Similarity Relations and Repetition-Freeness

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Outline

1. Background
2. Similarity relation
3. Relational repetition-freeness
4. Square-freeness and overlap-freeness
5. Cube-freeness and 3^+-freeness

Repetition-freeness – Background

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- similarity relations: Halava, Harju, Kärki

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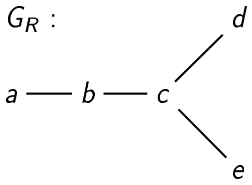
- A similarity relation R can be represented by an undirected graph $G_R = (V, E)$, where $V = \mathcal{A}$ and $(a, b) \in E$ iff $a \, R \, b$.

Similarity relation
Example



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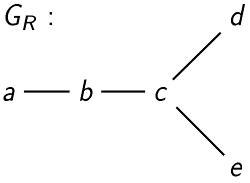

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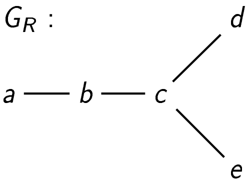

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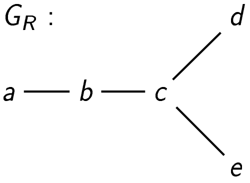

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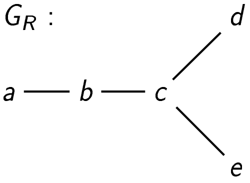

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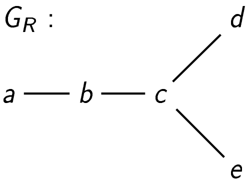

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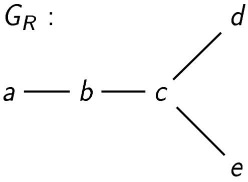

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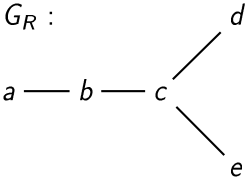

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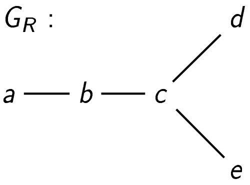

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- $(abcde, ccccc) \notin R$

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- $a\diamond cb\diamond \uparrow \diamond bcb\diamond$, $\quad a\diamond cb\diamond \not\uparrow b\diamond cba$
- Compatibility relation of partial words corresponds to the similarity relation $R_\uparrow = \langle\{(\diamond, a) \mid a \in \mathcal{A}\}\rangle$.

- Let $\mathcal{A}_n = \{0, 1, \ldots n-1\}$.
- *Chain relation* $\overline{R}_n$ is a similarity relation on $\mathcal{A}_n^*$ defined by $\overline{R}_n = \langle\{(i, i+1) \mid i = 0, 1, \ldots, n-2\}\rangle$
- *Cyclic relation* $\mathring{R}_n$ is a similarity relation on $\mathcal{A}_n^*$ defined by $\mathring{R}_n = \langle\{(i, i+1) \mid i = 0, 1, \ldots, n-2\} \cup \{(n-1, 0)\}\rangle$.

Similarity relation

Chain relations and cyclic relations

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- $543210 \; \mathring{R}_6 \; 432105$
- $(543210, 432105) \notin \overline{R}_6$

$G_{\overline{R}_6}$:

$0 - 1 - 2 - 3 - 4 - 5$

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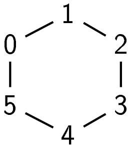

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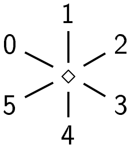

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- A *global R-overlap* is a local R-overlap $uu'vv'w$ such that also $u\,R\,w$.

- Let R be the chain relation $\overline{R}$ or the cyclic relation $\mathring{R}$. An R-repetition of order k is globally (resp. locally) *n-avoidable* if there exists an infinite word w over the alphabet $\mathcal{A}_n$ such that *each letter* of the alphabet $\mathcal{A}_n$ occurs *infinitely many times* in w and w is globally (resp. locally) (R_n, k)-free.

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- An $R_\uparrow$-repetition of order k is globally (resp. locally) *n-avoidable* if there exists an infinite word w over the alphabet $\mathcal{A}_n \cup \{\diamond\}$ such that *each letter* of the alphabet $\mathcal{A}_n \cup \{\diamond\}$ occurs *infinitely many times* in w and w is globally (resp. locally) $(R_\uparrow, k)$-free.

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Let R be the chain relation $\overline{R}$, the cyclic relation $\mathring{R}$ or the relation $R_{\uparrow}$.

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- The indices $\gamma(R,k^+)$ and $\lambda(R,k^+)$ are defined as above by replacing k by k^+.

