On factor complexity of infinite permutation

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Basic definitions

Definition

An ordered triple $\delta = \langle \mathbb{N}, <_\delta, < \rangle$, where $<_\delta$ and $<$ are linear orders on the set $\mathbb{N}$ is called *infinite permutation*.

To define an infinite permutation $\Leftrightarrow$ to define some linear order on set of positive integer numbers.

In what follows we will write $\delta(i) < \delta(j)$ instead $i <_\delta j$.

Example of infinite permutation

Example of infinite permutation

1. Let $a_n = (-1/2)^n$. Then a_n define the order $<_{\delta_1}$: $\delta(i) < \delta(j) \Leftrightarrow a_i < a_j$.
2. Let $b_n = 1000 + (-1/n)^n$. Then b_n define the order $<_{\delta_2}$: $\delta(i) < \delta(j) \Leftrightarrow b_i < b_j$.
3. The order $<_{\delta_3}$ defined by inequalities $\delta_3(2i) > \delta_3(2j+1)$, $\delta_3(2i) > \delta_3(2i+2)$, $\delta_3(2j+1) < \delta_3(2j+3)$

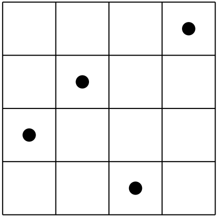
It is easy to see that orders $<_{\delta_1}$, $<_{\delta_2}$ and $<_{\delta_3}$ are equal. So permutations δ_1, δ_2 and δ_3 are equal too.

Factor complexity

Definition

Let δ be an infinite permutation. The finite permutation of length n such that $x(i) < x(j)$ if and only if $\delta(m+i-1) < \delta(m+j-1)$ is denoted by $\delta[m, m+n-1]$.

Example. Let $a_n = (-2)^n$ and a_n define the infinite permutation δ: $\delta(i) < \delta(j) \Leftrightarrow a_i < a_j$. Then $a_3 = -8$, $a_4 = 16$, $a_5 = -32$ and $a_6 = 64$. Permutation $\delta[3,6]$ can be illustrated as follows:



$\delta[3,6] = 2314$

Factor complexity

Definition

We say that finite permutation π of length n is *factor* (or *subpermutation*) of length n of the infinite permutation δ if $\pi = \delta[i, i+n-1]$ for some $i > 0$.

Example. Let $a_n = (-2)^n$ and a_n define the infinite permutation δ: $\delta(i) < \delta(j) \Leftrightarrow a_i < a_j$. By previous example $\delta[3,6] = 2134$. So 2134 is a factor of δ.

Definition

$Perm(n)$ is set of all factors of length n of the infinite permutation δ.

The factor complexity $\lambda(n)$ of a permutation δ is the cardinality of the set $Perm(n)$.

Results

In this paper we introduce a new class of infinite permutations which are constructed as some analogue of fixed point of morphism from combinatorics on words. We find a formula for their factor complexity. We also prove that permutations that we have introduced can not be generated by word over finite alphabet.

A fixed point of morphism

A fixed points of morphisms are one of the main source of constructing infinite words with different nontrival properties in combinatorics on words.

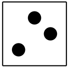
Example. Let $\varphi(0) = 011, \varphi(1) = 100.$
$0 \to \varphi(0) = 011 \to \varphi(011) = 011100100 \to$
$\varphi(011100100) = 011100100100011011100011011\ldots$

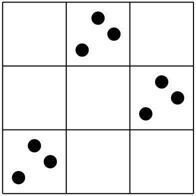
The main idea of constructing

When we construct a fixed point of morphism we substitute the letters on the blocks.

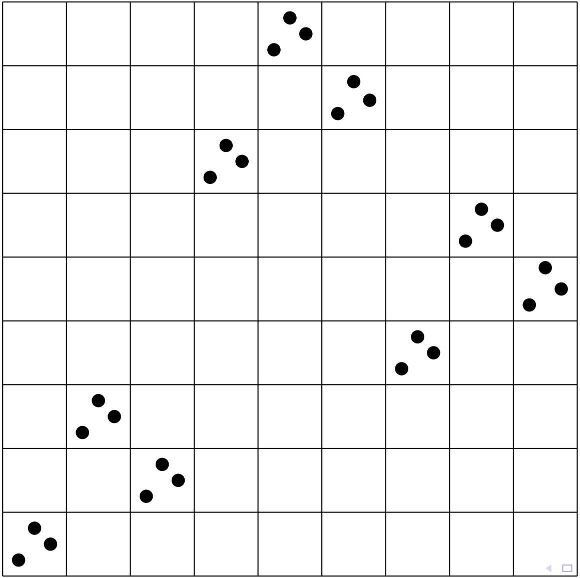
We would try apply the same idea to the permutations.

