Enumerating Abelian Returns to Prefixes of Sturmian Words

Zuzana Masáková, Edita Pelantová

Czech Technical University in Prague

WORDS 2013

Return words

$\mathbf{u} = u_0 u_1 u_2 \cdots$ infinite word over an alphabet $\mathcal{A}$ $(= \{0,1\})$

$w = u_i u_{i+1} \cdots u_{i+n-1}$ factor of $\mathbf{u}$ of length $|w| = n$

$\mathcal{L}(\mathbf{u})$ language of $\mathbf{u}$

i occurrence of w in $\mathbf{u}$

$|w|_a$ number of occurrences of the letter a in w

Return words

$\mathbf{u} = u_0 u_1 u_2 \cdots$ infinite word over an alphabet $\mathcal{A}$ $(= \{0, 1\})$

$w = u_i u_{i+1} \cdots u_{i+n-1}$ factor of $\mathbf{u}$ of length $|w| = n$

$\mathcal{L}(\mathbf{u})$ language of $\mathbf{u}$

i occurrence of w in $\mathbf{u}$

$|w|_a$ number of occurrences of the letter a in w

$\mathbf{u}$ is recurrent if every factor w has infinitely many occurrences.

If $n_1 < \cdots < n_i < n_{i+1} < \cdots$ are occurrences of w in $\mathbf{u}$,

then $v = u_{n_i} u_{n_i+1} \cdots u_{n_{i+1}-1}$ is a return word of w in u.

Abelian return words

w is abelian equivalent to w'

$w \sim_{\mathrm{ab}} w'$, if $|w|_a = |w'|_a$ for every $a \in \mathcal{A}$.

Abelian return words

w is abelian equivalent to w'

$w \sim_{\text{ab}} w'$, if $|w|_a = |w'|_a$ for every $a \in \mathcal{A}$.

If $n_1 < \cdots < n_i < n_{i+1} < \cdots$ are occurrences of factors $\sim_{\text{ab}} w$, i.e.

$$u_{n_i} u_{n_i+1} \cdots u_{n_i+n-1} \sim_{\text{ab}} w,$$

$$u_j u_{j+1} \cdots u_{j+n-1} \not\sim_{\text{ab}} w \text{ for } j \neq n_i,$$

then $v = u_{n_i} u_{n_i+1} \cdots u_{n_{i+1}-1}$ is an abelian return word to w in $\mathbf{u}$.

Sturmian words – many equivalent definitions

- Aperiodic words with exactly $n+1$ factors of each length n.

- Hedlund & Morse:

  Balanced aperiodic infinite words:

$$\big||w|_a - |w'|_a\big| \leq 1$$

  for every $w, w' \in \mathcal{L}(\mathbf{u})$, $|w| = |w'|$, and every $a \in \mathcal{A}$.

Sturmian words – many equivalent definitions

- Aperiodic words with exactly $n+1$ factors of each length n.

- Hedlund & Morse:

  Balanced aperiodic infinite words:

$$\big||w|_a - |w'|_a\big| \leq 1$$

  for every $w, w' \in \mathcal{L}(\mathbf{u})$, $|w| = |w'|$, and every $a \in \mathcal{A}$.

- Vuillon:

  Infinite words with exactly 2 return words for every $w \in \mathcal{L}(\mathbf{u})$.

- ...

Sturmian words and interval exchange

Irrational $\alpha \in (0,1)$, $T : [0,1) \to [0,1)$

$$T(x) = \begin{cases} x + 1 - \alpha & \text{if } x \in [0,\alpha) =: J_0\,, \\ x - \alpha & \text{if } x \in [\alpha,1) =: J_1\,. \end{cases}$$

Sturmian words and interval exchange

Irrational $\alpha \in (0,1)$, $T : [0,1) \to [0,1)$

$$T(x) = \begin{cases} x + 1 - \alpha & \text{if } x \in [0,\alpha) =: J_0\,, \\ x - \alpha & \text{if } x \in [\alpha,1) =: J_1\,. \end{cases}$$

For any $\rho \in [0,1)$ define $\mathbf{u}_{\alpha,\rho} = u_0 u_1 u_2 \cdots$ by

$$u_n = \begin{cases} 0 & \text{if } T^n(\rho) \in J_0\,, \\ 1 & \text{if } T^n(\rho) \in J_1\,. \end{cases}$$

Sturmian word with slope α and intercept ρ.

