The Lexicographic Cross-Section of the Plactic Monoid is Regular

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WORDS 2013 - Turku

A joint work with Christian Choffrut (LIAFA)

Question

Given a sequence, how long is its longest nondecreasing subword?

Schensted's Algorithm

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132541

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$\xleftarrow{1}$

$125, 124, 135, 134$

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$\boxed{1}\ \boxed{3} \qquad \xleftarrow{2}$

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$$\begin{array}{|c|}\hline 3 \\ \hline \end{array}$$
$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \quad \xleftarrow{5}$$

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$$\begin{array}{llll} \boxed{3} & & & \\ \boxed{1} & \boxed{2} & \boxed{5} & \quad \xleftarrow{4} \end{array}$$

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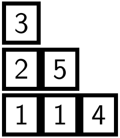

321 51 4

3 25 114

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- No results for either *plactic* or *plaxic*
  Latin *plaxus* under the etymology for the obsolete word *plash* (To bend down and interweave (stems partly cut through, branches, and twigs) so as to form a hedge or fence.): an unattested post-classical Latin form *plaxus*, alteration of classical Latin *plexus*, past participle of *plectere* (to plait, interweave, twine)

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- Cross-section of an equivalence relation is a set of elements such that each class has exactly one element in the cross-section.
- Denote Y(u) the Young tableau equivalent to $u \in \Sigma^*$.
  The plactic monoid admits a regular cross-section.
- AIM: Cross-section of the lexicographical representatives

Billiard Rules

Lemma (Billiard rule for rows)

Let $u \in \Sigma^*$ be a row and let $a < u[|u|]$. Then $ua \equiv bxay$, where $xby = u$ and b is the leftmost element strictly greater than a.

Example:

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Example:

$122\underline{3}45 \cdot 2$

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Billiard Rules (cont)

Lemma (Billiard rule for columns)

Let $u \in \Sigma^*$ be a column and let $a \leq u[1]$. Then $au \equiv xayb$, where $xby = u$ and b is the least element greater than or equal to a.

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$2 \cdot 54\underline{3}1$

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Billiard principle

Proposition

Let u and v be two columns defining a Young tableau, i.e., $u \succeq v$ Let w be a column such that $w[1] < u[|u|]$. Then $uvw \equiv uwv$.

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Corollary

Let u and v be two columns such that v is a subword of u. Then $uv \equiv vu$.

Maximal cross-section

PROPOSITION

The maximum representative of the congruence class a word $v \in \{1,2\}^*$ is part of, is of the form

$$v_{max} = 2^m 1^{|v|_1} 2^{|v|_2 - m},$$

where $m < |v|_1$ or $|v|_2 - m = 0$.

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Proof: [Arnold, Kanta, Krob, '97] $(21)^* 1^* 2^*$ is a cross-section.
For binary alphabets 21 commutes with both 1 and 2.
$2m$ is the maximal length Dyck subword in v.

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Theorem

The language determined by the maximum representatives of the congruence class determined by Young tableaux is not regular.

Minimal cross-section

Proposition

The minimum representative of the congruence class a word $v \in \{1,2\}^*$ *is part of, is of the form* $v_{min} = 1^*(21)^*2^*$.

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The minimum representative of the congruence class a word $v \in \{1,2,3\}^*$ *is part of, is of the form* $v_{min} = 1^*(21)^*(2^* + (31)^*)(321)^*(32)^*3^*$.

For $Y(u)$ we associate the *decomposition* $(n_1^{e_1}, \ldots, n_p^{e_p})$ where e_1 are its first columns of same height n_1, e_2 next columns of height n_2, …, etc., and the sequence $n_1, \ldots, n_p$ is strictly decreasing.

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Proposition

If $u = av$, for some $a \in \Sigma$, then $Y(v)$ has a decomposition of the form

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if $n_i > n_{i+1} + 1$, or, otherwise,

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Corollary

Let x be an element of the plactic monoid. There exist k factorizations of the form $x = ay$, where $a \in \Sigma$ and k is the number of different lengths for the columns of the associated Young tableau.

