Defect property in $\mathbb{Z}^2$

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Introduction

- Two-dimensional structures of various kinds can be viewed as generalizations of words.
- Code: subset X of a monoid such that every product of the elements decomposes uniquely over X.
- Two important properties related to word codes: defect and decipherability verification.
- Both lost for polyominoes and figures.
- We consider several kinds of figures:
  - labelled polyominoes
  - with/without designated start and end points
  - with catenation operation that uses/does not use a merging function to resolve possible conflicts.
- Decipherability verification is mostly decidable in this setting.
- Defect property fails miserably...

Definition (Undirected figure)

- Let $D \subseteq \mathbb{Z}^2$ be finite and connected and $l : D \to \Sigma$. A pair $f = (D, l)$ is called an *(undirected) figure* (over Σ) with
$$\begin{array}{lrcl} \textit{domain} & \mathrm{dom}(f) & = & D \\ \textit{labelling function} & \mathrm{label}(f) & = & l. \end{array}$$
- The set of all figures over Σ is denoted by $\Sigma^{\bowtie}$.
- Given $X \subseteq \Sigma^{\bowtie}$, the set of all figures tilable with (translated copies of) the elements of X is denoted by $X^{\bowtie}$.
- The terms *rectangles, squares* and *dominoes* refer to figures with respective domains. In particular, the domain of a domino is a $1 \times n$ or $n \times 1$ rectangle with $n \geq 1$.

Definition (Directed figure)

- Let $D \subseteq \mathbb{Z}^2$ be finite and connected, $b \in D$, $e \in D \cup \text{neighbours}(D)$ and $l : D \to \Sigma$. A quadruple $f = (D, b, e, l)$ is called a *directed figure* (over Σ) with

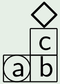
$$\begin{array}{lrcl} \textit{domain} & \text{dom}(f) & = & D \\ \textit{start point} & \text{begin}(f) & = & b \\ \textit{end point} & \text{end}(f) & = & e \\ \textit{labelling function} & \text{label}(f) & = & l. \end{array}$$

- *Translation vector* of f is defined as $\text{tran}(f) = \text{end}(f) - \text{begin}(f)$.
- Additionally, the *empty directed figure* ε is $(\emptyset, (0,0), (0,0), \emptyset)$.
- The set of all directed figures over Σ is denoted by $\Sigma^\diamond$.

Example

A directed figure and its graphical representation (a circle marks the start point and a diamond marks the end point of the figure).

$$
\begin{aligned}
D &= \{(0,0),(1,0),(1,1)\} \\
b &= (0,0) \\
e &= (1,2) \\
l &= \{(0,0) \mapsto a, (1,0) \mapsto b, (1,1) \mapsto c\}.
\end{aligned}
$$



Definition (Catenation of directed figures)

Let $x = (D_x, b_x, e_x, l_x)$ and $y = (D_y, b_y, e_y, l_y)$ be directed figures. If $D_x \cap \tau_{e_x - b_y}(D_y) = \emptyset$, a *catenation* of x and y is defined as

$$x \circ y = (D_x \cup \tau_{e_x - b_y}(D_y), b_x, \tau_{e_x - b_y}(e_y), l),$$

where

$$l(z) = \begin{cases} l_x(z) & \text{for } z \in D_x \\ \tau_{e_x - b_y}(l_y)(z) & \text{for } z \in \tau_{e_x - b_y}(D_y) \end{cases}$$

and τ_u denotes translation by a vector $u \in \mathbb{Z}^2$.
If $D_x \cap \tau_{e_x - b_y}(D_y) \neq \emptyset$, catenation of x and y is not defined.

