On Quasiperiodic Morphisms

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Plan

Quasiperiodicity

Definitions. Quasiperiodic words.

- A finite word w is quasiperiodic of quasiperiod $z \neq w$ if w can be covered by occurrences of z.
  - In other words, each letter of w falls within an occurrence of z.
  - We also say that w is u-quasiperiodic.

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3 quasiperiods: *aba*, *abaaba*, *abaababaaba*
The (smallest) quasiperiod is *aba*.

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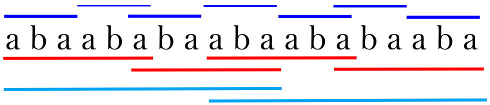

3 quasiperiods: *aba*, *abaaba*, *abaababaaba*
The (smallest) quasiperiod is *aba*.

- Every periodic word is quasiperiodic.

Quasiperiodicity

Definitions. Quasiperiodic words.

Lemma (Apostolico, Ehrenfeucht 1990, 1993)

All quasiperiods of a quasiperiodic finite word are quasiperiodic except the smallest quasiperiod which is a quasiperiod of all the others.

- Marcus (2004): And for infinite words?
  $\Rightarrow$ An infinite word can have infinitely many superprimitive (i.e. non quasiperiodic) quasiperiods (L., Richomme 2004)

Example

The infinite Fibonacci word F (Fixed point of $\varphi : a \mapsto ab,\ b \mapsto a$)

$$abaababaabaababaababaabaababaabaa\ldots$$

First superprimitive quasiperiods:

$$aba,\ abaab,\ abaababaa,\ abaababaabaabab$$

Plan

Origin of the study

Marcus' questions

Another Marcus' question: Are all Sturmian words quasiperiodic?

Origin of the study
Sturmian words

- Well-known properties
  - Infinite words
  - $n + 1$ factors of length n ($\Rightarrow$ binary alphabet $\{a, b\}$)
  - Balanced, non ultimately periodic
    - ⋆ An infinite word on $\{a, b\}$ is balanced if for all factors u and v with $|u| = |v|$, $||u|_a - |v|_a| = 1$.
    - ⋆ An infinite word w is ultimately periodic if $w = uv^\omega$ with u, v finite words.

> A Sturmian word w can be recursively decomposed over $\{L_a, R_a, L_b, R_b\}$.

- Normalized decompositions [Berthé, Holton, Zamboni 2006] $=$ without factors in $\{R_aR_b, R_bR_a, R_aL_a, R_bL_b\}$.

Morphic decomposition of Sturmian words

A Sturmian word w can be decomposed

- in factors $\{a, ab\}$ or $\{ab, b\}$ if w starts with a,
- in factors $\{a, ba\}$ or $\{b, ba\}$ if w starts with b.

- Sturmian morphisms (mapping any Sturmian word into a Sturmian word) composed of:

$$L_a : \begin{cases} a \mapsto a \\ b \mapsto ab \end{cases} \quad L_b : \begin{cases} a \mapsto ba \\ b \mapsto b \end{cases} \quad R_a : \begin{cases} a \mapsto a \\ b \mapsto ba \end{cases} \quad R_b : \begin{cases} a \mapsto ab \\ b \mapsto b \end{cases}$$

Morphic decomposition of Sturmian words

Example : Fibonacci word

$$w = abaababaabaababaababaabaababaaba\ldots$$

Morphic decomposition of Sturmian words

Example : Fibonacci word

$$w = abaababaabaababaababaabaababaaba\ldots \qquad L_a \left\{ \begin{array}{l} a \mapsto a \\ b \mapsto ab \end{array} \right.$$

Morphic decomposition of Sturmian words

Example : Fibonacci word

$$w = \color{red}{ab}\color{black}{aababaabaababaababaabaababaaba\ldots} \qquad \color{blue}{L_a \begin{cases} a \mapsto a \\ b \mapsto ab \end{cases}}$$

$$w = L_a(b$$

Morphic decomposition of Sturmian words

Example : Fibonacci word

$w = ab\textcolor{red}{a}ababaabaababaababaabaababaaba\ldots$ $\quad L_a \begin{cases} a \mapsto a \\ b \mapsto ab \end{cases}$