EXAMPLE

3		
2	5	
1	1	4

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3		
2	5	
1	1	4

1 32 541

3		
2	5	
1	1	4

1 32 541

$\xleftarrow{1}$

EXAMPLE

$\boxed{3}$
$\boxed{2}\boxed{5}$
$\boxed{1}\boxed{1}\boxed{4}$

1 32 541

$\boxed{1}$ $\xleftarrow{3}$

EXAMPLE

3		
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1 32 541

$\boxed{1}\ \boxed{3}$ $\xleftarrow{2}$

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$$\begin{array}{|c|c|c|}\cline{1-1} 3 \\ \cline{1-2} 2 & 5 \\ \hline 1 & 1 & 4 \\ \hline\end{array}$$

1 32 541

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1 32 541

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1 32 541

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1 32 541

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$$1\ \ 32\ \ 541$$

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$$\begin{array}{ccc} \boxed{3} & & \\ \boxed{2} & \boxed{5} & \\ \boxed{1} & \boxed{1} & \boxed{4} \end{array} \longrightarrow 1 \cdot \begin{array}{cc} \boxed{3} & \\ \boxed{2} & \boxed{5} \\ \boxed{1} & \boxed{4} \end{array}$$

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$$\begin{array}{|c|c|c|}\cline{1-1} 3 \\ \cline{1-2} 2 & 5 \\ \hline 1 & 1 & 4 \\ \hline\end{array} \;\rightarrow\; 1\cdot \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 2 & 5 \\ \hline 1 & 4 \\ \hline\end{array} \;\rightarrow\; 13\cdot \begin{array}{|c|c|}\cline{1-1} 5 \\ \cline{1-1} 2 \\ \hline 1 & 4 \\ \hline\end{array}$$

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$$\begin{array}{|c|c|c|}\cline{1-1} 3 \\ \cline{1-2} 2 & 5 \\ \hline 1 & 1 & 4 \\ \hline\end{array} \rightarrow 1\cdot \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 2 & 5 \\ \hline 1 & 4 \\ \hline\end{array} \rightarrow 13\cdot \begin{array}{|c|c|}\cline{1-1} 5 \\ \cline{1-1} 2 \\ \hline 1 & 4 \\ \hline\end{array} \rightarrow 132\cdot \begin{array}{|c|}\hline 5 \\ \hline 4 \\ \hline 1 \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\cline{1-1} 4 \\ \cline{1-2} 2 & 5 \\ \hline 1 & 3 & 4 \\ \hline\end{array}$$

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$$\begin{array}{|c|c|c|}\cline{1-1} 3 \\ \cline{1-2} 2 & 5 \\ \cline{1-3} 1 & 1 & 4 \\ \hline \end{array} \to 1\cdot \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 2 & 5 \\ \hline 1 & 4 \\ \hline \end{array} \to 13\cdot \begin{array}{|c|c|}\cline{1-1} 5 \\ \cline{1-1} 2 \\ \hline 1 & 4 \\ \hline \end{array} \to 132\cdot \begin{array}{|c|}\hline 5 \\ \hline 4 \\ \hline 1 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|}\cline{1-1} 4 \\ \cline{1-2} 2 & 5 \\ \cline{1-3} 1 & 3 & 4 \\ \hline \end{array} \to 1\cdot \cancel{\begin{array}{|c|c|}\cline{1-1} 4 \\ \hline 2 & 5 \\ \hline 3 & 4 \\ \hline \end{array}}$$

EXAMPLE

$$\begin{array}{|c|cc}\hline 3 \\ \hline 2 & \multicolumn{1}{|c|}{5} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{4} \\ \hline\end{array} \qquad 1\ 32\ 541 \qquad \begin{array}{ccc}\boxed{3} & & \\ \boxed{2} & \boxed{5} & \\ \boxed{1} & \boxed{1} & \boxed{4}\end{array}$$

$$\begin{array}{ccc}\boxed{3} & & \\ \boxed{2} & \boxed{5} & \\ \boxed{1} & \boxed{1} & \boxed{4}\end{array} \rightarrow 1\cdot \begin{array}{cc}\boxed{3} & \\ \boxed{2} & \boxed{5} \\ \boxed{1} & \boxed{4}\end{array} \rightarrow 13\cdot \begin{array}{cc}\boxed{5} & \\ \boxed{2} & \\ \boxed{1} & \boxed{4}\end{array} \rightarrow 132\cdot \begin{array}{c}\boxed{5} \\ \boxed{4} \\ \boxed{1}\end{array}$$

$$\begin{array}{ccc}\boxed{4} & & \\ \boxed{2} & \boxed{5} & \\ \boxed{1} & \boxed{3} & \boxed{4}\end{array} \rightarrow 2\cdot \cancel{\begin{array}{cc}\boxed{4} & \\ \boxed{5} & \boxed{4} \\ \boxed{1} & \boxed{3}\end{array}}$$

