Suffix conjugates of certain morphic subshifts

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Outline

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The shift orbit closure

- Let A be a finite alphabet.
- Let $f\colon A^* \to A^*$ be a morphism and

$$\mathbf{x} = f^{\omega}(a) = \lim_{n\to\infty} f^{n}(a) \qquad (a \in A).$$

- Let $\mathcal{X}_f$ be the shift orbit closure of $\mathbf{x}$. That is,

$$\mathcal{X}_f = \{\mathbf{y} \in A^{\mathbb{N}} : \ \mathrm{Fact}(\mathbf{y}) \subseteq \mathrm{Fact}(\mathbf{x})\}$$

A representation of points in $\mathcal{X}_f$

Lemma

If $\mathbf{y}_0 \in \mathcal{X}_f$*, then there exist* $a_0 \in A$ *and* $\mathbf{y}_1 \in A^{\mathbb{N}}$ *such that*

$$\mathbf{y}_0 = s_0 f(\mathbf{y}_1),$$

where s_0 *is a nonempty suffix of* $f(a_0)$ *and* $a_0\mathbf{y}_1 \in \mathcal{X}_f$*.*
Therefore,

$$\mathbf{y}_0 = s_0 f(s_1) f^2(s_2) \cdots f^n(s_n) \cdots$$

So, every $\mathbf{y} \in \mathcal{X}_f$ has a representation as a sequence $s_0, s_1, s_2 \ldots$

A representation of points in $\mathcal{X}_f$

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where s_0 is a nonempty suffix of $f(a_0)$ and $a_0\mathbf{y}_1 \in \mathcal{X}_f$. Therefore,

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So, every $\mathbf{y} \in \mathcal{X}_f$ has a representation as a sequence $s_0, s_1, s_2 \ldots$

Problem

Which sequences $s_0, s_1, s_2, \ldots$ actually represent points in $\mathcal{X}_f$?

Example

- Let $f = \varphi$ be the Fibonacci morphism $0 \mapsto 01$, $1 \mapsto 0$ and $\mathbf{f}$ the Fibonacci word

$$\mathbf{f} = \lim_{n\to\infty} \varphi^n(0) = 0100101001001\cdots$$

- Then

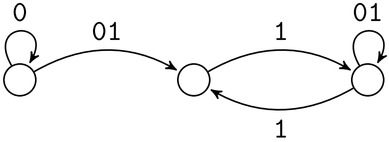
$$\begin{aligned}\mathbf{f} &= 01.0.01.010.01001.01001010.\ldots \\ &= 01.\varphi^1(1).\varphi^2(1).\varphi^3(1).\ldots.\varphi^n(1).\ldots\end{aligned}$$

so $01, 1, 1, 1, 1\ldots$ is a representation of $\mathbf{f}$.

- But no sequence of suffixes that starts with $0, 1, 1, \ldots$ can represent a point in $\mathcal{X}_\varphi$ because

$$0.\varphi(1).\varphi^2(1).\cdots = 0.0.01.\cdots$$

Suffix conjugate of the Fibonacci subshift



Outline

The subshift $(\mathcal{X}, T)$

- Let $f : A^* \to A^*$ be a morphism.
- A fixed point $f^\omega(\alpha) = \lim_{n\to\infty} f^n(\alpha)$ for some $\alpha \in A$.
- Let $\mathcal{X}$ be the shift orbit closure of $f^\omega(\alpha)$.
- The shift map $T \colon A^{\mathbb{N}} \to A^{\mathbb{N}}$ is

$$T(a\mathbf{x}) = \mathbf{x} \qquad (a \in A)$$

- Since T is continuous and $T(\mathcal{X}) \subset \mathcal{X}$, we get a topological dynamical system $(\mathcal{X}, T)$ called a subshift.

The mapping π

- Let $\mathrm{Suff}(f)$ be the set of nonempty suffixes of the f-images of letters:

$$\mathrm{Suff}(f) = \{s \in A^+ : \ f(a) = ps \quad \text{for some } a \in A\}.$$

- Denote $S = \{0, 1, \ldots, |\mathrm{Suff}(f)| - 1\}$ and let $c\colon S \to \mathrm{Suff}(f)$ be a bijection.
- If $\mathbf{s} = s_0 s_1 s_2 \cdots s_n \cdots \in S^{\mathbb{N}}$ with $s_i \in S$, then we write

$$\pi(\mathbf{s}) = c(s_0) f(c(s_1)) f^2(c(s_2)) \cdots f^n(c(s_n)) \cdots$$

- Thus π is a mapping $S^{\mathbb{N}} \to A^{\mathbb{N}}$.
- Define the suffix representation of $\mathcal{X}$:

$$\mathcal{S} = \left\{ \mathbf{s} \in S^{\mathbb{N}} \mid \pi(\mathbf{s}) \in \mathcal{X} \right\}.$$