Sturmian words and interval exchange

Irrational $\alpha \in (0,1)$, $T : [0,1) \to [0,1)$

$$T(x) = \begin{cases} x + 1 - \alpha & \text{if } x \in [0,\alpha) =: J_0\,, \\ x - \alpha & \text{if } x \in [\alpha,1) =: J_1\,. \end{cases}$$

For any $\rho \in [0,1)$ define $\mathbf{u}_{\alpha,\rho} = u_0 u_1 u_2 \cdots$ by

$$u_n = \begin{cases} 0 & \text{if } T^n(\rho) \in J_0\,, \\ 1 & \text{if } T^n(\rho) \in J_1\,. \end{cases}$$

Sturmian word with slope α and intercept ρ.

Language $\mathcal{L}(\mathbf{u}_{\alpha,\rho}) = \mathcal{L}(\alpha)$ depends only on α.

Abelian return words – Puzynina and Zamboni

Denote $\mathcal{AR}_{w,\mathbf{u}}$ the set of abelian return words of w in $\mathbf{u}$

Theorem (Puzynina & Zamboni) :

An aperiodic recurrent infinite word $\mathbf{u}$ is sturmian if and only if

$$\#\mathcal{AR}_{w,\mathbf{u}} \in \{2,3\} \text{ for any } w \in \mathcal{L}(\mathbf{u}).$$

Abelian return words – Puzynina and Zamboni

Denote $\mathcal{AR}_{w,\mathbf{u}}$ the set of abelian return words of w in $\mathbf{u}$

Theorem (Puzynina & Zamboni) :

An aperiodic recurrent infinite word $\mathbf{u}$ is sturmian if and only if

$$\#\mathcal{AR}_{w,\mathbf{u}} \in \{2,3\} \text{ for any } w \in \mathcal{L}(\mathbf{u}).$$

If $\#\mathcal{AR}_{w,\mathbf{u}} = 3$, then

$$\mathcal{AR}_{w,\mathbf{u}} = \{R_1, R_2, R_3\}, \text{ with } |R_1| + |R_2| = |R_3|.$$

Abelian return words – Puzynina and Zamboni

Denote $\mathcal{AR}_{w,\mathbf{u}}$ the set of abelian return words of w in $\mathbf{u}$

Theorem (Puzynina & Zamboni) :

An aperiodic recurrent infinite word $\mathbf{u}$ is sturmian if and only if

$$\#\mathcal{AR}_{w,\mathbf{u}} \in \{2,3\} \text{ for any } w \in \mathcal{L}(\mathbf{u}).$$

If $\#\mathcal{AR}_{w,\mathbf{u}} = 3$, then

$$\mathcal{AR}_{w,\mathbf{u}} = \{R_1, R_2, R_3\}, \text{ with } |R_1| + |R_2| = |R_3|.$$

Experiment (Břinda) :

$R_3 = R_1 R_2$ (for the Fibonacci word $\mathbf{f}$: $0 \mapsto 01$, $1 \mapsto 0$)

$$\mathbf{f}: \ 0100101001001010010100100101001001\cdots$$

Abelian return words – Rigo et al.

Denote $\quad \mathcal{APR}_{\mathbf{u}} = \bigcup \left\{ \mathcal{AR}_{w,\mathbf{u}} : w \text{ is a prefix of } \mathbf{u} \right\}.$

Theorem (Rigo, Salimov & Vandomme) :

$$\mathcal{APR}_{\mathbf{u}_{\alpha,\rho}} < +\infty \quad \iff \quad \rho \neq 0.$$

Abelian return words – Rigo et al.