EXAMPLE

$$\begin{array}{lll} \boxed{3} & & \\ \boxed{2} & \boxed{5} & \\ \boxed{1} & \boxed{1} & \boxed{4} \end{array} \qquad 1\ 32\ 541 \qquad \begin{array}{lll} \boxed{3} & & \\ \boxed{2} & \boxed{5} & \\ \boxed{1} & \boxed{1} & \boxed{4} \end{array}$$

$$\begin{array}{lll} \boxed{3} & & \\ \boxed{2} & \boxed{5} & \\ \boxed{1} & \boxed{1} & \boxed{4} \end{array} \to 1 \cdot \begin{array}{ll} \boxed{3} & \\ \boxed{2} & \boxed{5} \\ \boxed{1} & \boxed{4} \end{array} \to 13 \cdot \begin{array}{ll} \boxed{5} & \\ \boxed{2} & \\ \boxed{1} & \boxed{4} \end{array} \to 132 \cdot \begin{array}{l} \boxed{5} \\ \boxed{4} \\ \boxed{1} \end{array}$$

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$$\begin{array}{ccc} 3 & & \\ 2 & 5 & \\ 1 & 1 & 4 \end{array} \qquad 1\ 32\ 541 \qquad \begin{array}{ccc} 3 & & \\ 2 & 5 & \\ 1 & 1 & 4 \end{array}$$

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Relations on columns

We write $u \trianglelefteq v$ if the following conditions hold:

- for all $i = 1, \ldots, \min\{|u|, |v|\}$, the condition $u_i \leq v_i$ holds;
- furthermore, if $|u| < |v|$, then $u_{|u|} \leq v_{|u|+1}$ holds.

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$$432 \trianglelefteq 43 \text{ and } 43 \trianglelefteq 6541$$

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Theorem

The transitive closure of $\trianglelefteq$ is antisymmetric.

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Theorem

The transitive closure of $\trianglelefteq$ is antisymmetric.

$$z_0 \trianglelefteq \cdots \trianglelefteq z_p,$$

with $z_0 = z_p$, then for all $0 < i < p$, we have $z_i = z_0$.

Characterization of minimal cross-section

PROPOSITION

Let u, v be two columns such that, neither uv nor vu is a column. Then uv is lexicographically minimum in its class if and only if $u \trianglelefteq v$ holds.

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Theorem

Let $u = u_1 \cdots u_n$ be a factorization of maximal strictly decreasing strings. Then, u is the minimal lexicographical element of the cross-section if and only if $u_i \trianglelefteq u_j$, for all $1 \leq i < j \leq n$.

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$$\begin{array}{cccc} & & & 9 \\ 4 & & & 8 \\ 3 & 6 & 7 & 6 \\ 2 & 5 & 6 & 1 \end{array}$$

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$$
\begin{array}{cccc}
 & & & 9 \\
4 & & & 8 \\
3 & 6 & 7 & 6 \\
\color{red}{2} & 5 & 6 & 1
\end{array}
=
\begin{array}{cccc}
 & & & 9 \\
 & & 6 & 8 \\
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\end{array}
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$$\begin{array}{cccc} & & & 9 \\ 4 & & & 8 \\ 3 & 6 & 7 & 6 \\ 2 & 5 & 6 & 1 \end{array} = \begin{array}{cccc} & & & 9 \\ & 6 & & 8 \\ 4 & 5 & 7 & 6 \\ 3 & \color{red}{2} & 6 & 1 \end{array} = \begin{array}{cccc} & & & 9 \\ & & 7 & 8 \\ 4 & 6 & 6 & 6 \\ 3 & 5 & \color{red}{2} & 1 \end{array}$$

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\end{array}
=
\begin{array}{cccc}
 & & & 9 \\
 & & 7 & 8 \\
4 & 6 & 6 & 6 \\
3 & 5 & 2 & \color{red}{1}
\end{array}
=
\begin{array}{cccc}
 & & 7 & \\
 & & 6 & 9 \\
4 & 6 & 2 & 8 \\
3 & 5 & \color{red}{1} & 6
\end{array}
$$

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$$\begin{array}{cccc} & & & 9 \\ 4 & & & 8 \\ 3 & 6 & 7 & 6 \\ 2 & 5 & 6 & 1 \end{array} = \begin{array}{cccc} & & & 9 \\ & 6 & & 8 \\ 4 & 5 & 7 & 6 \\ 3 & 2 & 6 & 1 \end{array} = \begin{array}{cccc} & & & 9 \\ & & 7 & 8 \\ 4 & 6 & 6 & 6 \\ 3 & 5 & 2 & 1 \end{array} = \begin{array}{cccc} & & 7 & \\ & & 6 & 9 \\ 4 & 6 & \color{red}{2} & 8 \\ 3 & 5 & \color{red}{1} & 6 \end{array} = \begin{array}{cccc} & 6 & & \\ & 5 & & 9 \\ 4 & \color{red}{2} & 7 & 8 \\ 3 & \color{red}{1} & 6 & 6 \end{array}$$