Definition (*m*-catenation of directed figures)

Let $x = (D_x, b_x, e_x, l_x)$ and $y = (D_y, b_y, e_y, l_y)$ be directed figures. *Catenation* of x and y with respect to a *merging function* $m : \Sigma \times \Sigma \to \Sigma$ is defined as

$$x \circ_m y = (D_x \cup \tau_{e_x - b_y}(D_y), b_x, \tau_{e_x - b_y}(e_y), l),$$

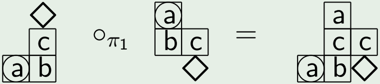
where

$$l(z) = \begin{cases} l_x(z) & \text{for } z \in D_x \setminus \tau_{e_x - b_y}(D_y) \\ \tau_{e_x - b_y}(l_y)(z) & \text{for } z \in \tau_{e_x - b_y}(D_y) \setminus D_x \\ m(l_x(z), \tau_{e_x - b_y}(l_y)(z)) & \text{for } z \in D_x \cap \tau_{e_x - b_y}(D_y). \end{cases}$$

Notice that when $x \circ y$ is defined, it is equal to $x \circ_m y$, regardless of the merging function m.

Example

Let π_1 be the projection onto the first argument.



Observation

- *$(\Sigma^\diamond, \circ_m)$ is a monoid if and only if m is associative.*
- *$(\Sigma^\diamond, \circ_m)$ is never free, since its basis must contain "unit figures", contradicting the freeness.*

Note

- *From now on let m be an arbitrary associative merging function.*
- *We write $X^\diamond$ and $X_m^\diamond$ to denote the set of all figures that can be composed by $\circ$ and $\circ_m$, respectively, from figures in $X \subseteq \Sigma^\diamond$.*

Codes and the defect property

Definition (Undirected figure codes)

$X \subseteq \Sigma^{\bowtie}$ is a *code*, if every element of $X^{\bowtie}$ admits exactly one tiling with the figures of X.

Definition (Directed figure codes and *m*-codes)

$X \subseteq \Sigma^{\diamond}$ is a *code*, if for any $x_1, \ldots, x_k$, $y_1, \ldots, y_l \in X$, $x_1 \circ \cdots \circ x_k = y_1 \circ \cdots \circ y_l$ implies $(x_1, \ldots, x_k) = (y_1, \ldots, y_l)$.
$X \subseteq \Sigma^{\diamond}$ is an *m-code*, if for any $x_1, \ldots, x_k$, $y_1, \ldots, y_l \in X$, $x_1 \circ_m \cdots \circ_m x_k = y_1 \circ_m \cdots \circ_m y_l$ implies $(x_1, \ldots, x_k) = (y_1, \ldots, y_l)$.

Theorem (Classical defect property for words)

Let $X \subseteq \Sigma^+$ be a non-code. There exists $Y \subseteq \Sigma^+$ such that $|Y| < |X|$ and $X^+ \subseteq Y^+$.

Defect property for undirected figures

- Results for undirected figures include:
  - counterexamples for several combinations of figure shapes and set sizes, i.e., non-codes composed of figures of a specified shape that cannot be tiled with fewer figures of the same shape,
  - two positive results: the defect theorem for two rectangles, squares and dominoes and for three dominoes.
- For sets of arbitrary cardinality, the property holds neither for unrestricted shapes, nor for rectangles, squares or dominoes.

Theorem (M 2007)

Let $X = \{k, l\} \subseteq \Sigma^{\bowtie}$ be a non-code containing two rectangles. Then there exists a common rectangular tiler for k, l, i.e., a rectangle $t \in \Sigma^{\bowtie}$ such that $k, l \in \{t\}^{\bowtie}$.

Corollary

Let $X = \{k, l\} \subseteq \Sigma^{\bowtie}$ be a non-code containing two squares. Then there exists a common square tiler for k, l, i.e., a square $t \in \Sigma^{\bowtie}$ such that $k, l \in \{t\}^{\bowtie}$.

Corollary

Let $X = \{k, l\} \subseteq \Sigma^{\bowtie}$ be a non-code containing two dominoes. Then there exists a common domino tiler for k, l, i.e., a domino $t \in \Sigma^{\bowtie}$ such that $k, l \in \{t\}^{\bowtie}$.

Theorem (M 2007)

Let $X \subseteq \Sigma^{\bowtie}$ be a set of three dominoes. If X is not a code then there exists a code $Y \subseteq \Sigma^{\bowtie}$ such that $|Y| < 3$ and Y tiles the dominoes of X.

