# A NOTE ON SQUARE-FREE SHUFFLES OF WORDS

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## Shuffles and repetitions

- Both popular items in words and languages.
- Prodinger, Urbanek (1979) considered *binary* words avoiding long squares using (perfect) shuffles.

  These cases for binary words were continued by Rampersad (2004) and Currie, Rampersad (2010).

A shuffle of two words $u$ and $v$:

$$u = u_1 u_2 \cdots u_n \quad \text{and} \quad v = v_1 v_2 \cdots v_n$$

with $u_i, v_i \in A^*$ :

$$u_1 v_1 u_2 v_2 \cdots u_n v_n \in u \sqcup\!\sqcup v\,.$$

It is a perfect shuffle if $u_i$ and $v_i$ are single letters:

$$aabb \sqcup\!\sqcup_P bbab = ababbabb$$

# Self-shuffle

- Charlier, Kamae, Puzynina, Zamboni (2013):
  infinite word $w$ obtained by shuffling it with itself:

$$w \in w \sqcup\!\sqcup w \,.$$

- [CKPZ] A neat proof for:

  *the Fibonacci word $f$ can be self-shuffled.*

  Indeed, $f = \varphi(f)$ for $\varphi(0) = 01,\ \varphi(1) = 0$.
  Hence $\varphi^2(a) = \varphi(a)a$ for $a \in \{0,1\}$. So

$$\begin{aligned} f &= a_1 a_2 \cdots = \varphi(a_1)\varphi(a_2)\cdots = \varphi^2(a_1)\varphi^2(a_2)\cdots \\ &= \varphi(a_1)a_1\varphi(a_2)a_2\cdots \in f \sqcup\!\sqcup f \,. \end{aligned}$$

- [CKPZ]: $0f$ *cannot* be self-shuffled.
- [CKPZ]: A longer proof for:
  *the Thue-Morse can be self-shuffled.*

**Theorem** [CKPZ]. *For self-shuffled infinite $w \in w \sqcup\!\sqcup w$, each sufficiently long prefix $u$ is Abelian bordered:*

$$u = xvx' \quad (x, x' \text{ have equal Parikh vectors})$$

# Now what?

**Theorem.** *There exist infinite square-free words $u, v$ over a 4-letter alphabet such that the perfect shuffle $u \sqcup\!\sqcup_P v$ is square-free.*

Of course, this is not self-shuffling.
In a 3-letter alphabet the above is not possible, but:

**Theorem.** *There exists an infinite square-free ternary word $u$ such that $u \sqcup\!\sqcup u$ contains a square-free word.*

# Lengths: Problem solved

**Theorem** [TH, Mike Müller (2013) – ArXiv].
*For each $n \geq 3$, there exists a square-free ternary word $u$ of length $n$ such that $u \sqcup\!\sqcup\, u$ contains a square-free word.*

# Notation in the paper

Instead of $⧢$ we used conducting sequences $\beta \in \{0,1\}^*$ that pick letters from the two given words: if $u$ and $v$ are finite words of length $n$, then

$$u ⧢ v = \left\{ \beta[u,v] \mid \beta \in \{0,1\}^{2n} \right\}.$$

For infinite words $u$ and $v$, the binary $\beta$ is also infinite. Also $\beta[u] = \beta[u,u]$.

**Example**. Let $u = 0100$, and $\beta = 00101110$. Then

$$\beta[u] = 01\,0\,0\,100\,0 \in u ⧢ v\,.$$

# Perfect Shuffles

**Example**. Let $u, v \in \{0,1,2\}^*$ be square-free of length $\geq 3$. Then the perfect shuffle $u \sqcup\!\sqcup_P v$ is *not square-free*.

Indeed, let without restriction

$$u = 01a_3 \cdots \qquad \text{and} \qquad v = b_1 b_2 b_3 \cdots .$$

Now $u \sqcup\!\sqcup_P v = 0b_1 1 b_2 a_3 b_3 \cdots$. To avoid squares, $b_1 = 2$ and $b_2 = 0$, and so $a_3 = 2$. Finally, $b_3 = 1$, and $u \sqcup\!\sqcup_P v$ starts with the square 021021; a contradiction.

# Four letters

**Theorem.** *There are square-free words* $u, v \in \{0,1,2,3\}^{\mathbb{N}}$ *such that the perfect shuffle* $u \sqcup\!\sqcup_P v$ *is square-free.*

*Sketch.* Dean (1965): Let $u$ be an infinite square-free word reduced in the free group of two generators: $u$ avoids the factors $02, 20, 13$ and $31$.

Let $u'$ be the dual of $u$ obtained by $0 \leftrightarrow 2$ and $1 \leftrightarrow 3$.
E.g., if $u = 0123$ then $u' = 2301$.
For any reduced square-free $u$, $u \sqcup\!\sqcup_P u'$ is square-free.
(Details omitted.) $\square$

# Dean words

The following morphism (due to Thue) is square-free

$$\begin{aligned}\alpha(0) &= 02102\\ \alpha(1) &= 010212\\ \alpha(2) &= 0121012\end{aligned}$$

Thus so is the uniform $\sigma\colon \{0,1,2\}^* \to \{0,1,2,3\}^*$

$$\begin{aligned}\sigma(0) &= 03210323\\ \sigma(1) &= 01032123\\ \sigma(2) &= 01210123\end{aligned}$$

Hence for each square-free $w \in \{0,1,2\}^*$, the image $\sigma(w)$ is square-free 'Dean word'.

# Single Ternary Words: $u \sqcup\!\sqcup\, u$

**Example**. A square-free $w \in u \sqcup\!\sqcup\, u$ may be obtained in several ways (using different conducting sequences).