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Proof

4			
3			9
2	6	7	8
1	5	6	6

Proof

$$
\begin{array}{cccc}
4 & & & \\
3 & & & 9 \\
2 & 6 & 7 & 8 \\
1 & 5 & 6 & 6
\end{array}
\quad \rightarrow \quad
\begin{array}{cccc}
4 & 6 & 7 & 9 \\
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2 & & & 6 \\
1 & & &
\end{array}
$$

Proof

$$
\begin{array}{cccc}
4 & & & \\
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\end{array}
\;\rightarrow\;
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$$\begin{array}{cccc} 4 & & & \\ 3 & & & 9 \\ 2 & 6 & 7 & 8 \\ 1 & 5 & 6 & 6 \end{array} \rightarrow \begin{array}{cccc} 4 & 6 & 7 & 9 \\ 3 & 5 & 6 & 8 \\ 2 & & & 6 \\ 1 & & & \end{array} \rightarrow \begin{array}{cccc} 4 & 6 & 7 & 9 \\ 3 & 5 & 6 & 8 \\ & & 2 & 6 \\ & & & 1 \end{array} = \begin{array}{cccc} \boxed{4} & \boxed{6} & \boxed{7} & \boxed{9} \\ \boxed{3} & \boxed{5} & \boxed{6} & \boxed{8} \\ & & \boxed{2} & \boxed{6} \\ & & & \boxed{1} \end{array}$$

contretableau

Proof

$$\begin{array}{|c|c|c|c|}\cline{1-1} 4 \\ \cline{1-2} 3 & 9 \\ \hline 2 & 6 & 7 & 8 \\ \hline 1 & 5 & 6 & 6 \\ \hline \end{array} = \begin{array}{cccc} 4 & & & \\ 3 & & & 9 \\ 2 & 6 & 7 & 8 \\ 1 & 5 & 6 & 6 \end{array} \rightarrow \begin{array}{cccc} 4 & 6 & 7 & 9 \\ 3 & 5 & 6 & 8 \\ 2 & & & 6 \\ 1 & & & \end{array} \rightarrow \begin{array}{cccc} 4 & 6 & 7 & 9 \\ 3 & 5 & 6 & 8 \\ & & 2 & 6 \\ & & & 1 \end{array} = \begin{array}{|c|c|c|c|}\hline 4 & 6 & 7 & 9 \\ \hline 3 & 5 & 6 & 8 \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 & 6 \\ \cline{3-4} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 \\ \cline{4-4} \end{array}$$

contretableau

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Corollary

The minimal representative of a class has the same column length distribution as its Young tableau.

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Given a word w of length n, there exists an $\mathcal{O}(n^{\frac{3}{2}})$ algorithm which finds the lexicographically minimal representative equivalent to w.

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Proposition

Given a word w of length n, there exists an $\mathcal{O}(n^{\frac{3}{2}})$ algorithm which finds the lexicographically minimal representative equivalent to w.

Theorem

The set of alphabetically minimal words of the plactic congruence over an arbitrary finite alphabet is regular.

1 Cross-sections

2 Further remarks

Transposition

Elements $x, y \in M$ are *transposed* if $x = uv$ and $y = vu$ for some $u, v \in M$.

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Elements $x, y \in M$ are *transposed* if $x = uv$ and $y = vu$ for some $u, v \in M$.

Theorem

Let x be an element of the plactic monoid over $\Sigma = \{a_1, \ldots, a_p\}$. Define $\ell_i = |x|_{a_i}$ for $i = 1, \ldots, p$. Then $(x, a_1^{\ell_1} \cdots a_p^{\ell_p}) \in T^p$.

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Take the word uvw with v increasing and uw not empty and assume that

$$\max\{a \mid v \in \Sigma^* a \Sigma^*\} < b = \min\{a \mid uw \in \Sigma^* a \Sigma^*\}.$$

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Take the word uvw with v increasing and uw not empty and assume that

$$\max\{a \mid v \in \Sigma^* a \Sigma^*\} < b = \min\{a \mid uw \in \Sigma^* a \Sigma^*\}.$$

Then by the Billiard Rules we have

$$vwu \equiv u'v'w'$$

where $v' = vb^r$ with $r = |uw|_b$.

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Conjugacy

Elements $x, y \in M$ are *conjugates* if exists $z \in M$ such that $xz = zy$.

Proposition

In the plactic monoid we always have $C = T^{2(||A||-1)}$.

THANK YOU!