$w = L_a(ba$

Morphic decomposition of Sturmian words

Example : Fibonacci word

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Quasiperiodic Sturmian words

Theorem [LR2007]

A Sturmian word w is not quasiperiodic if and only if it can be infinitely decomposed over $\{L_a, R_b\}$ or over $\{L_b, R_a\}$.

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- Idea of the proof

$$L_aL_b : \begin{cases} a \mapsto aba \\ b \mapsto ab \end{cases} \qquad L_bL_a : \begin{cases} a \mapsto ba \\ b \mapsto bab \end{cases}$$

$$L_aR_a = R_aL_a : \begin{cases} a \mapsto a \\ b \mapsto aba \end{cases} \qquad L_bR_b = R_bL_b : \begin{cases} a \mapsto bab \\ b \mapsto b \end{cases}$$

$$R_aR_b : \begin{cases} a \mapsto aba \\ b \mapsto ba \end{cases} \qquad R_bR_a : \begin{cases} a \mapsto ab \\ b \mapsto bab \end{cases}$$

All these morphisms are quasiperiodic on Sturmian words.
Moreover any non-erasing morphism preserves quasiperiodicity.

Quasiperiodic Sturmian words

Sturmian morphisms and quasiperiodicity

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  $$L_aL_b,\ L_bL_a,\ L_aR_a = R_aL_a,\ L_bR_b = R_bL_b,\ R_aR_b, \text{ and } R_bR_a.$$

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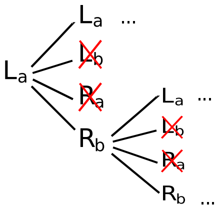

Plan

Strongly quasiperiodic morphisms

A morphism f is *strongly quasiperiodic* if
for all non-quasiperiodic word w, $f(w)$ is quasiperiodic.

Can we decide whether a morphism is strongly quasiperiodic or not?

- Yes!

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- Solution for infinite words (we can do samely for finite words)

Strongly quasiperiodic morphisms

Step 1: restricting quasiperiods

$\mathcal{Q}(f) = \{q \mid \forall \text{ letter } \alpha, |q| \leq 2|f(\alpha)| \text{ and } q \text{ is a factor of } f(\alpha)^3\}$

Example

$f(a) = aba$, $f(b) = ab$

$|q| \leq 4$, q factor of $ababab$ and $abaabaaba$

$$\mathcal{Q}(f) = \{a, b, ab, ba, aba\}$$

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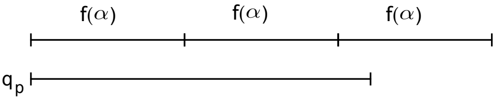
Restriction of quasiperiods

If f is a strongly quasiperiodic on infinite words morphism,
then, for any infinite word $\mathbf{w}$, the (smallest) quasiperiod of $f(\mathbf{w}) \in \mathcal{Q}(f)$

- for p prefix of $\mathbf{w}$, $f(p\alpha^\omega)$ is q_p-quasiperiodic
- $q_p \in \mathcal{Q}(f)$:
  q_p is a factor of $f(\alpha)^\omega \Rightarrow f(\alpha)$ "period" of q_p. So if $|q_p| \geq 2|f(\alpha)|$, q_p is not superprimitive $\Rightarrow q_p$ is a factor of $f(\alpha)^3$.



