Subword complexity in free groups

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From the monoid to the group

A alphabet (always finite in this talk)

$A^* = \{$words written with letters in $A\}$ is the free monoid on A (product by concatenation).

A^* embeds into $F(A)$, the free group on A.

The free group on A

To describe $F(A)$, we need the alphabet $A^{\pm 1} = A \cup A^{-1}$. An element of $F(A)$ is a *reduced* word on $A^{\pm 1}$ (no aa^{-1} or $a^{-1}a$)

The product is concatenation/reduction. Every element has an inverse: $(aba^{-2}b^3)^{-1} = b^{-3}a^2b^{-1}a^{-1}$.

Automorphisms of free groups

A key difference between A^* and $F(A)$: the group is much more symmetric.

A free group F has a large group of automorphisms $\mathrm{Aut}(F)$.

Example: $a \mapsto ab^{-1}$, $b \mapsto a^{-1}$ is an automorphism (with inverse $a \mapsto b^{-1}$, $b \mapsto a^{-1}b^{-1}$).

To apply an automorphism to a word, apply it to each letter and reduce: $a^{-1}ba \mapsto ba^{-1}a^{-1}ab^{-1} = ba^{-1}b^{-1}$.

The study of $\mathrm{Aut}(F)$ and $\mathrm{Out}(F) = \mathrm{Aut}(F)/\,\mathrm{Inn}(F)$ (where $\mathrm{Inn}(F)$ is the group of inner automorphisms $i_x : g \mapsto xgx^{-1}$) is a very active topic.

For instance, Bestvina-Feighn-Handel proved a Tits alternative: every subgroup $H \subset \mathrm{Out}(F)$ either contains a nonabelian free group or has an abelian subgroup of finite index.

In particular, either H contains α, β satisfying no relation, or $\alpha^N \beta^N = \beta^N \alpha^N$ for all $\alpha, \beta \in H$ (with N depending only on the rank of F).

Abstract free groups

Other point of view on the symmetry of free groups: like a vector space, a free group F may exist as an abstract object, with no preferred basis A (e.g. a subspace/subgroup).

An element of F is represented by a word once we choose a basis. Different bases give words that differ by an automorphism ("transition matrix").

We are interested in *intrinsic* properties of elements of F.

Two equivalent viewpoints:

• view F as abstract, look for properties independent of the basis A chosen to make F concrete

• view F as $F(A)$, look for properties of words which are invariant under automorphisms.

I'll focus on *length* and *complexity*.

Length

Given a basis A of a free group F, the (word) length $\ell_A(g)$ of any $g \in F$ is the length of the reduced word in $A^{\pm 1}$ representing g, e.g. $\ell_A(aba^{-2}b) = 5$.

If B is another basis, ℓ_A and ℓ_B are Lipschitz-equivalent: $\exists\, M$ (depending on A, B) such that

$$\frac{1}{M} \leq \frac{\ell_A(g)}{\ell_B(g)} \leq M \qquad \forall g$$

(applying an automorphism does not change length of words too much).

In particular, statements like $\ell(g_n) = O(n^2)$, or $\dfrac{\ell(g_n)}{\ell(h_n)} \to \infty$, make sense.

A digression: groups as geometric objects

If G is an arbitrary group, with a finite generating set A, define $\ell_A(g)$ as the minimal length of a word in $A^{\pm 1}$ representing g.

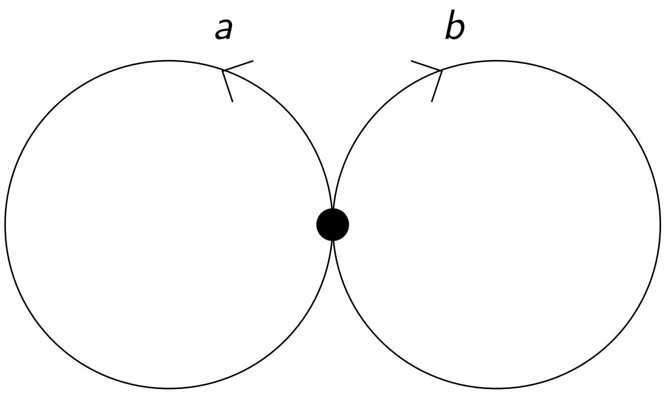
Different generating sets give Lipschitz-equivalent length functions ($\to$ growth of groups).

