Another Generalization of Abelian Equivalence: Binomial Complexity of Infinite Words

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Definitions

$\Sigma_s = \{0, 1, \ldots, s-1\}$ we call an **alphabet**.
$\Sigma^* = \{\varepsilon\} \cup \bigcup_{n\in\mathbb{N}} \Sigma^n$ — the set of finite words over Σ.

$x = x_0 x_1 \ldots \in \Sigma^{\mathbb{N}_0}$ — an **infinite word** over Σ.

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$u \in \Sigma^n$ is a **factor** [A. Thue 1906] of $x = x_0 x_1 \ldots$ **occurring** at position i in $x = x_0 x_1 \ldots$ iff $u = x_i x_{i+1} \ldots x_{i+n-1}$. In the case $i = 0$, the word u is a **prefix** of x.

The language of factors (of length n) of x is denoted by F_x ($F_x(n)$) and the language of prefixes — by P_x.
A **morphism** $\varphi : \Sigma^* \to \Delta^*$ is a mapping satisfying to $\varphi(vw) = \varphi(v)\varphi(w)$ for each pair of words v, w.

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$f_x(n) = |\{x_k x_{k+d} \ldots x_{k+(n-1)d} \ : \ k \in \mathbb{N}_0 \, , d \in \mathbb{N}\}|$.

Maximal pattern complexity [T. Kamae, L.Q. Zamboni 2002]
$p_x^*(n) = \sup_{t_1 < t_2 < \ldots < t_n} |\{x_{k+t_0} x_{k+t_1} \ldots x_{k+t_n} \ : \ k \in \mathbb{N}_0\}|$.

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Abelian complexity [G. Richomme et al. 2009]
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k-**abelian complexity** [J. Karhumaki et al. 2012]
$f_x^{k,ab}(n) = |F_x(n)/\sim_{k,ab}|$, where $v \sim_{k,ab} w$ iff $|v|_u = |w|_u$ for each $u \in \Sigma^{\leqslant k}$.

Key definitions

$u \in \Sigma^n$ is a **subword** of $x = x_0 x_1 \dots$ iff $u = x_I = x_{i_1} x_{i_2} \dots x_{i_{|u|}}$ for some set $I = \{i_1 < i_2 < \dots i_{|u|}\} \subset \mathbb{N}_0$.

Binomial coefficient [Lothaire's "Combinatorics on Word" Section 6.3] of u in v is $\binom{v}{u} = |\{I \subseteq [0, |v|) \ : \ v_I = u\}|$.

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v, w are ***k*-binomial equivalent** $v \sim_k w$ iff $\binom{v}{u} = \binom{w}{u}$ for each $u \in \Sigma^{\leqslant k}$.

Examples

$\binom{a^n}{a^k} = \binom{n}{k} \quad , \binom{u}{a^k} = \binom{|u|_a}{k}$

$01100 \sim_2 10010 \quad \binom{01100}{01} = |\{01100, 01100\}| = 2 = \binom{10010}{01}$

$01100 \not\sim_3 10010 \quad \binom{01100}{110} = |\{01100, 01100\}| = 2 \neq \binom{10010}{110} = 1$

Classes of k-binomial equivalence

For a word v define the class $\mathbf{B}^{(k)}(v) \in \Sigma^*/\sim_k$ s.t. $v \in \mathbf{B}^{(k)}(v)$.

Observation

$$\binom{vw}{u} = \sum_{ps=u,\ p,s\in\Sigma^*} \binom{v}{p}\binom{w}{s}.$$

Corollary

For a given k, the operation $\circ$ satisfying to the property $\mathbf{B}^{(k)}(v) \circ \mathbf{B}^{(k)}(w) = \mathbf{B}^{(k)}(vw)$ for each v, w is well defined.

Monoid of k-binomial equivalence classes

For a word v and a word $u = u_0 u_1 \dots u_{k-1}$ of length k define the $(k+1) \times (k+1)$ matrix

$$M_u(v) = \left(\begin{array}{cccccc} 1 & \binom{v}{u_0 u_1 \dots u_{k-1}} & \binom{v}{u_0 u_1 \dots u_{k-2}} & \dots & \binom{v}{u_0 u_1} & \binom{v}{u_0} \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & \binom{v}{u_{k-1}} & 1 & \dots & 0 & 0 \\ 0 & \binom{v}{u_{k-2} u_{k-1}} & \binom{v}{u_{k-2}} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \binom{v}{u_2 u_3 \dots u_{k-1}} & \binom{v}{u_2 u_1 \dots u_{k-2}} & \dots & 1 & 0 \\ 0 & \binom{v}{u_1 u_2 \dots u_{k-1}} & \binom{v}{u_1 u_1 \dots u_{k-2}} & \dots & \binom{v}{u_1} & 1 \end{array} \right).$$

The statement $M_u(vw) = M_u(v) M_u(w)$ holds for each v, w.

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The statement $M_u(vw) = M_u(v) M_u(w)$ holds for each v, w.
Parikh matrices should be there.

