On the characterization of all β with $(\pm\beta)$-integers encoded by conjugated morphisms

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WORDS 2013, Turku
17 September 2013

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b-expansions

Let $b \in \mathbb{R}$, $|b| > 1$, $x \in [\ell, \ell+1)$, $\ell \in \mathbb{R}$. The (b, ℓ)-expansion of x is the sequence $(x_i)_{i \geq 1}$, where

$$x_i := \lfloor b T^{i-1}(x) - \ell \rfloor ,$$
$$T(x) := bx - \lfloor bx - \ell \rfloor .$$

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We write $d(x) = d_{b,\ell}(x) := x_1x_2x_3\cdots$ and it holds

$$x = \frac{x_1}{b} + \frac{x_2}{b^2} + \frac{x_3}{b^3} + \cdots .$$

b-expansions $\cdots$ admissibility

$(x_i)_{i\in\mathbb{N}}$ is (b,ℓ)-admissible if $d(x) = x_1x_2\cdots$ for some $x \in [\ell, \ell+1)$

b-expansions $\cdots$ admissibility

$(x_i)_{i\in\mathbb{N}}$ is (b,ℓ)-admissible if $d(x) = x_1x_2\cdots$ for some $x \in [\ell, \ell+1)$

Theorem

For fixed b,ℓ it holds: $(x_i)_{i\in\mathbb{N}}$ is (b,ℓ)-admissible iff

$$d(\ell) \preceq x_k x_{k+1} x_{k+2} \cdots \prec d^*(\ell+1) = lim_{\varepsilon\to 0+} d(\ell+1-\varepsilon)\,,$$

for all $k \geq 1$, where

- $b = \beta > 1 \rightarrow$ *lexicographic ordering* $\preceq_{\mathrm{lex}}$
- $b = -\beta < -1 \rightarrow$ *alternate ordering* $\preceq_{\mathrm{alt}}$

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$$u_1 u_2 \cdots \prec_{\mathrm{alt}} v_1 v_2 \cdots \text{ if } (-1)^k (u_k - v_k) < 0,\ k = \min\{i \geq 1, u_i \neq v_i\}$$

($\pm\beta$)-expansions of reals

Rényi β-expansions (1957):

- $b = \beta > 1$, $[\ell, \ell+1) = [0, 1)$
- $x_i \in \{0, 1, \cdots, \lceil \beta \rceil - 1\}$
- Any $x \in \mathbb{R}$ has a unique β-expansion $(x)_\beta = (\pm) x_k x_{k-1} \cdots x_1 x_0 \bullet x_{-1} x_{-2} \cdots$.

$(\pm\beta)$-expansions of reals

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Ito-Sadahiro $(-\beta)$-expansions (2009):

- $b = -\beta < -1$, $[\ell, \ell+1) = \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$
- $x_i \in \{0, 1, \cdots, \lfloor\beta\rfloor\}$
- For any $x \in \mathbb{R}$ we can define a unique $(-\beta)$-expansion $\langle x\rangle_{-\beta} = x_k x_{k-1} \cdots x_1 x_0 \bullet x_{-1} x_{-2} \cdots$.

$(\pm\beta)$-integers

β-integers:

$$\mathbb{Z}_\beta^+ := \{x \in \mathbb{R}^+ \ : \ (x)_\beta = x_k \cdots x_1 x_0 \bullet 0^\omega\} = \bigcup_{n \geq 0} \beta^n T_\beta^{-n}(0)$$

$$\mathbb{Z}_\beta := \mathbb{Z}_\beta^+ \cup (-\mathbb{Z}_\beta^+)$$

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$(-\beta)$-integers:

$$\mathbb{Z}_{-\beta} := \{x \in \mathbb{R} \ : \ \langle x \rangle_{-\beta} = x_k \cdots x_1 x_0 \bullet 0^\omega\} = \bigcup_{n \geq 0} (-\beta)^n T_{-\beta}^{-n}(0)$$

$(\pm\beta)$-integers $\cdots$ encoding by infinite word

Let $\beta > 1$ be a Pisot number (algebraic integer, conjugates $|\cdot| < 1$). Then:

$\mathbb{Z}_\beta^+$: is an infinite set,
contains distances of lengths $\Delta_0, \ldots, \Delta_k$, all ≤ 1,
can be encoded by infinite word $\mathbf{u}_\beta \in \{0, \ldots, k\}^{\mathbb{N}}$,
there exists a morphism φ fixing $\mathbf{u}_\beta$.