- $\lambda(R, k) \geq \gamma(R, k)$

Relational repetition-freeness
Avoidability index

- $\lambda(R, k) \geq \gamma(R, k)$
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- $\lambda(\overline{R}, k) \geq \gamma(\overline{R}, k) \geq 3$

Relational repetition-freeness

Avoidability index

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- $\lambda(\mathring{R}, k) \geq \gamma(\mathring{R}, k) \geq 4$
- If R-repetitions of order k are n-avoidable for $R = \overline{R}$, $R = \mathring{R}$ or $R = R_\uparrow$, then they are also $(n+1)$-avoidable.

Avoidability results

	$\overline{R}$		$\mathring{R}$		$R_{\uparrow}$	
	γ	λ	γ	λ	γ	λ
2	6	6	7	7	-	-
2^{+}	4	4	5	5	3	-
3	3	4	4	5	2	-
3^{+}	3	3	4	4	2	2

Square-freeness and overlap-freeness

	$\overline{R}$		$\mathring{R}$		$R_{\uparrow}$	
	γ	λ	γ	λ	γ	λ
2	6	6	7	7	-	-
2^+	4	4	5	5	3	-
3	3	4	4	5	2	-
3^+	3	3	4	4	2	2

- One cannot avoid *trivial squares* $a\diamond$ or $\diamond a$, where a is a letter.
- One cannot avoid *local cubes* $a\diamond b$, where a and b are letters.

Square-freeness and overlap-freeness

	$\overline{R}$		$\mathring{R}$		$R_\uparrow$	
	γ	λ	γ	λ	γ	λ
2	6	6	7	7	-	-
2^+	4	4	5	5	3	-
3	3	4	4	5	2	-
3^+	3	3	4	4	2	2

- There exists uncountably many infinite words with an infinite number of holes over a three-letter alphabet such that they do not contain any squares other than the trivial ones (Halava,Harju,Kärki'08; Blanchet-Sadri,Mercaş,Scott'09).

Square-freeness and overlap-freeness

	$\overline{R}$		$\mathring{R}$		$R_{\uparrow}$	
	γ	λ	γ	λ	γ	λ
2	6	6	7	7	-	-
2^+	4	4	5	5	3	-
3	3	4	4	5	2	-
3^+	3	3	4	4	2	2

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- $\gamma(R_{\uparrow}, 2^+) > 2$ (BMS'09; HHK,Séébold'09)

Square-freeness and overlap-freeness

	$\overline{R}$		$\mathring{R}$		$R_{\uparrow}$	
	γ	λ	γ	λ	γ	λ
2	6	6	7	7	-	-
2^+	4	4	5	5	3	-
3	3	4	4	5	2	-
3^+	3	3	4	4	2	2

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- $\gamma(R_{\uparrow}, 2^+) > 2$ (BMS'09; HHK,Séébold'09)

Theorem (Blanchet-Sadri,Mercaș,Scott'09)

$\gamma(R_{\uparrow}, 2^+) = 3$

Square-freeness and overlap-freeness

	$\overline{R}$		$\mathring{R}$		$R_{\uparrow}$	
	γ	λ	γ	λ	γ	λ
2	6	6	7	7	-	-
2^+	4	4	5	5	3	-
3	3	4	4	5	2	-
3^+	3	3	4	4	2	2

Theorem (K.'12)

$$\gamma(\overline{R}, 2) = \lambda(\overline{R}, 2) = 6, \qquad \gamma(\mathring{R}, 2) = \lambda(\mathring{R}, 2) = 7,$$
$$\gamma(\overline{R}, 2^+) = \lambda(\overline{R}, 2^+) = 4, \qquad \gamma(\mathring{R}, 2^+) = \lambda(\mathring{R}, 2^+) = 5.$$

Cube-freeness and 3^+-freeness

	$\overline{R}$		$\mathring{R}$		$R_\uparrow$	
	γ	λ	γ	λ	γ	λ
2	6	6	7	7	-	-
2^+	4	4	5	5	3	-
3	3	4	4	5	2	-
3^+	3	3	4	4	2	2

Theorem (Manea, Mercaș'07)

There exist infinitely many cube-free infinite partial words over a binary alphabet containing an infinite number of holes.