Monoid of k-binomial equivalence classes

For a word v and a number k define the matrix

$$M_k(v) = \begin{pmatrix} M_{u^{(1)}}(v) & 0 & \dots & 0 \\ 0 & M_{u^{(2)}}(v) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & M_{u^{(|\Sigma|^k)}}(v) \end{pmatrix},$$

where the sequence $u^{(1)}, u^{(2)}, \dots$ is the lexicographical of Σ^k.
The statement $M_k(vw) = M_k(v)M_k(w)$ holds for each v, w.

Conclusion

The monoid $\langle \mathfrak{B}, \circ \rangle$ is isomorphic to a submonoid of matrices.

The number of k-binomial equivalence classes

- For 2-abelian equivalence $|\Sigma_2^n/\sim_{2,ab}| = n^2 - n + 2$
- And, in general, $|\Sigma_s^n/\sim_{k,ab}| = \Theta(n^{(s-1)s^{k-1}})$

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Lemma

- $|\Sigma_2^n / \sim_2| = \frac{n^3+5n+6}{6}$
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The family of words of the $1^i0^j10^l1^m$ has a representative in each 2-binomial class.

$$\sum_{u\in\Sigma^n} \binom{v}{u} = \binom{|v|}{n}$$

k-binomial complexity

k-binomial complexity of a word $x = x_0 x_1 \dots$ is $\mathbf{b}_x^{(k)}(n) = |F_x(n)/\sim_k|$ the number of its factors of length of n distinct up to k-binomial equivalence.

- $\mathbf{b}_x^{(k)}(n) \leqslant \mathbf{b}_x^{(k+1)}(n)$
- $f_x^{ab}(n) \leqslant \mathbf{b}_x^{(k)}(n) \leqslant f_x(n)$

An infinite word $x = x_0 x_1 \ldots$ is **Sturmian** iff x is aperiodic and for any two factors v, w of equal length $||v|_1 - |w|_1| \leqslant 1$.

Example

Consider the morphism $\varphi : 0 \mapsto 01, 1 \mapsto 0$ and its fixed point $0100101001001\ldots$

k-binomial complexity, Sturmian case

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- $f_x(n) \equiv n+1$, $f_x^{ab}(n) \equiv 2$, $p_x^*(n) \equiv 2n$

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Theorem

For a Sturmian word x, $\mathbf{b}_x^{(k)}(n) \equiv n+1$ for each $k > 1$.

Let $00 \in F_x$ for a Strumian x.
There exists a unique k and a Sturmian word y s.t. for any $v \in F_x$

$$v = 0^r 1 0^{k+\epsilon_0} 1 0^{k+\epsilon_1} 1 \dots 0^{k+\epsilon_{n-1}} 1 0^s ,$$

where $r, s \leqslant k+1$ and $\epsilon = \epsilon_0 \epsilon_1 \dots \epsilon_{n-1} \in F_y$.

k-binomial complexity, Sturmian case, proof sketch

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Define $S(\epsilon) = \sum_{i=0}^{n-1} (n-i)\epsilon_i$.
Consider $\binom{v}{01} = r(n+1) + S(\epsilon) + k\frac{n(n+1)}{2}$

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Let $n \geq 1$. If $\epsilon \neq \epsilon'$ are factors of length n occurring in a Sturmian word, then $S(\epsilon) \not\equiv S(\epsilon') \pmod{n+1}$.

k-binomial complexity, Sturmian case, proof sketch

$S(\epsilon, m) = \sum_{i=0}^{m-1}(n-i)\epsilon_i$ and $S(\epsilon) = S(\epsilon, n-1)$

Lemma

Let $n \geq 1$. If $\epsilon \neq \epsilon'$ are factors of length n occurring in a Sturmian word y, then $S(\epsilon) \not\equiv S(\epsilon') \pmod{n+1}$.

Define $\Delta(m) := |\epsilon_0\epsilon_1 \cdots \epsilon_{m-1}|_1 - |\epsilon'_0\epsilon'_1 \ldots \epsilon'_{m-1}|_1 \in \{-1, 0, 1\}$
Note that Δ is either non-negative or non-positive due to balanceness of a y.

$$
\begin{array}{rcl}
 & & S(\epsilon, j+1) - S(\epsilon', j+1) = \\
\Delta(j+1) > \Delta(j) & \Rightarrow & S(\epsilon, j) - S(\epsilon', j) + (n-j-1) \\
\Delta(j+1) = \Delta(j) & \Rightarrow & S(\epsilon, j) - S(\epsilon, j) \\
\Delta(j+1) < \Delta(j) & \Rightarrow & S(\epsilon, j) - S(\epsilon', j) - (n-j-1).
\end{array}
$$

In view of these observations, we have $0 < S(\epsilon) - S(\epsilon') < n+1$

k-binomial complexity, Thue-Morse case

Consider the morphism $\varphi : 0 \mapsto 01, 1 \mapsto 10$
Its fixed point $011010011001\ldots$ is the Thue-Morse word.

- $f_x(n)$ grows linearly.
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A morphism φ is **balanced** if $\varphi(a) \simeq_{ab} \varphi(b)$ for all symbols $a, b \in \Sigma$.

