Infinite words with well distributed occurrences

Ľ. Balková, M. Bucci, J. Hladký, A. De Luca, S. Puzynina

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Program

1. PRNGs based on infinite words
2. Words with well distributed occurrences
3. Statistical tests

Linear congruential generator (LCG)

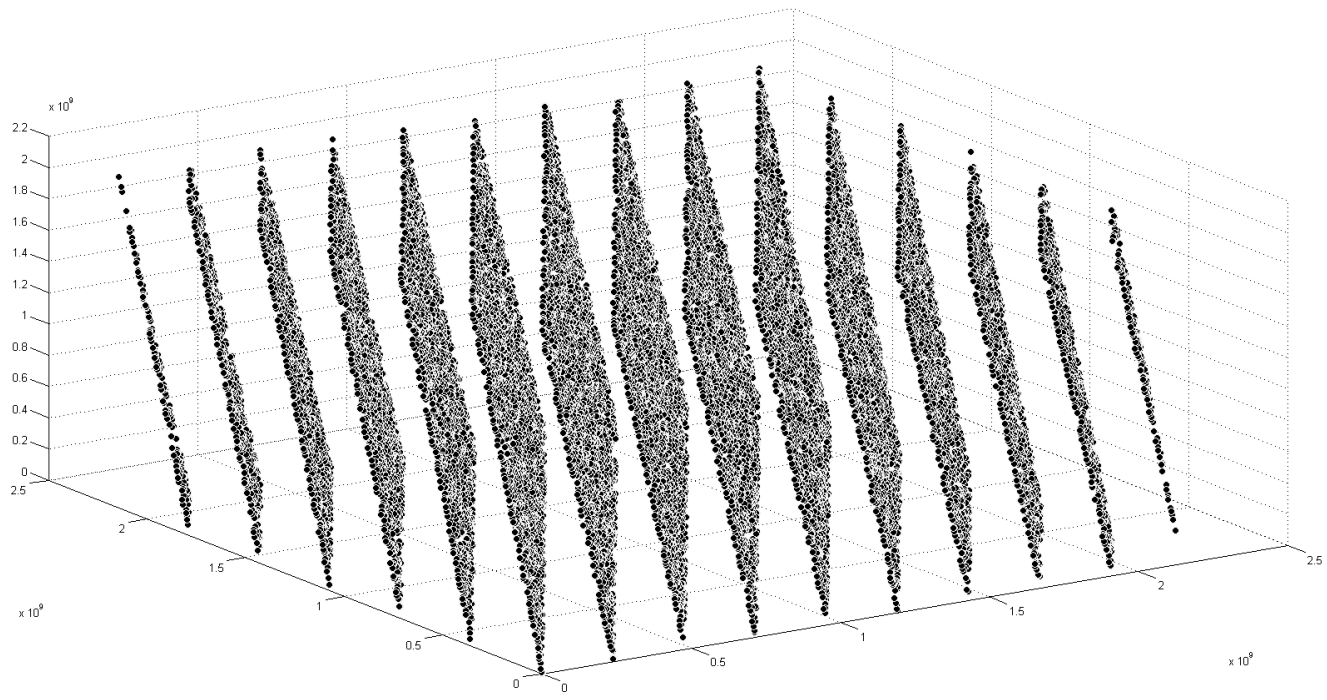
- defined by a recurrent relation

$$x_{n+1} = (ax_n + c) \mod m$$

$0 < m$	modulo
$0 \leq a < m$	multiplier
$0 \leq c < m$	shift
$0 \leq x_0 < m$	seed

- weaknesses of LCGs:
  - periodicity
  - lattice structure

Lattice structure of LCGs



RANDU: $a = 65539$, $m = 2^{31}$, $c = 0$

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Binary words

- we are given two LCGs with modulo $m \geq 2$ and output $\{0, 1, \ldots, m-1\}$

$$X = (x_n)_{n \geq 0} \text{ a } Y = (y_n)_{n \geq 0}$$

and an infinite word $\mathbf{u} = u_1 u_2 u_3 \ldots$ over $\{0, 1\}$

- then the PRNG $Z = (z_n)_{n \geq 0}$ based on $\mathbf{u}$ is obtained by the following algorithm:
  1. we read letters of $\mathbf{u}$
  2. if we read in $\mathbf{u}$ a zero for the i-th time, then we add to the sequence Z the i-th element of X
  3. if we read in $\mathbf{u}$ a one for the i-th time, then we add to the sequence Z the i-th element of Y