Denote $\quad \mathcal{APR}_{\mathbf{u}} = \bigcup \big\{ \mathcal{AR}_{w,\mathbf{u}} : w \text{ is a prefix of } \mathbf{u} \big\}.$

Theorem (Rigo, Salimov & Vandomme) :

$$\mathcal{APR}_{\mathbf{u}_{\alpha,\rho}} < +\infty \quad \iff \quad \rho \neq 0.$$

Fibonacci word $\mathbf{f}$: $\quad \mathcal{APR}_{\mathbf{f}} = \{0, 1, 01, 10, 001\}.$

Our results

- If $\#\mathcal{AR}_{w,\mathbf{u}} = 3$, then $\mathcal{AR}_{w,\mathbf{u}} = \{R_1, R_2, R_1 R_2\}$.
- $\#\mathcal{APR}_{\mathbf{u}}$ dependingly on ρ and α.
- Algorithm for listing elements of $\mathcal{APR}_{\mathbf{u}}$.
- $\mathcal{APR}_{\mathbf{c}_\alpha}$ where $\mathbf{c}_\alpha = \mathbf{u}_{\alpha,1-\alpha}$

Factors of sturmian words

α, $T^{-1}(\alpha)$, ..., $T^{-n+1}(\alpha)$ defines a partition of $[0,1)$,

$$[0,1) = \bigcup \left\{ J_w : w \in \mathcal{L}_n(\alpha) \right\}.$$

Factors of sturmian words

$\alpha,\ T^{-1}(\alpha),\ \ldots,\ T^{-n+1}(\alpha)$ defines a partition of $[0,1)$,

$$[0,1) = \bigcup \left\{ J_w : w \in \mathcal{L}_n(\alpha) \right\}.$$

For $w \in \mathcal{L}_n(\alpha)$, we have $|w|_1 \in \left\{ \lceil n\alpha \rceil, \lceil n\alpha \rceil - 1 \right\}$.

w is heavy if $|w|_1 = \lceil n\alpha \rceil$,

w is light if $|w|_1 = \lceil n\alpha \rceil - 1$.

Factors of sturmian words

$\alpha,\ T^{-1}(\alpha),\ \ldots,\ T^{-n+1}(\alpha)$ defines a partition of $[0,1)$,

$$[0,1) = \bigcup \left\{ J_w : w \in \mathcal{L}_n(\alpha) \right\}.$$

For $w \in \mathcal{L}_n(\alpha)$, we have $|w|_1 \in \left\{ \lceil n\alpha \rceil, \lceil n\alpha \rceil - 1 \right\}$.

w is heavy if $|w|_1 = \lceil n\alpha \rceil$,

w is light if $|w|_1 = \lceil n\alpha \rceil - 1$.

Lemma: w is light $\iff$ $J_w \subset \left[0, T^{-n+1}(\alpha)\right)$.

First return map to $\mathbf{I} \subset [0, 1)$

Return time to I by $r : I \to \{1, 2, 3, \ldots\}$

$$r(x) = \min\{n \in \mathbb{N},\ n \geq 1\ :\ T^n(x) \in I\}\,.$$

First return map to $\mathbf{I} \subset [0, 1)$

Return time to I by $r : I \to \{1, 2, 3, \dots\}$

$$r(x) = \min\{n \in \mathbb{N},\ n \geq 1 \ :\ T^n(x) \in I\}\,.$$

First return map $T_I : I \to I$,

$$T_I(x) = T^{r(x)}(x)\,.$$

First return map to $\mathbf{I} \subset [0, 1)$

Return time to I by $r : I \to \{1, 2, 3, \dots\}$

$$r(x) = \min\{n \in \mathbb{N},\ n \geq 1 \ :\ T^n(x) \in I\}\,.$$

First return map $T_I : I \to I$,

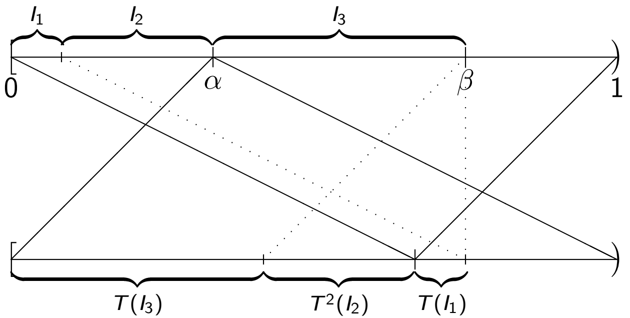
$$T_I(x) = T^{r(x)}(x)\,.$$

I-itinerary of x under T

$$R(x) = \text{ the prefix of length } r(x) \text{ of } \mathbf{u}_{\alpha,x}.$$

Example

Let $I = [0, \beta)$, where $\max\{\alpha, 1-\alpha\} \leq \beta \leq 1$.