Let, $u = 012102010212$ and choose

$$\begin{aligned} \beta_1 =& 000000000001111111101111 \\ \beta_2 =& 000000110100100111101111 \end{aligned}$$

Then $\beta_1[u] = 012102010210121020120212 = \beta_2[u]$.

**Example**. A shuffled word $w \in u \sqcup\!\sqcup\, u$ can be square-free even if $u$ is not.

Let $u = (012)^2$ then the following words in $u \sqcup\!\sqcup\, u$ are square-free:

$$\beta_0[u] = 012010212012, \text{ where } \beta_0 = 000001011111$$
$$\beta_1[u] = 010201210212, \text{ where } \beta_1 = 001001101011$$
$$\beta_2[u] = 010210120212, \text{ where } \beta_2 = 001010011011$$

# Main result

**Theorem.** *There exist infinite square-free* $u \in \{0,1,2\}^{\mathbb{N}}$ *such that* $u \sqcup\!\sqcup u$ *has a square-free word.*

*Sketch.* The words to be shuffled will be images of the 12-uniform morphism $\rho\colon \{0,1,2,3\}^* \to \{0,1,2\}^*$:

$$\begin{aligned}
\rho(0) &= 010210120212\\
\rho(1) &= 012101202102\\
\rho(2) &= 012102010212\\
\rho(3) &= 012102120102\,.
\end{aligned}$$

$$
\begin{aligned}
\rho(0) &= 010210120212 \\
\rho(1) &= 012101202102 \\
\rho(2) &= 012102010212 \\
\rho(3) &= 012102120102\,.
\end{aligned}
$$

- The images $\rho(i)$ are square-free, but

  the morphism $\rho$ is not square-free: the images of $12, 20$ and $30$ have squares.

  For instance, $\rho(20) = 01210 \cdot (201021)^2 \cdot 0120212$.

- We need an extra morphism $\alpha$ to fix this problem.

# Nevertheless

Each $\rho(i)$ can be shuffled to obtain a square-free word: $\sigma(i) \in \rho(i) \sqcup\!\sqcup\, \rho(i)$.

$$
\begin{aligned}
\sigma(0) &= 010210120102120210120212 \\
\sigma(1) &= 012101202101210201202102 \\
\sigma(2) &= 012102010210121020120212 \\
\sigma(3) &= 012102120102101202120102
\end{aligned}
$$

# The fix: $\alpha$

$$\begin{aligned}\alpha(0) &= 1013\\ \alpha(1) &= 1023\\ \alpha(2) &= 1032\end{aligned}$$

- For all $w \in \{0,1,2\}^*$, the images $\alpha(w)$ avoids the ‘forbidden’ words $12, 20$ or $30$.
- The morphism $\alpha$ is surely square-free.

## Combine:

$$
\begin{aligned}
B(i) &= \rho\alpha(i) \\
S(i) &= \sigma\alpha(i)
\end{aligned}
$$

The images of the words $B(i)$ are:

$B(0)$ =012101202102010210120212012101202102012102120102

$B(1)$ =012101202102010210120212012102010212012102120102

$B(2)$ =012101202102010210120212012102120102012102010212

The lengths of these words are 48.

The images of the shuffled words are of length 96:

$$
\begin{aligned}
S(0) =& 012101202101210201202102010210120102120210120212\\
&\cdot 012101202101210201202102012102120102101202120102\\
S(1) =& 012101202101210201202102010210120102120210120212\\
&\cdot 012102010210121020120212012102120102101202120102\\
S(2) =& 012101202101210201202102010210120102120210120212\\
&\cdot 012102120102101202120102012102010210121020120212
\end{aligned}
$$

By Crochemore's criterion and a computer check:
the morphisms $B$ and $S$ are square-free.

Finally:

$$S(w) \in B(w) \sqcup B(w)$$

for infinite square-free words $w$. (Details omitted.) $\square$

**Corollary**. *There exist infinite square-free ternary words that are Abelian periodic:*

$$w_1 w_2 \cdots \qquad \text{where } w_i \sim w_j$$

(same Parikh vectors).

A simpler solution to the corollary was given earlier using stem factorizations.

# Open questions

The main question involves self-shuffling.

**Problem 1.** *Does there exist an infinite square-free word $w$ such that $w \in w \sqcup\!\sqcup\, w$?*

**Problem 2.** *Which words $w \in u \sqcup\!\sqcup\, u$ can be obtained in more than one way from a single $u$?*

**Example**. $u \sqcup\!\sqcup u$ can contain several square-free words.

E.g., $u = 01021201$ gives rise to 3 square-free words, but $v = 01201021$ only one.

There are also square-free words that do not shuffle to any square-free word.

**Example**. Square-free words in $u \sqcup\!\sqcup u$ seem to be relatively sparse compared to the number of all square-free words. How sparse?

(Note that the letters have even parity in $u \sqcup\!\sqcup u$.)

| Length L | # of all square-free | # shuffles $\beta[u]$ | # $\lvert u \rvert = L/2$ |
|---|---|---|---|
| 4 | 18 | 0 | 0 |
| 6 | 42 | **6** | **6** |
| 8 | 78 | 12 | 6 |
| 10 | 144 | 30 | 12 |
| 12 | 264 | 24 | 18 |
| 14 | 456 | 42 | 30 |
| 16 | 798 | 78 | 42 |
| 18 | 1392 | 138 | 36 |
| 20 | 2388 | 228 | 54 |
| 22 | 4146 | 396 | 138 |
| 24 | 7032 | 588 | 168 |
| 26 | 11892 | 1008 | 234 |