G may be viewed as a metric space, with distance $d_A(g,h) = \ell_A(g^{-1}h)$.

The large-scale geometry of G is independent of A. This is a key idea of geometric group theory (Gromov).

Back to free groups: geometric interpretation of length

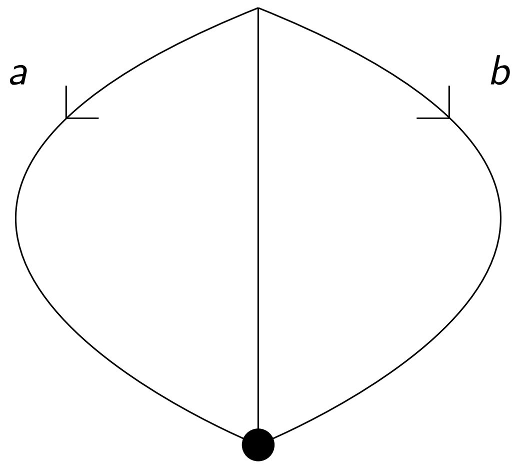
The length of g is the length of the non-backtracking loop representing it.



a, b have length 1

$a^2, ab, a^{-1}b$ have length 2.

Using a different graph with the same fundamental group gives a Lipschitz-equivalent length function.

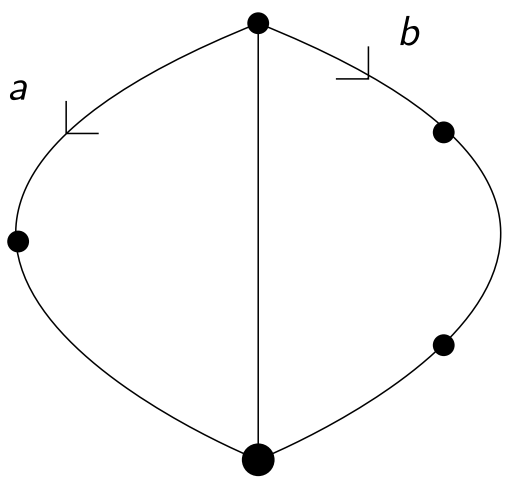


$\ell(a) = \ell(b) = 2$

$\ell(ab) = \ell(ba^{-1}) = 4$

$\ell(a^{-1}b) = 2$

One may also assign variable lengths to edges (e.g. subdivide).



$\ell(a) = 3$, $\ell(b) = 4$

$\ell(ab) = \ell(ba^{-1}) = 7$

$\ell(a^{-1}b) = 5$

Complexity

A right-infinite word on an alphabet has a complexity function $p(n)$, counting subwords of length n.

Right-infinite reduced words on $A^{\pm 1}$ form the boundary of $F(A)$.

A free group F has an intrinsic boundary ∂F (space of ends, Gromov boundary), like a plane has a "circle at infinity". It describes the "ways to go to infinity" in F.

(It is a Cantor set compactifying F, with an action of $\mathrm{Aut}(F)$ by homeomorphisms.)

If A is a basis of F, any $X \in \partial F$ is represented by a right-infinite reduced word X_A in $A^{\pm 1}$. If B is another basis, X_A and X_B differ by an automorphism of F.

X_A has a complexity function $p_A(n)$, so a given $X \in \partial F$ has a complexity function $p_A(n)$ depending on the basis A.