$x \in I_1$	$r(x) = 1$	$T_I(x) = x + 1 - \alpha$	$R(x) = 0$
$x \in I_2$	$r(x) = 2$	$T_I(x) = x + 1 - 2\alpha$	$R(x) = 01$
$x \in I_3$	$r(x) = 1$	$T_I(x) = x - \alpha$	$R(x) = 1$

Abelian return words as I-itineraries

Lemma:

Let $w \in \mathcal{L}_n(\alpha)$. Put

$$I = \begin{cases} \big[0, T^{-n+1}(\alpha)\big) & \text{if } w \text{ is light,} \\ \big[T^{-n+1}(\alpha), 1\big) & \text{otherwise.} \end{cases}$$

Then

$$v \text{ is an abelian return to } w$$

$$\Updownarrow$$

$$v \text{ is an } I\text{-itinerary of some } x \in I \text{ under } T.$$

Abelian returns to prefixes

Theorem:

Let $\alpha, \rho \in [0,1)$, α irrational. Then satisfies

$$\mathcal{APR}_{\mathbf{u}_{\alpha,\rho}} = \mathcal{R}_\rho^\alpha \cup \mathcal{R}'^\alpha_\rho\,,$$

where

$$\mathcal{R}_\rho^\alpha = \bigcup_{\rho \le \beta < 1} \left\{ R : R = R(x) \text{ is the } [0,\beta)\text{-itinerary for an } x \in [0,\beta) \right\},$$

$$\mathcal{R}'^\alpha_\rho = \bigcup_{0 < \gamma \le \rho} \left\{ R : R = R(x) \text{ is the } [\gamma,1)\text{-itinerary for an } x \in [\gamma,1) \right\}.$$

Abelian returns to prefixes

Theorem:

Let $\alpha, \rho \in [0,1)$, α irrational. Then satisfies

$$\mathcal{APR}_{\mathbf{u}_{\alpha,\rho}} = \mathcal{R}_\rho^\alpha \cup \mathcal{R}'^{\alpha}_\rho \,,$$

where

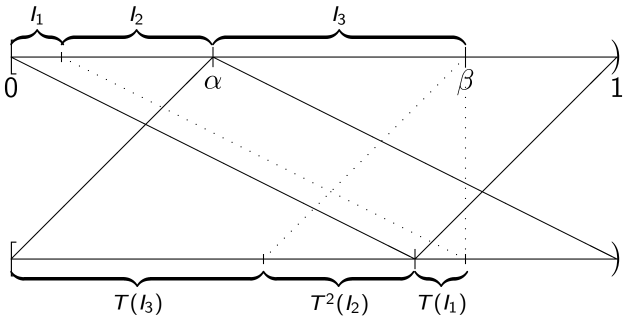
$$\mathcal{R}_\rho^\alpha = \bigcup_{\rho \le \beta < 1} \left\{ R : R = R(x) \text{ is the } [0,\beta)\text{-itinerary for an } x \in [0,\beta) \right\},$$

$$\mathcal{R}'^{\alpha}_\rho = \bigcup_{0 < \gamma \le \rho} \left\{ R : R = R(x) \text{ is the } [\gamma,1)\text{-itinerary for an } x \in [\gamma,1) \right\}.$$

Note: $\mathcal{R}'^{\alpha}_\rho = E\left(\mathcal{R}_{1-\rho}^{1-\alpha}\right)$ where $E : 0 \leftrightarrow 1$.

Return time vs. itineraries – example 1

Let $I = [0, \beta)$, where $\max\{\alpha, 1-\alpha\} \leq \beta \leq 1$.



$x \in I_1$	$r(x) = 1$	$T_I(x) = x + 1 - \alpha$	$R(x) = 0$
$x \in I_2$	$r(x) = 2$	$T_I(x) = x + 1 - 2\alpha$	$R(x) = 01$
$x \in I_3$	$r(x) = 1$	$T_I(x) = x - \alpha$	$R(x) = 1$

Return time vs. itineraries – example 2

Let $\alpha = \frac{1}{\tau}$, where $\tau = \frac{1}{2}(1+\sqrt{5})$,

$$T(x) = \begin{cases} x + 1 - \frac{1}{\tau} & \text{if } x \in [0, \frac{1}{\tau})\,, \\ x - \frac{1}{\tau} & \text{if } x \in [\frac{1}{\tau}, 1)\,. \end{cases}$$

Let $I = [\frac{1}{\tau^3}, \frac{1}{\tau} + \frac{1}{\tau^4})$.