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$\mathbb{Z}_{-\beta}$: is an infinite set $\Leftrightarrow \beta \geq \tau = (1 + \sqrt{5})/2$,
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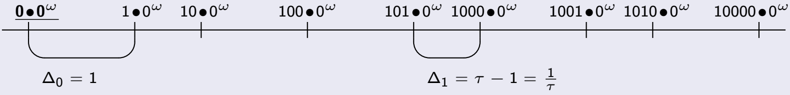
- $\varphi(xy) = \varphi(x)\varphi(y)$, $\psi(xy) = \psi(y)\psi(x)$
- φ^2, ψ^2 $\cdots$ both morphisms, can be compared
- β not Pisot $\Rightarrow$ possibly infinite number of types of Δ_i, Δ'_i

$(\pm\beta)$-integers $\cdots$ examples

Let $\beta = \tau$, the Golden ratio.

Example ($\mathbb{Z}_\tau^+$)



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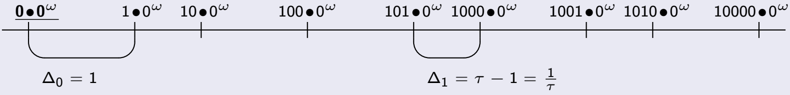

0

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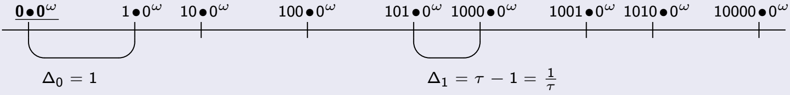

$0 \to 01$

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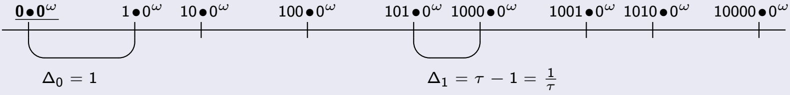

$0 \to 01 \to 010$

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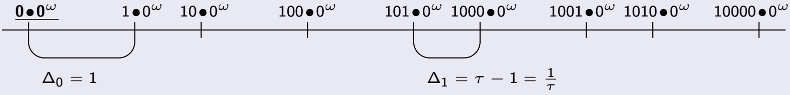

$0 \to 01 \to 010 \to 01001$

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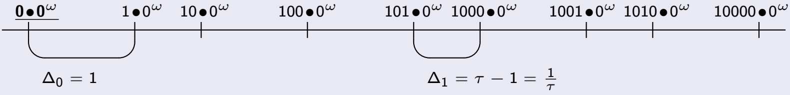

$$0 \to 01 \to 010 \to 01001 \to 01001\textcolor{red}{010}$$

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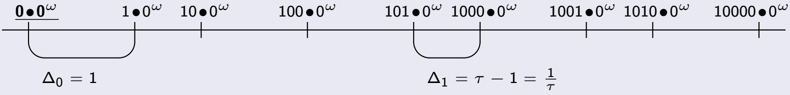

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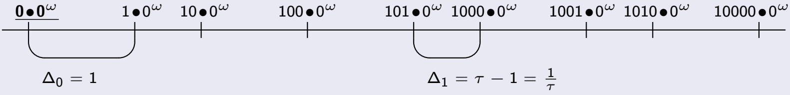

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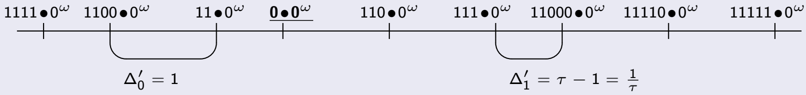

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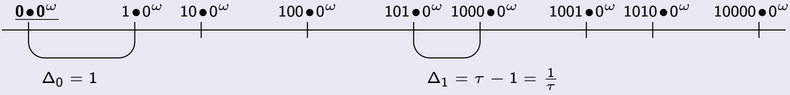

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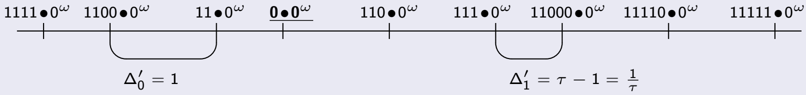

$1|0$

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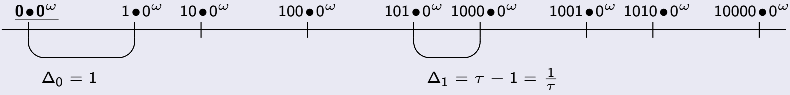

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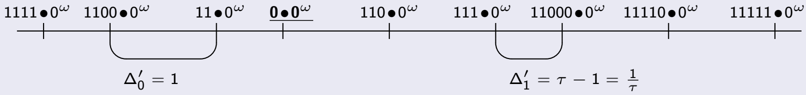

$$1|0 \to \color{red}{0}1|0$$

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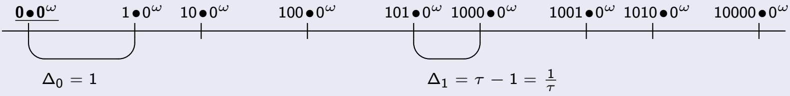