Example

- Let $f = \varphi$ with $\varphi(0) = 01$ and $\varphi(1) = 0$.
- Let $f^\omega(\alpha) = \mathbf{f}$ be the Fibonacci word.
- Then $\mathrm{Suff}(f) = \{0, 1, 01\}$ and $S = \{0, 1, 2\}$.
- Let $c \colon S \to \mathrm{Suff}(f)$ with $c(0) = 0$, $c(1) = 1$, and $c(2) = 01$.
- If $\mathbf{s} \in S^{\mathbb{N}}$ with $\mathbf{s} = 021\cdots$, then

$$\begin{aligned}\pi(\mathbf{s}) &= c(0).\varphi(c(2)).\varphi^2(c(1))\cdots \\ &= 0.\varphi(01).\varphi^2(1)\cdots \\ &= 0.010.01\cdots\end{aligned}$$

Outline

Circular morphisms

Definition

A morphism $f\colon A^* \to A^*$ is circular on $U \subseteq A^*$ if f is injective on U and every sufficiently long $w \in A^*$ satisfies:

1. There exists a pair $(w_1, w_2) \in A^* \times A^*$ such that
$$w = w_1 w_2$$
2. For all $v_1, v_2 \in A^*$,
$$v_1 w v_2 \in f(U) \implies v_1 w_1 \in f(U) \quad \text{and} \quad w_2 v_2 \in f(U).$$

Example

- The Fibonacci morphism $\varphi\{0,1\}^* \to \{0,1\}^*$ with

$$\varphi(0) = 01 \qquad \text{and} \qquad \varphi(1) = 0$$

  is circular on $\{0,1\}^*$.

- The Thue-Morse morphism $\mu\colon \{0,1\}^* \to \{0,1\}^*$ with

$$\mu(0) = 01 \qquad \text{and} \qquad \mu(1) = 10$$

  is not circular on $\{0,1\}^*$ but it is circular on Fact($\mathbf{t}$), where $\mathbf{t}$ is the Thue-Morse word.

Mossé's Theorem

Theorem (Mossé, 1991)

If f is a primitive morphism with an aperiodic fixed point $\mathbf{x}$, then f is circular on $Fact(\mathbf{x})$.

The morphism class $\mathcal{N}$

Definition

Let $f\colon A^* \to A^*$ be a morphism and let $\mathbf{x} = f^\omega(a)$ $(a \in A)$ be aperiodic. Then we say that f is in class $\mathcal{N}$ if

1. f is circular on $\mathrm{Fact}(\mathbf{x})$.
2. the set $f(A)$ is a suffix code.
3. for every $a \in A$, we have $|f^n(a)| \to \infty$ as $n \to \infty$.

Remark

If f is primitive, $\mathbf{x}$ aperiodic, and $f(A)$ a suffix code, then $f \in \mathcal{N}$ by Mossé's Theorem.

Example

Example

- The Fibonacci morphism $\varphi(0) = 01$, $\varphi(1) = 0$ is primitive.
- $\varphi\big(\{0,1\}\big) = \{01, 1\}$ is a suffix code.
- $\mathbf{f} = \varphi^\omega(0)$ is aperiodic.
- Thus $\varphi \in \mathcal{N}$.

Example

- The Thue-Morse morphism $\mu(0) = 01$, $\mu(1) = 10$ is primitive.
- $\mu\big(\{0,1\}\big) = \{01, 10\}$ is a suffix code.
- $\mu^\omega(0)$ is aperiodic.
- Thus $\mu \in \mathcal{N}$.

Motivating class $\mathcal{N}$

Recall the suffix representation

$$\mathcal{S} = \{ \mathbf{s} \in S^{\mathbb{N}} \mid \pi(\mathbf{s}) \in \mathcal{X} \}.$$

and the shift map $T(a\mathbf{x}) = \mathbf{x} \qquad (a \in A)$.

Lemma

If $f \in \mathcal{N}$, then $T(\mathcal{S}) \subseteq \mathcal{S}$. Thus $(\mathcal{S}, T)$ is a subshift.

Lemma

If $f \in \mathcal{N}$, then $\pi : \mathcal{S} \to \mathcal{X}$ is injective.