Iterations of permutation

 $\tau = 132$



Iterations of permutations



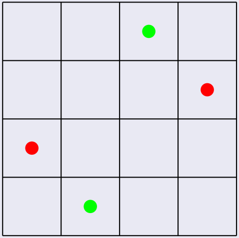
Color permutations

Definition

Let $\Sigma = \{1, 2, ..., q\}$ be q–letter alphabet. A pair $x = (\sigma(x), c(x))$, where $\sigma(x)$ is a finite permutation of length n and $c(x) = c(x)_1 c(x)_2 \dots c(x)_n$ is a word over Σ, is called *finite color permutation* of length n.

Example.

$\Sigma = \{1, 2\}$, $x = (2143, 1221)$



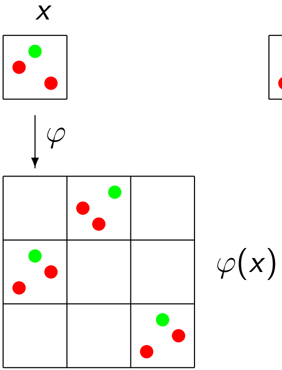
Basic definitions

Definition of $\varphi(x)$.

Let $\tau_1, \tau_2, \dots, \tau_q$ be color permutations of length l. Let x be a finite color permutation of length n. Now we define the image of x under action of φ. We define color permutation $\varphi(x)$ of length nl as follows: if $i \in [(k-1)l+1, kl]$ and $j \in [(m-1)l+1, ml]$ for some k and m ($k \neq m$), then $\gamma_{\varphi(x)}(i;j) = \gamma_x(k;m)$; if $i, j \in [(k-1)l+1, kl]$ for some k, then $\gamma_{\varphi(x)}(i;j) = \gamma_{\tau_{c(x)_k}}(\bar{i};\bar{j})$, where $\bar{i}, \bar{j}$ are residues of numbers i and j modulo l; if $i \in [(k-1)l+1, kl]$ then $c(\varphi(x))_i = c(\tau_{c(x)_k})_{\bar{i}}$ where $\bar{i}$ is a residue of i modulo l.

Example of $\varphi(x)$

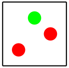
Example. Let $\Sigma = \{1, 2\}$, $x = (231, 121)$, $\tau_1 = (132, 121)$ and $\tau_2 = (312, 112)$.



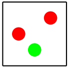
Construction of $\delta = \lim_{n\to\infty} \varphi^n(\tau)$

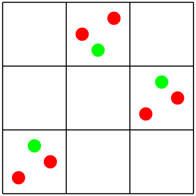
Example. Let $\Sigma = \{1, 2\}$, $\tau_1 = (132, 121)$ and $\tau_2 = (213, 121)$. Then color permutations τ, $\varphi(\tau)$ and $\varphi^2(\tau)$ can be schematically illustrated as follows:

$\tau_1 = \tau$



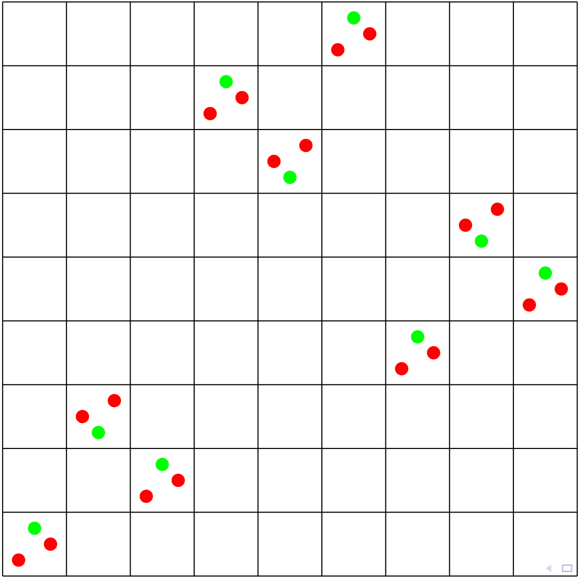
τ_2





$\varphi(\tau)$

Construction of $\delta = \lim_{n\to\infty} \varphi^n(\tau)$



$\varphi^2(\tau)$

The definition of $\delta = \lim_{n\to\infty} \varphi^n(\tau)$

Definition

Consider color permutations $\tau_1, \tau_2, \ldots, \tau_q$ of length $l > 2$ such that $c(\tau_1)_1 = 1$. Let $\tau = \tau_1$. We define infinite colored permutation $\delta = \lim_{n\to\infty} \varphi^n(\tau)$ by the following way. We define color permutation on the set $\{1, 2, \ldots, l\}$ is equal τ. We define color permutation on the set $\{1, 2, \ldots, l^2\}$ is equal $\varphi(\tau)$, and so on, we define every time color permutation on the set $\{1, 2, \ldots, l^s\}$ is equal $\varphi^{s-1}(\tau)$. Thus, we define the color infinite permutation δ on the set of all positive integer numbers.

Note that permutation $\varphi(\delta)$ is δ. In a sense, δ is a fixed point under action of φ.

Conditions for φ

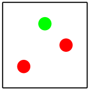
A finite permutation x of length n is called *monotonic* if $x = 12\dots n$ or $x = n(n-1)\dots 1$, otherwise x is called *nonmonotonic*. We say that colored permutation x of length n is *monotonic* (*nonmonotonic*) if $\sigma(x)$ is *monotonic* (*nonmonotonic*).

Let color permutations $\tau_1, \tau_2, \dots, \tau_q$ of length $l > 2$ which satisfy the following properties:

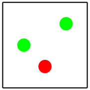
1. Colored permutations $\tau_1, \tau_2, \dots, \tau_q$ are nonmonotonic.
2. Permutations $\sigma(\tau_1), \sigma(\tau_2), \dots, \sigma(\tau_q)$ are distinct.
3. Letters $c(\tau_1)_1, c(\tau_2)_1, \dots, c(\tau_q)_1$ are distinct.
4. Letters $c(\tau_1)_l, c(\tau_2)_l, \dots, c(\tau_q)_l$ are distinct.
5. $c(\tau_1)_1 = 1$.

Example. Let $\Sigma = \{1, 2\}$, $\tau_1 = (132, 121)$ and $\tau_2 = (213, 212)$.

τ_1



τ_2



Then τ_1 and τ_2 satisfy properties 1-5:

1. τ_1 and τ_2 are nonmonotonic.
2. Permutations $\sigma(\tau_1) = 132$ and $\sigma(\tau_2) = 213$ are distinct.
3. Letters $c(\tau_1)_1 = 1$ and $c(\tau_2)_1 = 2$ are distinct.
4. Letters $c(\tau_1)_3 = 1$ and $c(\tau_2)_3 = 2$ are distinct.
5. $c(\tau_1)_1 = 1$.

In this paper we study properties of infinite colored permutation δ, where $\delta = \lim_{n\to\infty} \varphi^n(\tau)$ and colored permutations $\tau_1, \tau_2, \ldots, \tau_q$ satisfy properties 1-5. We find the factor complexity $\lambda(n)$ of infinite permutation $\sigma(\delta)$.

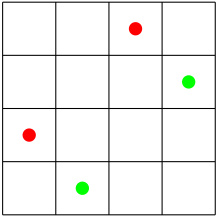
Equivalent permutations

Definition

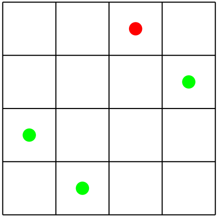
Let $ColPerm(n)$ be the set of all factors of length n of δ.
Define an equivalence relation $\sim$ on $ColPerm(n)$ as follows: let $x \in ColPerm(n)$ and $y \in ColPerm(n)$; then $x \sim y$ if and only if $\sigma(x) = \sigma(y)$ and $c(x)[2, n-1] = c(y)[2, n-1]$.