$x \in I_1$	$r(x) = 1$	$T_I(x) = x + \frac{1}{\tau^2}$	$R(x) = 0$
$x \in I_2$	$r(x) = 3$	$T_I(x) = x + \frac{1}{\tau^4}$	$R(x) = 010$
$x \in I_3$	$r(x) = 2$	$T_I(x) = x - \frac{1}{\tau^3}$	$R(x) = 01$
$x \in I_4$	$r(x) = 2$	$T_I(x) = x - \frac{1}{\tau^3}$	$R(x) = 10$

where

$$I_1 = [\tfrac{1}{\tau^3}, \tfrac{1}{\tau^2}),\ I_2 = [\tfrac{1}{\tau^2}, \tfrac{1}{\tau^2} + \tfrac{1}{\tau^5}),\ I_3 = [\tfrac{1}{\tau^2} + \tfrac{1}{\tau^5}, \tfrac{1}{\tau}),\ I_4 = [\tfrac{1}{\tau}, \tfrac{1}{\tau} + \tfrac{1}{\tau^4})\,.$$

First return map for Sturmian words

Interval exchange:

Partition $J_0 \cup J_1 \cup \cdots \cup J_{k-1} = [0, 1)$,
$t_0, t_1, \ldots, t_{k-1} \in \mathbb{R}$ such that

$$S(x) = x + t_j \ \text{ for } x \in J_j, \ \text{ is a bijection } S : [0, 1) \to [0, 1).$$

Theorem:

First return map T_I to $I \subset [0, 1)$ induced by a 2iet
$T : [0, 1) \to [0, 1)$ is always an exchange of two or three intervals.

First return map for Sturmian words

If $\alpha = [0, a_1, a_2, \dots]$, denote for $k \geq 0$, $1 \leq s \leq a_{k+1}$,

$$\delta_{k,s} := \left|(s-1)(p_k - \alpha q_k) + p_{k-1} - \alpha q_{k-1}\right|,$$

where $\frac{p_k}{q_k}$ are convergents of α.

First return map for Sturmian words

If $\alpha = [0, a_1, a_2, \dots]$, denote for $k \geq 0$, $1 \leq s \leq a_{k+1}$,

$$\delta_{k,s} := \left|(s-1)(p_k - \alpha q_k) + p_{k-1} - \alpha q_{k-1}\right|,$$

where $\frac{p_k}{q_k}$ are convergents of α.
Note: $\delta_{k,s}$ frequencies of sturmian factors (Berthé 1996)

First return map for Sturmian words

If $\alpha = [0, a_1, a_2, \dots]$, denote for $k \geq 0$, $1 \leq s \leq a_{k+1}$,

$$\delta_{k,s} := \left|(s-1)(p_k - \alpha q_k) + p_{k-1} - \alpha q_{k-1}\right|,$$

where $\frac{p_k}{q_k}$ are convergents of α.
Note: $\delta_{k,s}$ frequencies of sturmian factors (Berthé 1996)

Theorem:

Let $I = [c, c+\delta)$, where $0 \leq c < c + \delta \leq 1$. Then T_I is

- a 2iet, if $\delta = \delta_{k,s}$ for some $k \geq 0$, $1 \leq s \leq a_{k+1}$;
- a 3iet with permutation (321), otherwise.

Three $[0, \beta)$-itineraries

Theorem

Let $I = [0, \beta) \subset [0, 1)$. There exist words R and R', $R \prec_{\text{lex}} R'$, such that the I-itinerary $R(x)$ of every $x \in I$ under T satisfies

$$R(x) \in \{R, R', RR'\}.$$

Three $[0, \beta)$-itineraries

Theorem

Let $I = [0, \beta) \subset [0, 1)$. There exist words R and R', $R \prec_{\mathrm{lex}} R'$, such that the I-itinerary $R(x)$ of every $x \in I$ under T satisfies

$$R(x) \in \{R, R', RR'\}\,.$$

Corollary:

For every factor w of a sturmian word $\mathbf{u}$ there exist factors w_1, w_2 such that $\mathcal{AR}_{w,\mathbf{u}} \in \{w_1, w_2, w_1 w_2\}$.

