ 2 \cdot 3 \cdot 1,\ 3 \cdot 1 \cdot 2,\ 3 \cdot 2 \cdot 1\}$.
- Let $\beta_n := 1 \cdot 1 \cdot 2 \cdot 2 \cdots n \cdot n$.

Upper Bound $(5n^2 + 2n)$

- The set $S'_n := S_n \cup \{\beta_n\}$ is periodicity forcing.
- From literature, there exists a test set $T_n \subset S_n$ of S_n consisting of at most $5n$ words.
- $T_n \cup \{\beta_n\}$ is a test set for S'_n.

Upper Bound $(5n^2 + 2n)$

- Let α be the result of concatenating every word in T'_n.
- Note that all words in T'_n have the same ratio of letters.
- Thus, any morphisms agreeing on α agrees on T'_n, and therefore S'_n.
- So α is periodicity forcing.
- T'_n consists of $5n$ words of length n and one word of length $2n$, so $|\alpha| = 5n^2 + 2n$.

Arbitrary Factors

Theorem:

For any word β, there exists a periodicity forcing word α which contains β as a factor.

Arbitrary Factors

- Achieved by constructing a morphism from the target factor, a periodicity forcing word over the same alphabet, and some of its conjugates.
- Another measure of how general the set of periodicity forcing words is.
- The proof can be generalised for prefixes/suffixes effortlessly.

Conclusions

- The set of periodicity forcing words can be generated from a subset using morphisms.
- The shortest periodicity forcing words grow at a rate of $O(n^2)$.
- Periodicity forcing words may contain any factor/prefix/suffix.

Questions?

Thank You