Example. $x \sim y$

x



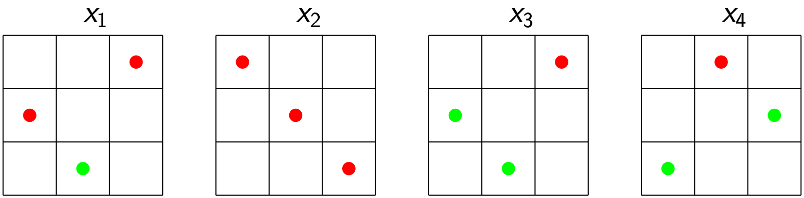
y



Definition

The set of all color permutations x from $ColPerm(n)$ such that $x \sim y$ for some $y \in ColPerm(n)$ and $y \neq x$ is denoted $A(n)$.

Example. Let $ColPerm(3) = \{x_1, x_2, x_3, x_4\}$.



Then $A(3) = \{x_1, x_3\}$.

Recall that we want to find the factor complexity $\lambda(n)$ of infinite permutation $\sigma(\delta)$. Let $N = (2l-2)l+1$. Let d be the number such that $ld - l + 1 > N$. Consider the following partition of $[dl+1, +\infty)$:

$[dl+1, (d+1)l]$
$[(d+1)l+1, (d+2)l]$
$[(d+2)l+1, (d+3)l]$

.....

$[(dl-1)l+1, dl^2]$
$[dl^2+1, (d+1)l^2]$
$[(d+1)l^2+1, (d+2)l^2]$
$[(d+1)l^2+1, (d+2)l^2]$
$[(dl-1)l^2+1, dl^3]$
$[dl^3+1, (d+1)l^3]$

.....

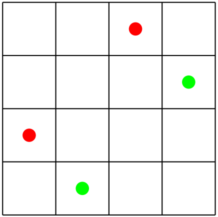
So for any $n > N$ there exists $k \in \{d, ..., ld-1\}$ such that inequality $kl^s + 1 \leq n \leq (k+1)l^s$ holds for some integer $s > 0$.

Definition

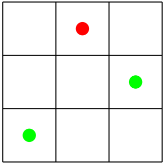
$A(k+2) = A_1 \bigcup A_2 \bigcup \ldots \bigcup A_m$ — partition of $A(k+2)$ into equivalence classes with respect $\sim$.

Definition. The finite color permutation x of length n such that $\sigma(x) = \sigma(x)[m, m+n-1]$ and $c(x)_i = c(x)_{m+i-1}$ for all i is denoted by $x[m, m+n-1]$.

Example.



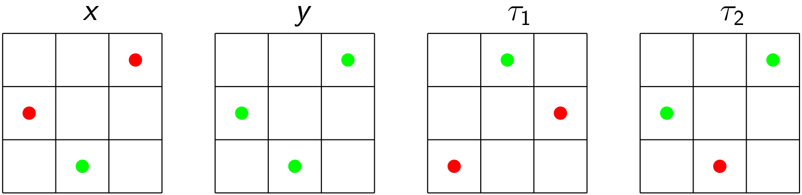
x



$x[2,4]$

Permutation $x[m, m]$ of length 1 is denoted by $x[m]$.

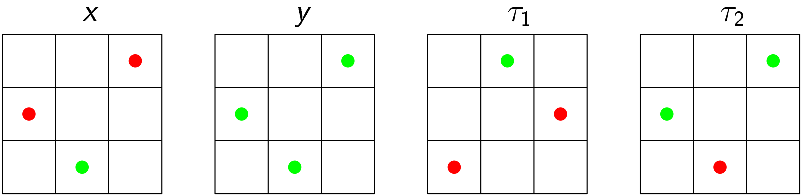
Definition. $P(i) = \{(x[1], x[k+2]) | x \in A_i\}$. Then we define sequence $P_i(s)$ as follows: $P_i(0) = P(i)$ and
$P_i(t+1) = \{\varphi(x)[l], \varphi(y)[1]) | (x, y) \in P_i(t)\}$.
Example. Let $A_i = \{x, y\}$. Then $P_i(0) = \{(1, 1), (1, 1)\}$.