The mappings λ and G

Definition

Let $\lambda\colon S^* \to S^*$ be the morphism given by

$$\lambda(s) = \begin{cases} c^{-1}(f(c(s))) & \text{if } c(s) \in A; \\ c^{-1}(f(a))c^{-1}(u) & \text{if } c(s) = au \text{ with } a \in A \text{ and } u \in A^+ \end{cases}$$

Write

$$S_1 = \{s \in S : \ c(s) \in A\}.$$

Definition

Let $G\colon S^{\mathbb{N}} \to S^{\mathbb{N}}$ be defined by

$$G(\mathbf{s}) = \begin{cases} \lambda(ps)\mathbf{t} & \text{if } \mathbf{s} = ps\mathbf{t} \text{ with } p \in S_1^* \text{ and } s \in S \setminus S_1 \\ \lambda(\mathbf{s}) & \text{if } \mathbf{s} \in S_1^{\mathbb{N}}. \end{cases}$$

The Suffix Conjugate

Let $H\colon \mathcal{S} \to \mathcal{S}$ be the mapping defined by $H = T \circ G$.

Theorem (Currie, Rampersad, S. 2013)

If $f \in \mathcal{N}$, then $\pi \circ H = T \circ \pi$ and $\pi\colon (\mathcal{S}, H) \to (\mathcal{X}, T)$ is a conjugacy.

$$\begin{tikzcd} \mathcal{S} \arrow[r, "H"] \arrow[d, "\pi"] & \mathcal{S} \arrow[d, "\pi"] \\ \mathcal{X} \arrow[r, "T"] & \mathcal{X} \end{tikzcd}$$

Definition

If $f \in \mathcal{N}$, we call $(\mathcal{S}, H)$ the suffix conjugate of $(\mathcal{X}, T)$

Outline

The mapping H_φ

Lemma

The mapping $H_\varphi \colon \mathcal{S}_\varphi \to \mathcal{S}_\varphi$ *of the suffix conjugate* $(\mathcal{S}_\varphi, H_\varphi)$ *of the Fibonacci subshift* $(\mathcal{X}_\varphi, T)$ *is given by*

$$H_\varphi(\mathbf{s}) = \begin{cases} 1\mathbf{z} & \text{if } \mathbf{s} = 2\mathbf{z}; \\ \lambda(x2)\mathbf{z} & \text{if } \mathbf{s} = ax2\mathbf{z} \text{ with } a \in \{0,1\},\ x \in \{0,1\}^*; \\ \lambda(\mathbf{z}) & \text{if } \mathbf{s} = a\mathbf{z} \text{ with } a \in \{0,1\} \text{ and } \mathbf{z} \in \{0,1\}^{\mathbb{N}}, \end{cases}$$

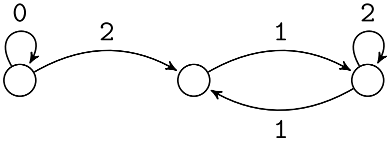
where λ *is the morphism*

$$\lambda(1) = 0, \quad \lambda(0) = 2, \quad \text{and} \quad \lambda(2) = 21.$$

The language of the suffix conjugate

Theorem (Currie, Rampersad, S. 2013)

The language $\mathcal{L}(\mathcal{S}_\varphi)$ of the suffix conjugate $(\mathcal{S}_\varphi, H_\varphi)$ of the Fibonacci subshift $(\mathcal{X}_\varphi, T)$ is regular. An infinite word $\mathbf{s} \in S^{\mathbb{N}}$ is in $\mathcal{S}_\varphi$ if and only if it is the label of an infinite walk on the graph below.



Example

Let $\mathbf{f} \in \mathcal{X}_\varphi$ be the Fibonacci word. Then

- $\mathbf{s} = 2121^\omega \quad \xrightarrow{\pi} \quad 010\mathbf{f}$

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- $H_\varphi^4(\mathbf{s}) = 1^\omega \xrightarrow{\pi} T\mathbf{f}$

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- $H_\varphi^4(\mathbf{s}) = 1^\omega \xrightarrow{\pi} T\mathbf{f}$
- $H_\varphi^5(\mathbf{s}) = 0^\omega \xrightarrow{\pi} T^2\mathbf{f}$

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- $H_\varphi^4(\mathbf{s}) = 1^\omega \xrightarrow{\pi} T\mathbf{f}$
- $H_\varphi^5(\mathbf{s}) = 0^\omega \xrightarrow{\pi} T^2\mathbf{f}$
- $H_\varphi^6(\mathbf{s}) = 2^\omega \xrightarrow{\pi} T^3\mathbf{f}$