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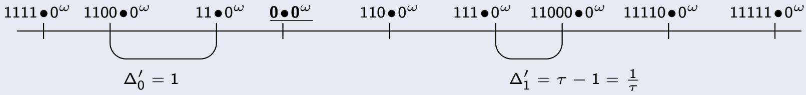

$$1|0 \to 01|0 \to 01|0{\color{red}01}$$

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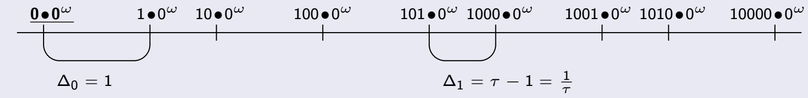

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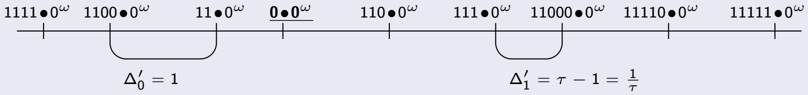

$$1|0 \to 01|0 \to 01|001 \to 00101|001$$

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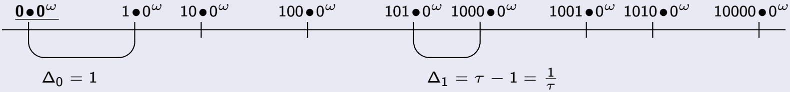

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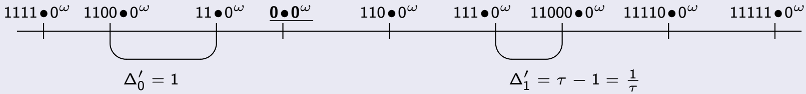

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Theorem (Ch. Frougny)

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- if $z = \sum_{i=0}^{k} a_i \beta^i$, then $z \in \mathbb{Z}_\beta^+$, even if $a_k \cdots a_0 0^\omega$ is not admissible
- this class of β $\cdots$ confluent Pisot numbers

Similarity of $\mathbb{Z}_\beta^+$ and $\mathbb{Z}_{-\beta}$

For whose β are $\mathbb{Z}_\beta^+$ and $\mathbb{Z}_{-\beta}$ "similar"?

What is "similar"?

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Note:

- two infinite words have the same language iff morphisms fixing them are conjugated

Conjugacy of morphisms

Definition

Let $\mathcal{A}$ be an alphabet (finite or infinite) and $\pi, \rho : \mathcal{A}^* \to \mathcal{A}^*$ be morphisms on $\mathcal{A}$. We say that π and ρ are conjugated, if there exists a word $w \in \mathcal{A}^*$ such that either

$$w\pi(a) = \rho(a)w, \text{ for all } a \in \mathcal{A}, \text{ or } \pi(a)w = w\rho(a), \text{ for all } a \in \mathcal{A}.$$

We denote $\pi \sim \rho$.

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Example

Let $\beta = \tau$, $\mathcal{A} = \{0, 1\}$.

- $\mathbf{u}_\tau$ *fixed point of* φ^2: $\varphi^2(0) = 010, \varphi^2(1) = 01$
- $\mathbf{v}_{-\tau}$ *fixed point of* ψ^2: $\psi^2(0) = 001, \psi^2(1) = 01$
- $\varphi^2(a)(01) = (01)\psi^2(a)$

Main result

Theorem (D., Z. Masáková, T. Vávra)

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1. *Words* $\mathbf{u}_\beta$ *and* $\mathbf{v}_{-\beta}$ *have the same language* ($\varphi^2 \sim \psi^2$).

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3. *β is a Pisot number with minimal polynomial $x^d - mx^{d-1} - \dots - mx^2 - mx - n$ with $m \geq n \geq 1$ for d odd and $m = n \geq 1$ for d even.*

Even degree remark

Let $\beta > 1$ be a Pisot number with minimal polynomial $x^d - mx^{d-1} - \ldots - mx^2 - mx - n$ with d even and $m > n \geq 1$.