Proposition

If A and B are bases of F, the complexities of X are equivalent: there exists M (depending on A, B) such that

$$p_A(n) \leq M\ p_B(Mn + M).$$

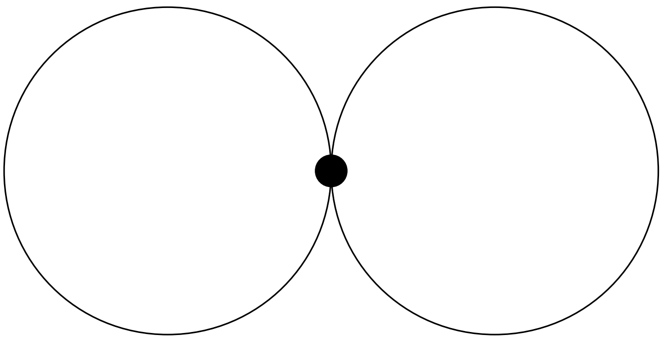
so the following make sense:

X has linear (quadratic, exponential...) complexity

$\frac{p(n)}{n} \to 0 \implies$ *p bounded, and $X = \lim_n g^n$ for some $g \in F$*
(sublinear complexity implies eventually periodic)

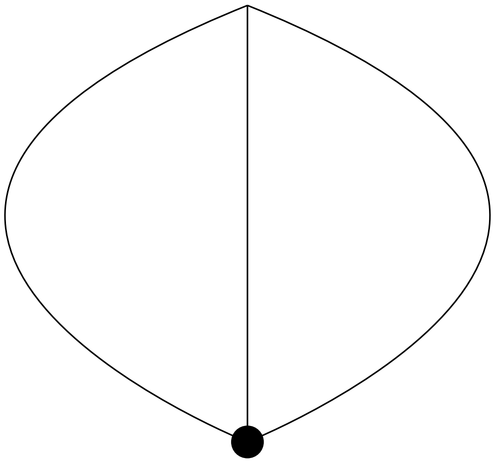
Geometric interpretation

Geometrically, X is an infinite path on the graph, and $p(n)$ is the number of subpaths of length n.



Complexity may be computed (up to equivalence) on any graph with the same fundamental group.

One may assign variable lengths to edges (e.g. subdivide).



Automorphisms of free groups

Automorphisms of free groups, e.g.

$a \mapsto ab^{-1}$, $b \mapsto a^{-1}$

are :

- more general than substitutions (negative letters allowed)
- less general (invertibility required)

Common generalization: endomorphisms of free groups.

Here are 3 nice facts about substitutions:

- the length of $\varphi^n(a)$ grows like $n^d\lambda^n$ as $n \to +\infty$
  e.g. $a \mapsto ab \mapsto aba \mapsto abaab \mapsto \dots$
  $b \mapsto a$
- after replacing φ by a power, $\lim_n \varphi^n(a)$ is a fixed infinite word X (if $\varphi^n(a)$ does grow)
- the complexity of X is severely restricted (Pansiot's theorem)

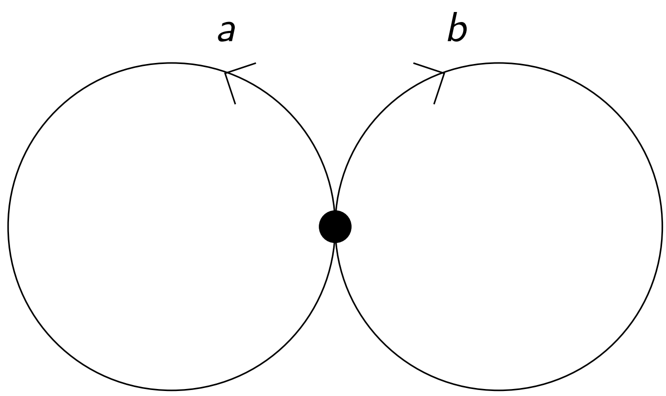
None of these facts is obvious for automorphisms of free groups, because of cancellation.

The key tool for studying automorphisms of free groups is *train tracks* (Bestvina, Feighn, Handel), a way to control cancellation.

Price to pay: forget words, think about graphs!

Train tracks

Any automorphism φ of F is represented by a map on a graph with 1 vertex, with vertex going to vertex and edges going to edge-paths. For instance, $a \mapsto a^2b^{-1}$, $b \mapsto a^{-1}$.