The proof is directly combinatorial, resorting to the defect theorem for words in its basic cases. It also highlights a property of 3-domino non-codes: they are either trivial or they are essentially word non-codes.

Example (Harju and Karhumäki 2004)

A non-code containing three squares that cannot be tiled with two squares. Note, however, that it can be tiled with two rectangles.

$$\boxed{a} \qquad \begin{array}{|c|c|}\hline a & a \\ \hline b & c \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline b & c \\ \hline a & a \\ \hline\end{array}$$

Example (Harju and Karhumäki 2004)

A non-code containing four dominoes that cannot be tiled with three dominoes.

$$\begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array} \qquad \begin{array}{|c|}\hline c \\ \hline d \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline a & c \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline b & d \\ \hline\end{array}$$

Example (Huova 2009)

A non-code containing three rectangles that cannot be tiled with two rectangles.

a
b
c

c	b	a

a	c	b	a
b	a	c	b
c	b	a	c

Example (Huova 2009)

A non-code containing two figures that cannot be tiled with one figure.

a	a	a	
	b	b	b

a	a	
	b	b

Note

Status of the defect property for various combinations of figure shapes and set sizes:

Figures/set size	*2*	*3*	≥ 4
Squares	$+$	$-$	$-$
Dominoes	$+$	$+$	$-$
Rectangles	$+$	$-$	$-$
Unrestricted	$-$	$-$	$-$

Defect property for directed figures

- Here we show several counterexamples to disprove the defect theorem for directed figures.
- They show that the property fails even for very simple sets.
- On the other hand, restricting figures e.g. to word-like shapes, with appropriately chosen start and end points, obviously guarantees the defect property.
- Note that when $|\Sigma| = 1$ we do not mark the labels.

Example (Kolarz and M 2009)

Let $\Sigma = \{a\}$, $m = \{(a, a) \mapsto a\}$ and

$X = \{x =$ , $y =$ $\}$.

For each $n \geq 1$

$(xy)^n =$ 

hence X is not an m-code. However, there exists no Y such that $|Y| < 2$ and $X_m^\diamond \subseteq Y_m^\diamond$.

Thus, the defect effect does not even hold for two squares.

Even a singleton set can be a non-code:

Example (Kolarz and M 2009)

Let $\Sigma = \{a\}$, $m = \{(a, a) \mapsto a\}$ and

$X = \{$  $\}$

For each $n \geq 1$

$$x^n = x$$

hence X is not an m-code. Obviously, there exists no Y such that $|Y| < 1$ and $X_m^\diamond \subseteq Y_m^\diamond$.

The defect theorem does not even hold for non-codes that do not allow a word x with $\mathrm{tran}(x) = (0,0)$ to be composed:

Example (Kolarz and M 2009)

Let $\Sigma = \{a, b, c\}$, $m = \{(a, \cdot) \mapsto a, (\cdot, a) \mapsto a, \ldots\}$ (remaining values can be set arbitrarily) and

$$X = \{x = \boxed{a \mid b \mid a},\ y = \boxed{a \mid c \mid a}\}.$$

Then

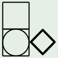
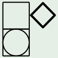
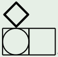
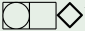
$$xy = yx = \boxed{a \mid a \mid a \mid a}$$

hence X is not an m-code, but there exists no Y such that $|Y| < 2$ and $X_m^\diamond \subseteq Y_m^\diamond$.

The following example shows a non-code X which is "fully non-overlapping:" for any two figures in $X^\diamond$ their $\circ$-catenation exists, since there are no overlapping parts.

Example

Let $\Sigma = \{a\}$, $m = \{(a,a) \mapsto a\}$ and

$X = \{w =$ , $x =$ , $y =$ , $z =$ $\}$.

X is not an m-code, as $wx = yz$. Moreover, because of the non-overlapping property, it is not a code, either. However, there exists no Y such that $|Y| < 4$ and $X^\diamond \subseteq Y^\diamond$.

Question:

- What restriction on sets of figures guarantees the defect property?
- It holds for sets of figures that are homomorphic images of sets of words. This is a very strong restriction...
- Is there any simple geometric characterization?

Thank you!