Main result

Recall

$$\mathcal{R}_\rho^\alpha = \bigcup_{\rho \le \beta < 1} \{R : R = R(x) \text{ is the } [0,\beta)\text{-itinerary for an } x \in [0,\beta)\}$$

Corollary:

Let $k, s \in \mathbb{N}$ be minimal such that $\rho \geq \delta_{k,s}$. Then

$$\#\mathcal{R}_\rho^\alpha = 1 + a_1 + a_2 + \cdots + a_k + s\,.$$

Main result

Recall

$$\mathcal{R}_\rho^\alpha = \bigcup_{\rho \le \beta < 1} \{ R : R = R(x) \text{ is the } [0,\beta)\text{-itinerary for an } x \in [0,\beta) \}$$

Corollary:

Let $k, s \in \mathbb{N}$ be minimal such that $\rho \ge \delta_{k,s}$. Then

$$\#\mathcal{R}_\rho^\alpha = 1 + a_1 + a_2 + \cdots + a_k + s\,.$$

Note: $\#\mathcal{R}_0^\alpha = +\infty$ and hence also $\#\mathcal{APR}_{\mathbf{u}_{\alpha,0}} = +\infty$.

Main result

Recall $\mathcal{APR}_{\mathbf{u}_{\alpha,\rho}} = \mathcal{R}_{\rho}^{\alpha} \cup E\big(\mathcal{R}_{1-\rho}^{1-\alpha}\big)$.

Theorem:

Let $\alpha, \rho \in (0,1)$, $\alpha = [0, a_1, a_2, \dots]$ irrational, $a_1 \geq 2$, $\mathbf{u} = \mathbf{u}_{\alpha,\rho}$.

(i) Let $\rho \in (\alpha, 1-\alpha)$. Then

$$\#\mathcal{APR}_{\mathbf{u}} \in \{a_1 + 3, a_1 + 4\}\,.$$

(ii) Let $\rho \notin (\alpha, 1-\alpha)$. Then

$$\#\mathcal{APR}_{\mathbf{u}} = 2 + a_1 + \cdots + a_k + s\,,$$

where $k \geq 0$ and $1 \leq s \leq a_{k+1}$ are minimal such that $\min\{\rho, 1-\rho\} \geq \delta_{k,s}$.

$\mathcal{APR}_{\mathbf{u}}$ for characteristic sturmian words

Characteristic Sturmian words $\mathbf{c}_\alpha = \mathbf{u}_{\alpha,1-\alpha}$.

Theorem:

Let $\alpha = [0, a_1, a_2, \cdots]$ be an irrational in $(0, 1)$. Then

$$\mathcal{APR}_{\mathbf{c}_\alpha} = \begin{cases} \{1, 10, 0, 01, 001, \ldots, 0^{a_2+1}1\} & \text{if } \alpha > \frac{1}{2}, \\ \{0, 01, 1, 10, 110, \ldots, 1^{a_1}0\} & \text{otherwise.} \end{cases}$$

$\mathcal{APR}_{\mathbf{u}}$ for characteristic sturmian words

Characteristic Sturmian words $\quad \mathbf{c}_\alpha = \mathbf{u}_{\alpha,1-\alpha}$.

Theorem:

Let $\alpha = [0, a_1, a_2, \cdots]$ be an irrational in $(0,1)$. Then

$$\mathcal{APR}_{\mathbf{c}_\alpha} = \begin{cases} \{1, 10, 0, 01, 001, \ldots, 0^{a_2+1}1\} & \text{if } \alpha > \frac{1}{2}, \\ \{0, 01, 1, 10, 110, \ldots, 1^{a_1}0\} & \text{otherwise.} \end{cases}$$

Applying $\alpha = \frac{1}{\tau} = [0, 1, 1, \ldots]$, we obtain

$$\mathcal{APR}_{\mathbf{f}} = \{0, 1, 01, 10, 001\}.$$

Thank for your attention