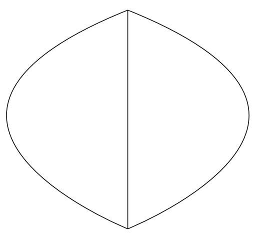


But this map may not be well-adapted (no control over cancellation).

Analogy: given an endomorphism of a vector space, there are bases in which the matrix is especially simple (diagonal, triangular...).

A (relative, completely split,...) train track map is a "nice" representative H of φ on a graph, which may have several vertices if φ is not a substitution (the topology of the graph is dictated by φ).

Silly example: $a \mapsto b$, $b \mapsto a^{-1}b^{-1}$ is best viewed as permuting the edges of



A key fact about train track maps:

Proposition

Given a finite path γ, there exists q such that, when applying powers of H to $H^q(\gamma)$, cancellation only occurs in:

- *Nielsen subpaths*
  $a \mapsto ab,\ b \mapsto a^{-1}$
  $bab^{-1}a^{-1} \mapsto a^{-1}abab^{-1}a^{-1} = bab^{-1}a^{-1} \mapsto \dots$
- *exceptional subpaths*
  $a \mapsto ac,\ b \mapsto c^{-2}b,\ c \mapsto c$
  $ab \mapsto acc^{-2}b = ac^{-1}b \mapsto \dots \mapsto ac^{-q}b \mapsto acc^{-q}c^{-2}b$
- *connecting subpaths*

Results

Cancellation only occurs in slowly growing subpaths, so:

Theorem

If $\varphi \in \mathrm{Aut}(F)$ and $g \in F$, the length of $\varphi^n(g)$ grows like $n^d\lambda^n$ as $n \to +\infty$ (with $d \geq 0$ and $\lambda \geq 1$).

$\varphi^n(g)$ grows like some $H^n(\gamma)$, with γ a loop in a train track

We know how $\varphi^n(g)$ grows, but what happens to it?

Theorem (L.-Lustig)

After replacing φ by a power, the following holds: given $g \in F$, either $\varphi(g) = g$ or $\lim_n \varphi^n(g) = X$ with $X \in \partial F$ fixed by φ. (same behavior as for a substitution)

The proof uses train tracks and geometric machinery, such as real trees.

Pansiot's theorem

Any $\varphi \in \mathrm{Aut}(F)$ of infinite order has φ-periodic infinite words X. What about their complexity?

Recall Pansiot's theorem:

If φ is a substitution on A, and $X = \lim \varphi^n(a)$ is a φ-fixed right-infinite word, its complexity function is Lipschitz-equivalent to one of the following 5 functions:

- 1 (bounded)
- n (linear)
- n^2 (quadratic)
- $n \log n$
- $n \log \log n$

Examples (with invertible substitutions and $X = \lim \varphi^n(a)$):

- $a \mapsto ab$, $b \mapsto b$ $\qquad 1$
- $a \mapsto ab$, $b \mapsto a$ $\qquad n$
- $a \mapsto abc$, $b \mapsto bc$, $c \mapsto c$ $\qquad n^2$
- $a \mapsto a^2bc$, $b \mapsto a$, $c \mapsto cd$, $d \mapsto c$ $\qquad n \log n$
- $a \mapsto abc$, $b \mapsto a$, $c \mapsto cd$, $d \mapsto c$ $\qquad n \log \log n$

These examples show the stratified structure of a train track: edges of the graph are grouped into strata (colors), the set of strata is linearly ordered: red $<$ blue $<$ black, and the image of an edge contains no edge belonging to a higher stratum.

This allows inductive arguments.

Theorem (Hilion-L.)

Pansiot's theorem extends to the complexity of fixed infinite words $X = \lim_n \varphi^n(g)$ in ∂F, for $\varphi \in \mathrm{Aut}(F)$.

Besides combinatorial arguments, the proof is based on the proposition controlling cancellation in iterated images of paths in train tracks.

The complexity of X depends on the nature of the strata involved.