Properties of PRNGs based on infinite words

- PRNG keeps aperiodicity
- PRNG based on a subclass of "cut and project" sequences do not have the lattice structure

Guimond L.-S., Patera J., Patera J., *Statistical properties and implementation of aperiodic pseudorandom number generators*, Applied Numerical Mathematics **46(3-4)** (2003), 295–318

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Absence of the lattice structure

- PRNGs based on words with well distributed occurrences do not have the lattice structure
- an aperiodic infinite word $\mathbf{u}$ over $\{0,1\}$ has *well distributed occurrences (the WDO property)* if for every $m \in \mathbb{N}$ and for every factor w the following condition is satisfied: denote $i_1, i_2, \ldots$ occurrences of w in $\mathbf{u}$, then

$$\left\{ (|u_1 u_2 \ldots u_{i_j}|_0, |u_1 u_2 \ldots u_{i_j}|_1) \mod m \;\middle|\; j \in \mathbb{N} \right\} = \mathbb{Z}_m^2$$

- example: the Thue-Morse word does not have the WDO property

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Sturmian words

- Sturmian words have the WDO property
- they form a larger class than the subclass of “cut and project” sequences considered in the paper

Multiliteral alphabets

- if we combine more LCGs, then the obtained PRNG does not have the lattice structure if the corresponding infinite word $\mathbf{u}$ has the WDO property:
- an aperiodic word $\mathbf{u}$ over $\{0, 1, \ldots, d-1\}$ has the *WDO property* if for every $m \in \mathbb{N}$ and every factor w the following condition is satisfied: denote $i_1, i_2, \ldots$ occurrences of w in $\mathbf{u}$, then

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Arnoux-Rauzy words

- an infinite word $\mathbf{u}$ is called *Arnoux-Rauzy* over $\{0, 1, \ldots, d-1\}$ if its language is closed under reversal and $\mathbf{u}$ has one LS factor of every length and this factor has exactly d left extensions
- AR words have the WDO property
- example: if we replace $d-1$ with 0 in an AR word, the WDO property is kept

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- excellent results in tests (in comparison to LCGs)
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Questions

- Why PRNGs based on infinite words with the WDO property succeed in statistical tests?
- Is there any relation between other combinatorial properties of infinite words (complexity, palindromes, factor frequencies, return words etc.) and statistical properties of arising PRNGs?

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Generator	Time(10^{10})	BigCrush	PractRand
LCG($2^{47}-115, 71971110957370, 0$)	281	14	1TB
LCG($2^{63}-25, 2307085864, 0$)	277	2	>16TB
LCG($2^{59}, 13^{13}, 0$)	14.1	19	128MB
LCG($2^{63}, 5^{19}, 1$)	14.4	19	8GB
LCG($2^{63}, 9219741426499971445, 1$)	14.4	19	8GB
LCG($2^{64}, 2862933555777941757, 1$)	14.0	18	32GB
LCG($2^{64}, 3202034522624059733, 1$)	14.1	14	16GB
LCG($2^{64}, 3935559000370003845, 1$)	14.0	13	8GB

Generator	Time(10^{10})	BigCrush	PractRand	Combination
Fibonacci	25.7	(8×) 0 (0) (1×) 0 (1)	(9×) 2TB	Total 9 combinations {1,0}
Fibonacci2	17.3	(22×) 0 (0) (4×) 0 (1) (1×) 0 (2)	(25×) 1TB (2×) 0.5TB	Total 27 combinations {0,1,1} {0,2,1} {0,1,2} {0,2,2} {0,0,0} {0,0,2}
Tribonacci	25.7	(22×) 0 (0) (4×) 0 (1) (1×) 0 (2)	(16×) 2TB (7×) 4TB (2×) 8TB (2×) 1TB	Total 27 combinations {1,1,0} {2,1,0} {1,2,0} {2,2,0} {0,0,0} {0,0,0} {0,1,1} {0,2,1} {0,1,2} {0,2,2} {0,1,0} {0,2,0} {0,0,2} {0,0,1} {1,0,0} {2,0,0}

Thank you for attention