Corollary

For any $k \geq 3$, we have

$$\gamma(R_\uparrow, k) = 2, \quad \gamma(\overline{R}, k) = 3 \quad \textit{and} \quad \gamma(\mathring{R}, k) = 4 \ .$$

Cube-freeness and 3^+-freeness

	$\overline{R}$		$\mathring{R}$		$R_\uparrow$	
	γ	λ	γ	λ	γ	λ
2	6	6	7	7	-	-
2^+	4	4	5	5	3	-
3	3	4	4	5	2	-
3^+	3	3	4	4	2	2

- For any letters x and y, the factor $x1y$ is a local $\overline{R}_3$-cube

Cube-freeness and 3^+-freeness

	$\overline{R}$		$\mathring{R}$		$R_\uparrow$	
	γ	λ	γ	λ	γ	λ
2	6	6	7	7	-	-
2^+	4	4	5	5	3	-
3	3	4	4	5	2	-
3^+	3	3	4	4	2	2

- For any letters x and y, the factor $x1y$ is a local $\overline{R}_3$-cube
- Local $\overline{R}$-overlaps are 4-avoidable

Cube-freeness and 3^+-freeness

	$\overline{R}$		$\mathring{R}$		$R_\uparrow$	
	γ	λ	γ	λ	γ	λ
2	6	6	7	7	-	-
2^+	4	4	5	5	3	-
3	3	4	4	5	2	-
3^+	3	3	4	4	2	2

- For any letters x and y, the factor $x1y$ is a local $\overline{R}_3$-cube
- Local $\overline{R}$-overlaps are 4-avoidable

$$\Rightarrow \lambda(\overline{R}, 3) = 4$$

Cube-freeness and 3^+-freeness

	$\overline{R}$		$\mathring{R}$		$R_\uparrow$	
	γ	λ	γ	λ	γ	λ
2	6	6	7	7	-	-
2^+	4	4	5	5	3	-
3	3	4	4	5	2	-
3^+	3	3	4	4	2	2

- Assume that there exists a locally $(\mathring{R}_4, 3)$-free word w containing infinitely many occurrences of each letter in $\mathcal{A}_4$.

	$\overline{R}$		$\mathring{R}$		$R_\uparrow$	
	γ	λ	γ	λ	γ	λ
2	6	6	7	7	-	-
2^+	4	4	5	5	3	-
3	3	4	4	5	2	-
3^+	3	3	4	4	2	2

- Assume that there exists a locally $(\mathring{R}_4, 3)$-free word w containing infinitely many occurrences of each letter in $\mathcal{A}_4$.
- Let $\mathcal{A} = \{0, 2\}$ or $\mathcal{B} = \{1, 3\}$.

Cube-freeness and 3^+-freeness

	$\overline{R}$		$\mathring{R}$		$R_\uparrow$	
	γ	λ	γ	λ	γ	λ
2	6	6	7	7	-	-
2^+	4	4	5	5	3	-
3	3	4	4	5	2	-
3^+	3	3	4	4	2	2

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- WLOG: There is a factor 201331 or 201311 in w.
  It is followed only by letters in $\mathcal{B}$: a contradiction.

Cube-freeness and 3^+-freeness

	$\overline{R}$		$\mathring{R}$		$R_\uparrow$	
	γ	λ	γ	λ	γ	λ
2	6	6	7	7	-	-
2^+	4	4	5	5	3	-
3	3	4	4	5	2	-
3^+	3	3	4	4	2	2

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- Local $\mathring{R}$-overlaps are 5-avoidable

Cube-freeness and 3^+-freeness

	$\overline{R}$		$\mathring{R}$		$R_\uparrow$	
	γ	λ	γ	λ	γ	λ
2	6	6	7	7	-	-
2^+	4	4	5	5	3	-
3	3	4	4	5	2	-
3^+	3	3	4	4	2	2

- Assume that there exists a locally $(\mathring{R}_4, 3)$-free word w containing infinitely many occurrences of each letter in $\mathcal{A}_4$.
- Let $\mathcal{A} = \{0, 2\}$ or $\mathcal{B} = \{1, 3\}$.
- WLOG: There is a factor 201331 or 201311 in w. It is followed only by letters in $\mathcal{B}$: a contradiction.
- Local $\mathring{R}$-overlaps are 5-avoidable

$$\Rightarrow \lambda(\mathring{R}, 3) = 5$$

	$\overline{R}$		$\mathring{R}$		$R_{\uparrow}$	
	γ	λ	γ	λ	γ	λ
2	6	6	7	7	-	-
2^+	4	4	5	5	3	-
3	3	4	4	5	2	-
3^+	3	3	4	4	2	2

Theorem

For $k > 3$, we have

$$\lambda(R_{\uparrow}, k) = 2, \quad \lambda(\overline{R}, k) = 3 \quad \textit{and} \quad \lambda(\mathring{R}, k) = 4.$$

The proof is based on a modification of the Thue-Morse word $\tau^{\omega}(a)$, where $\tau^5(a)$ is replaced by the word $abbabaabbaaba\diamond babaababbaabbabaab$.

Avoidability results

	$\overline{R}$		$\mathring{R}$		$R_{\uparrow}$	
	γ	λ	γ	λ	γ	λ
2	6	6	7	7	-	-
2^+	4	4	5	5	3	-
3	3	4	4	5	2	-
3^+	3	3	4	4	2	2

Thank you for your attention!