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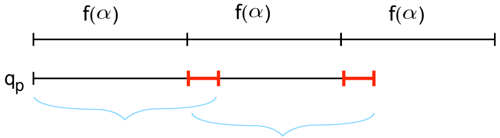
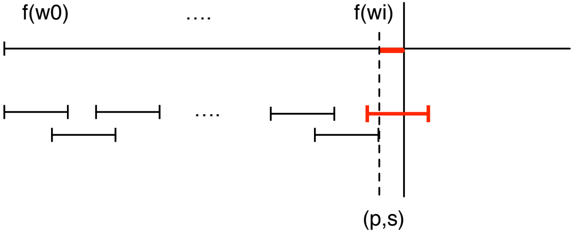


- Since $\mathcal{Q}(f)$ is finite, $\exists q$ such that for an infinity of prefixes p, $q_p = q$.
  $\Rightarrow f(\mathbf{w})$ q-quasiperiodic

Strongly quasiperiodic morphisms

Step 2: an automaton

f: $f(a) = ab$, $f(b) = aba$. $\mathcal{Q}(f) = \{a, b, ab, ba, aba\}$



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$\mathcal{A}_{aba}$:

$$\longrightarrow (\varepsilon, \varepsilon)$$

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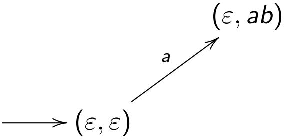

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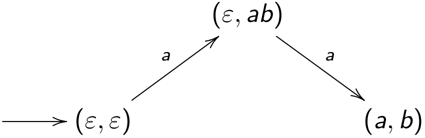

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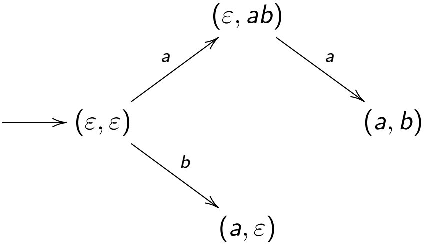

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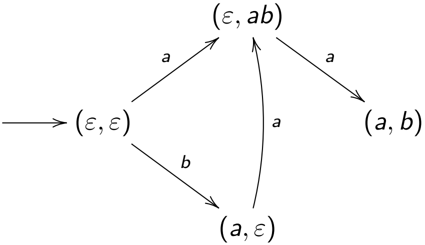

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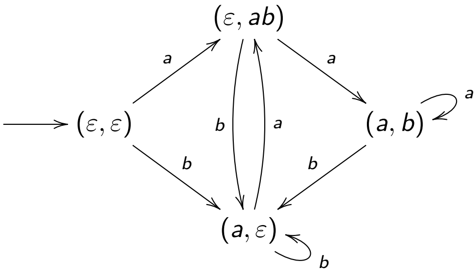

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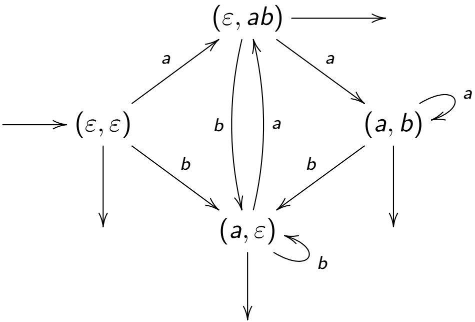

Strongly quasiperiodic morphisms

Step 2: an automaton

Automaton $\mathcal{A}_q$:

- States: pairs (p, s) with ps proper prefix of q with meaning: after reading $f(w)$,
  - $f(w) = xps$
  - xp is q-quasiperiodic
  - ps longest prefix of q that is suffix of $f(w)$

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  Accordingly to meaning of states

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Property

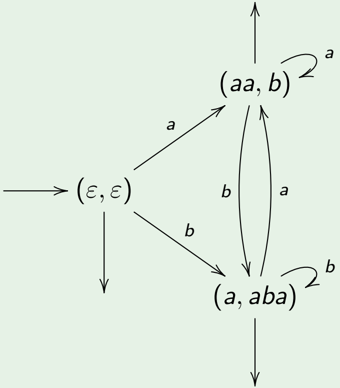
An infinite word $f(\mathbf{w})$ is q-quasiperiodic if and only if all its prefixes are recognized by $\mathcal{A}_q$.