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- $H_\varphi^4(\mathbf{s}) = 1^\omega \xrightarrow{\pi} T\mathbf{f}$
- $H_\varphi^5(\mathbf{s}) = 0^\omega \xrightarrow{\pi} T^2\mathbf{f}$
- $H_\varphi^6(\mathbf{s}) = 2^\omega \xrightarrow{\pi} T^3\mathbf{f}$
- $H_\varphi^7(\mathbf{s}) = 12^\omega \xrightarrow{\pi} T^4\mathbf{f}$

Points in the orbit of the Fibonacci word

- An infinite word $\mathbf{y} \in A^{\mathbb{N}}$ is a tail (or suffix) of $\mathbf{x} \in A^{\mathbb{N}}$ if $T^n\mathbf{x} = \mathbf{y}$ for some $n \geq 0$.
- The orbit of $\mathbf{x} \in A^{\mathbb{N}}$ is the set

$$\{\mathbf{y} \in A^{\mathbb{N}} : \mathbf{x} \text{ is a tail of } \mathbf{y} \text{ or vice versa}\}$$

Theorem (Currie, Rampersad, S. 2013)

$T^2\mathbf{f}$ splits the orbit of the Fibonacci word $\mathbf{f}$ in the following way: Let $\mathbf{s} \in \mathcal{S}_\varphi$.

- *$T^2\mathbf{f}$ is a proper tail of $\pi(\mathbf{s}) \iff 1^\omega$ is a tail of $\mathbf{s}$.*
- *$\pi(\mathbf{s})$ is a proper tail of $T^2\mathbf{f} \iff 2^\omega$ is a tail of $\mathbf{s}$.*

Incidentally, $\pi(0^\omega) = T^2\mathbf{f}$.

Outline

The mapping H_μ

Lemma

The mapping $H_\mu \colon \mathcal{S}_\mu \to \mathcal{S}_\mu$ of the suffix conjugate $(\mathcal{S}_\mu, H_\mu)$ of the Thue-Morse subshift $(\mathcal{X}_\mu, T)$ is given by

$$H_\mu(\mathbf{s}) = \begin{cases} 1\mathbf{z} & \text{if } \mathbf{s} = 2\mathbf{z}; \\ 0\mathbf{z} & \text{if } \mathbf{s} = 3\mathbf{z}; \\ \lambda(\mathbf{z}) & \text{if } \mathbf{s} = a\mathbf{z} \text{ with } a \in \{0,1\} \text{ and } \mathbf{z} \in \{0,1\}^{\mathbb{N}}; \\ \lambda(x2)\mathbf{z} & \text{if } \mathbf{s} = ax2\mathbf{z} \text{ with } a \in \{0,1\},\ x \in \{0,1\}^*; \\ \lambda(x3)\mathbf{z} & \text{if } \mathbf{s} = ax3\mathbf{z} \text{ with } a \in \{0,1\},\ x \in \{0,1\}^*, \end{cases}$$

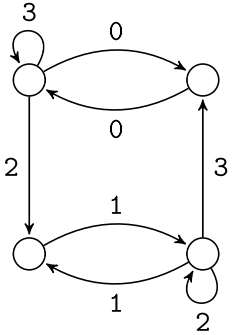
where λ is the morphism given by

$$\lambda(0) = 2, \quad \lambda(1) = 3, \quad \lambda(2) = 21, \quad \text{and} \quad \lambda(3) = 30.$$

The language of $\mathcal{S}_\mu$

Theorem (Currie, Rampersad, S. 2013)

The language $\mathcal{L}(\mathcal{S}_\mu)$ of the suffix conjugate $(\mathcal{S}_\mu, H_\mu)$ of the Thue-Morse subshift $(\mathcal{X}_\mu, T)$ is regular. An infinite word $\mathbf{s} \in S^{\mathbb{N}}$ is in $\mathcal{S}_\mu$ if and only if it is the label of an infinite walk on the graph below.



Points in the orbit of the Thue-Morse word

Theorem (Currie, Rampersad, S. 2013)

Let $\mathbf{s} \in \mathcal{S}_\mu$. *Then*

(i) $\pi(\mathbf{s})$ *is a proper tail of* $T^2\mathbf{t}$ *if and only if* 3^ω *is a tail of* $\mathbf{s}$.

(ii) $T\mathbf{t}$ *is a proper tail of* $\pi(\mathbf{s})$ *if and only if* 1^ω *is a tail of* $\mathbf{s}$.

Thank you!