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1. *For $\mathrm{p}(x) = x^3 - mx^2 - mx - 1$:*
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3. *In all other cases, $\mathbb{Z}_{-\beta}$ contains distances > 1.*

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- what we get $\to$ Rauzy fractal

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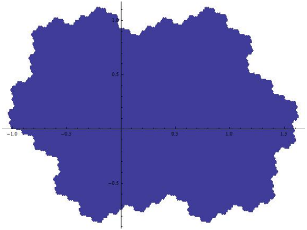

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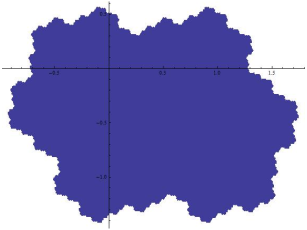

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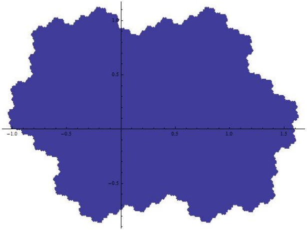

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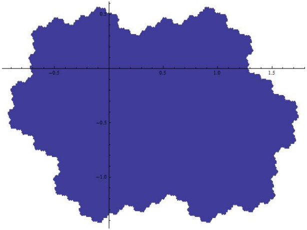

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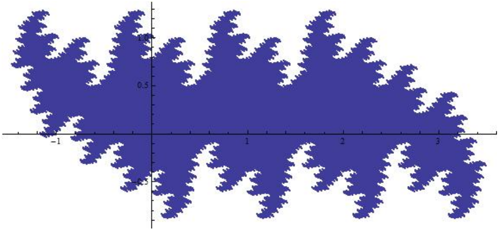

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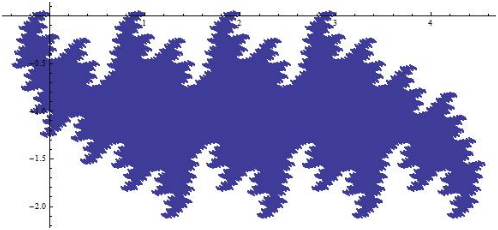

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Case 2: same distances, $\varphi^2 \not\sim \psi^2$

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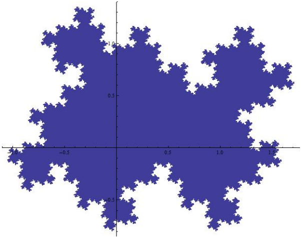

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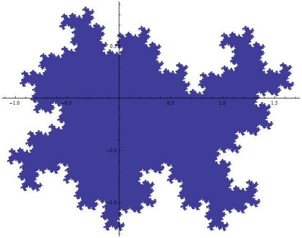

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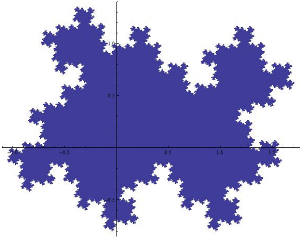

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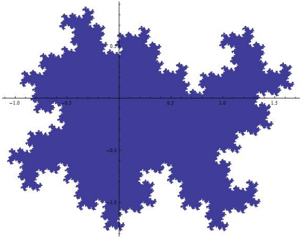

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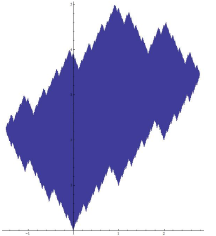

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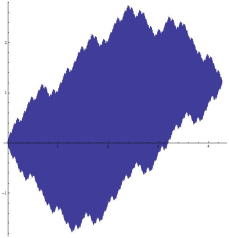

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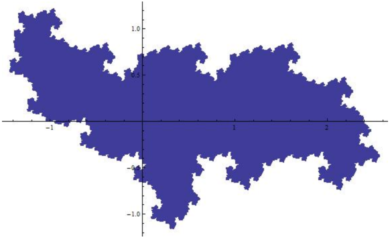

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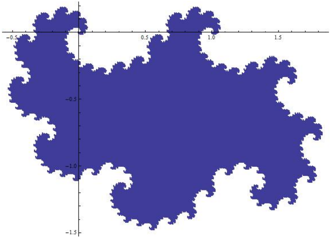

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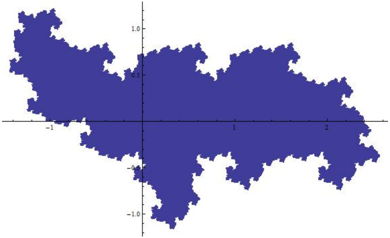

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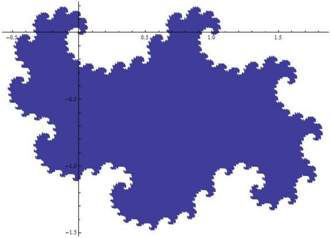

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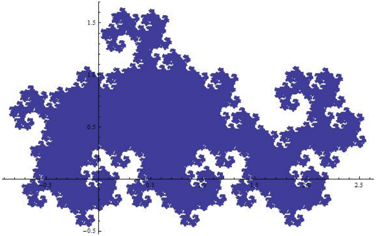

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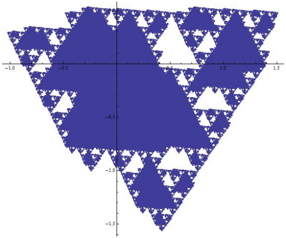

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