Examples

Example

f: $f(a) = aabaab$, $f(b) = aabaaaba$ and $f(c) = aabaababaabaa$.

$\mathcal{A}_{aabaa}$:



Characterization of strongly quasiperiodic morphisms

Characterization

A morphism f is strongly quasiperiodic on infinite words if and only if

$$A^* = \bigcup_{q \in \mathcal{Q}(f)} \mathcal{L}(\mathcal{A}_q.)$$

where $\mathcal{L}(\mathcal{A}_q)$ is the language recognized by the automaton $\mathcal{A}_q$.

Example

Let f be the morphism defined by $f(a) = abaaba$, $f(b) = baabaaba$. Here follow automata $\mathcal{A}_{aba}$ and $\mathcal{A}_{baaba}$.

$$\begin{tikzcd} & {} & {} \\ {} \arrow[r] & (\varepsilon, \varepsilon) \arrow[u] \arrow[r, "a"] & (a, \varepsilon) \arrow[u] \arrow[loop right, "{a,b}"] \end{tikzcd} \qquad \begin{tikzcd} & {} & {} \\ {} \arrow[r] & (\varepsilon, \varepsilon) \arrow[u] \arrow[r, "b"] & (ba, \varepsilon) \arrow[u] \arrow[loop right, "{a,b}"] \end{tikzcd}$$

Characterization of strongly quasiperiodic morphisms

Corollary: one can decide whether a morphism is strongly quasiperiodic.

Observation: until now "strongly quasiperiodic on infinite words"
Can be adapted to "strongly quasiperiodic on finite words"

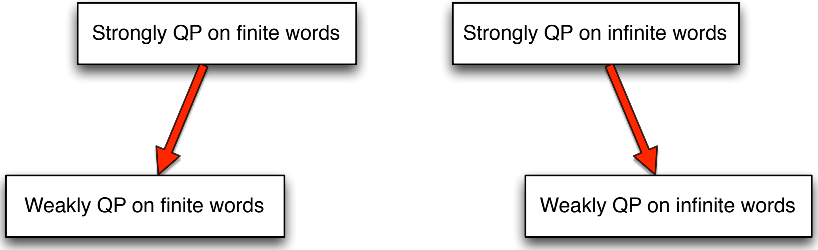
Plan

Basic relations

A morphism f is *weakly quasiperiodic* if
there exists a non-quasiperiodic word w such that $f(w)$ is quasiperiodic.

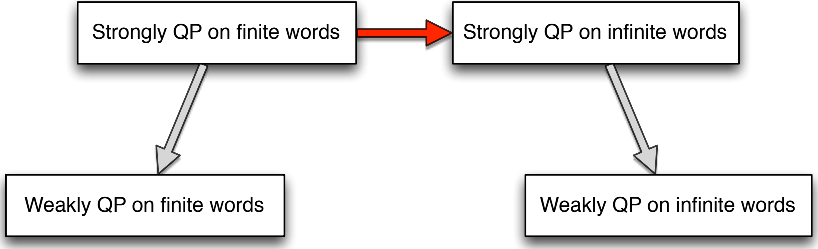
What do we know about relations between strongly and weakly quasiperiodic morphisms?

Basic relations



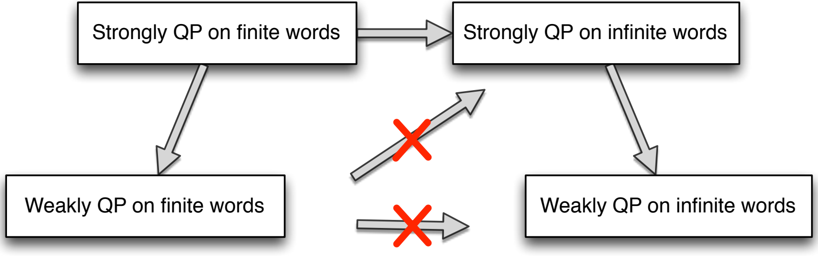
Trivial from definition.