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Each $v \in F_x(n)$ admits a decomposition $v = p\varphi^k(u)s$, where p, s are from finite set.

$$\mathbf{B}^{(k)}(v) = \mathbf{B}^{(k)}(p) \circ \mathbf{B}^{(k)}(\varphi^k(u_0)) \circ \mathbf{B}^{(k)}(\varphi^k(u_1)) \circ \ldots \circ \mathbf{B}^{(k)}(s)$$

Complexity in Thue-Morse case, sketch of a good proof

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If φ is a balanced morphism then $\mathbf{B}^{(k)}(\varphi^k(a)) = \mathbf{B}^{(k)}(\varphi^k(b))$ for all a, b.

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$\varphi(a) \simeq_{ab} \varphi(b)$ for all symbols $a, b \in \Sigma$. Prove by induction.

For $a \in \Sigma$ define the word $v_0 v_1 \dots v_l = \varphi(a)$. Denote the set of monotonous increasing subsequences of size p of a finite set A by $\mathcal{IS}(A, p)$. For an element $f \in \mathcal{IS}(A, p)$ denote by $f(i)$ the i's element of the sequence f starting numeration from 1.

$$\binom{\varphi^{m+1}(a)}{u} = \binom{\varphi^m(v)}{u} = \sum_{0 \leqslant i \leqslant l} \binom{\varphi^m(v_i)}{u} +$$

$$+ \sum_{e^{(1)} e^{(2)} \dots e^{(p)} = u,\ e^{(i)} \in \Sigma^+} \ \sum_{f \in \mathcal{IS}([0,l],p)} \prod_{i=1}^{p} \binom{\varphi^m(v_{f(i)})}{e^{(i)}}$$

Complexity in Thue-Morse case, sketch of a bad proof

Let $v = \varphi(u)$ for the Thue-Morse φ. Lets count $\binom{v}{01}$.

$\binom{v}{01} = |u|_0 + \binom{|u|}{2}$.

**		*		*			
0	1	1	0	1	0	0	1

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$v = \varphi(u') = \varphi^2(u)$ and we are calculating $\binom{v}{011}$

* * *				
[*] [*] [*]	[* *] [*]		[*] [* *]	
$\binom{\|u'\|}{3}$	[[* *]] [[*]]	[[**][*]]	[[*]][[**]]	[[*][**]]
	$2\binom{\|u\|}{2}$	$\|u\|_0$	$0\binom{\|u\|}{2}$	$0\|u\|_0$

For any u considering a deep enough partition of v the coefficient $\binom{v}{u}$ may be expressed.

Bounded complexities

Let $x = x_0 x_1 \ldots$, complexity f_x^{ab} is bounded iff there exists Δ and, for each letter a, the frequency λ_a s.t.

$$\left| |v|_a - |v|\lambda_a \right| \leqslant \Delta$$

for each factor v of x.

Bounded $\mathbf{b}_x^{(2)}$ implies bounded f_x^{ab}, what additional properties does x have?

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Bounded $\mathbf{b}_x^{(2)}$ implies bounded f_x^{ab}, what additional properties does x have?

Lemma

If $\mathbf{b}_x^{(2)}$ for a recurrent x is bounded, then frequencies of letters are rational.

Rationality of frequencies, proof sketch

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Let $pvp \in F_x$, for long p and v. Define $n = |pv|$. Let $\mathbf{b}_x^{(2)} \leqslant c$. For words $u^{(i)} = p_i p_{i+1} \dots p_{|p|-1} v p_0 p_1 \dots p_{i-1}$ we have:

$$\binom{u^{(i+1)}}{01} = \begin{cases} \binom{u^{(i)}}{01} - |pv|_1, & p_i = 0\ ; \\ \binom{u^{(i)}}{01} + |pv|_0, & p_i = 1\ . \end{cases}$$

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There are i, j s.t. $\binom{u^{(i)}}{01} = \binom{u^{(j)}}{01}$ and $i, j \leqslant c$. It implies existence of $c_0, c_1 \leqslant c$ such that $c_0|pv|_0 + c_1|pv|_1 = 0$.

$$c_0(n\lambda_0 + \delta) + c_1(n(1 - \lambda_0) - \delta) = 0 ,$$

where $\delta \leqslant \Delta$ is some real number.

$$\lambda_0 = \frac{c_1}{c_1 - c_0} - \frac{\delta}{n} .$$

Summary and open problems

Family of new equivalence relations and corresponding complexities introduced using word binomial coefficients.
Classes form a monoid isomorphic to a submonoid of matrixes.
Complexities of Sturmian words grows as $n + 1$. For the Thue-Morse and similar words complexities are bounded.

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2. Are 2-binomial squares avoidable on 3 letters? Is it true for the fixed point of $\varphi : 0 \mapsto 012, 1 \mapsto 02, 2 \mapsto 1$.

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2. Are 2-binomial squares avoidable on 3 letters? Is it true for the fixed point of $\varphi : 0 \mapsto 012, 1 \mapsto 02, 2 \mapsto 1$. "Yes" by M. Rao, M. Rigo.
3. What about Toeplitz words?

Thank you!