Then $\varphi(1)[3] = \varphi(1)[1] = 1$ and $\varphi(1)[3] = \varphi(1)[1] = 1$.
So $P_i(1) = \{(1, 1), (1, 1)\}$. One can analogously obtain that $P_i(s) = \{(1, 1), (1, 1)\}$ for any s.

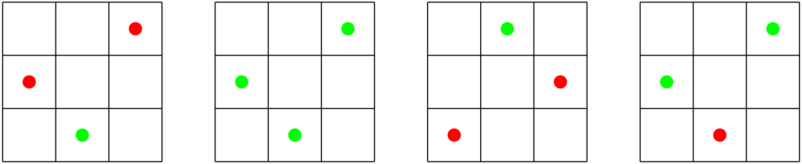
Definition. Define function $\lambda(s, i, r_1, r_2)$: if $r_1, r_2 \leq l$, then $\lambda(s, i, r_1, r_2)$ is the number of distinct pairs $(\sigma(\varphi(x)[l - r_1 + 1, l]), \sigma(\varphi(y)[1, r_2])$ such that $(x, y) \in P_i(s - 1)$; else $\lambda(s, i, r_1, r_2) = 0$.

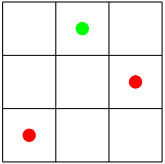
Example. Let $A_i = \{x, y\}$ and $r_1 = r_2 = 2$.



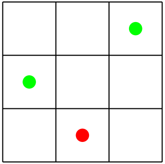
Since $r_1 = r_2 = 2$ and $l = 3$, we have that $\lambda(s, i, r_1, r_2)$ is the number of distinct pairs $(\sigma(\varphi(x)[2, 3]), \sigma(\varphi(y)[1, 2])$ such that $(x, y) \in P_i(s - 1)$.

x $\quad y \quad$ τ_1 $\quad \tau_2$





$\varphi(1)$



$\varphi(1)$

By previous example we have $P_i(s-1) = \{(1,1),(1,1)\}$.
Then $\sigma(\varphi(1)[1,2]) = \sigma(\varphi(1)[2,3]) = 12$ and
$\sigma(\varphi(1)[2,3]) = \sigma(\varphi(1)[1,2]) = 21$. It is clear that
$(12,21) \neq (21,12)$.
So in this case $\lambda(s,i,r_1,r_2) = 2$ for $r_1 = r_2 = 2$.

Definition

Let

$$g(s,r) = \sum_{r_1+r_2=r,i} \lambda(s,i,r_1,r_2)$$

for $r \leq 2l$.

Definition

Let $\chi(n)$ be the number of distinct color factors of length n of δ.

The main theorem

Let d be the number such that $ld - l + 1 > N$. Let $k \in \{d, ..., ld - 1\}$ such that inequality $kl^s + 1 \leq n \leq (k+1)l^s$ holds for some integer $s > 0$.

Theorem 1

Let

$$\delta = \lim_{n \to \infty} \varphi^n(\tau).$$

Let $n = kl^s + r$, where $r \in [1; l^s]$, $k \in \{d, ..., ld - 1\}$ and $n > N$. Then factor complexity $\lambda(n)$ of infinite permutation $\sigma(\delta)$ is calculated as follows:

1. If $r > 2l$, then $\lambda(n) = (r-1)\chi(k+2) + (l^s - r + 1)\chi(k+1)$.
2. If $r \leq 2l$, then $\lambda(n) = g(s, r) + (r-1)\chi(k+2) + (l^s - r + 1)\chi(k+1) - (r-1)|A(k+2)|$.

Connections with permutations generated by words

Let $\omega = \omega_1\omega_2\omega_3\dots$ be an infinite word over the alphabet Σ. A word ω corresponds to the binary real number $R_\omega(i) = 0, \omega_i\omega_{i+1}\dots = \sum_{k\geq 0} \omega_{i+k}2^{-(k+1)}$. An infinite permutation x such that $x(i) < x(j)$ if and only if $R_\omega(i) < R_\omega(j)$ is denoted by δ_ω. We say that infinite nonperiodic word ω *generate* infinite permutation x if $x = \delta_\omega$.

Theorem 2

Permutation $\sigma(\delta)$, where

$$\delta = \lim_{n\to\infty} \varphi^n(\tau)$$

can not be generated by word over finite alphabet.

Thank you for your attention!