Basic relations



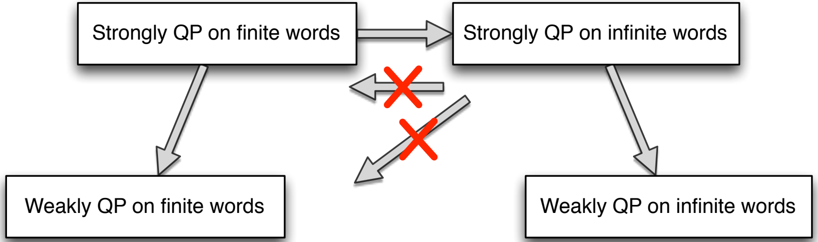
Sketch of the proof.
f is strongly quasiperiodic on finite words. Let q_α be the quasiperiod of $f(\alpha)$.
For any word u, there exists an integer k such that $f(\alpha^k u \alpha^k)$ is q_α-quasiperiodic.
$\Rightarrow$ $f(\alpha u)$ is a prefix of a q_α-quasiperiodic word.
$\Rightarrow$ For any infinite word $\mathbf{w}$, $f(\alpha \mathbf{w})$ is a q_α-quasiperiodic word

Basic relations



$$f(a) = aa$$
$$f(b) = bb$$

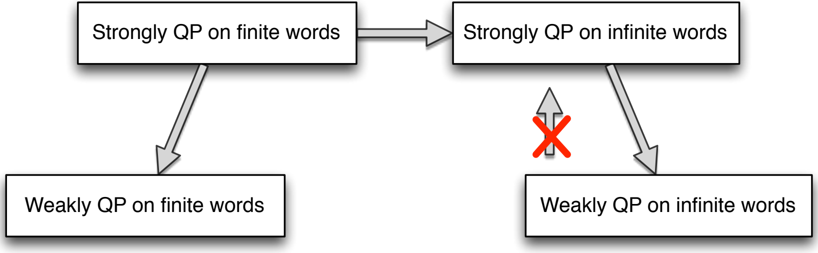
Basic relations



$$f(a) = abaababaabababababaab$$
$$f(b) = abaabaabababababaab.$$

Sketch of the proof: $f(w)$ is *aba*-quasiperiodic for any infinite word w. If u is $q-$quasiperiodic, $|q| \geq |f(a)| = |f(b)|$. Then $q = f(q')$ for some word q' and u is q'-quasiperiodic.

Basic relations



$f(a) = ba$
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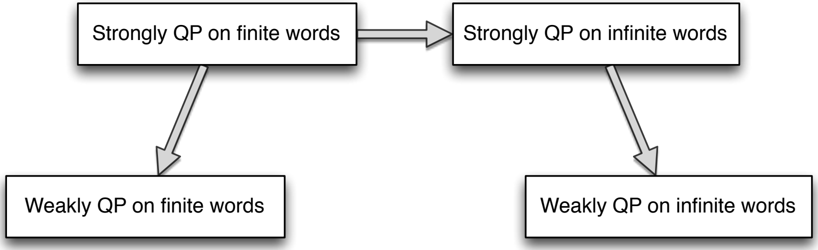

Open question

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  - non-prefix or non-suffix
  - non preserving primitive words

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  Equivalently:
  characterization of non-quasiperiodicity preserving morphisms?

- Sufficient conditions:
  - non-injectivity
  - non-prefix or non-suffix
  - non preserving primitive words

- Intermediary questions:
  - for weakly quasiperiodic on infinite words morphisms, is it true that there exist u, v such that uv^ω non-quasiperiodic and $f(uv^\omega)$ quasiperiodic?
  - if true, any bound on $|u|$, $|v|$?

Thank you for your attention