Weakly Unambiguous Morphisms with Respect to Sets of Patterns with Constants

Aleksi Saarela

Department of Mathematics and Statistics
and FUNDIM Centre,
University of Turku, Finland

WORDS 2013
Turku

Outline

Basic Definitions

Let Σ be an alphabet of *constants* and Ξ an alphabet of *variables*.

- A word $\alpha \in \Xi^+$ is called a (constant-free) *pattern*.
- The set of variables in α is denoted by $\mathrm{var}(\alpha)$.
- The *empty word* is denoted by ε.
- A *morphism* is a mapping $h : \Xi^* \to \Sigma^*$ such that

$$h(\alpha\beta) = h(\alpha)h(\beta) \qquad \text{for all } \alpha, \beta \in \Xi^*.$$

- A morphism h is *non-erasing* (w.r.t. α) if $h(x) \neq \varepsilon$ for all $x \in \mathrm{var}(\alpha)$.
- A non-erasing morphism h is *length-increasing* (w.r.t. α) if $|h(x)| \geq 2$ for at least one $x \in \mathrm{var}(\alpha)$.

Unambiguity

A morphism h is *(strongly) unambiguous w.r.t. a pattern* α if there is no other morphism g such that $h(\alpha) = g(\alpha)$.

"Other" should be interpreted w.r.t. α.

Two questions that have been studied are:

- For which patterns does there exist an unambiguous morphism?
- For which patterns does there exist a non-erasing unambiguous morphism?

(Morphisms h for which $h(\alpha) = \varepsilon$ do not count.)

Examples

Example 1

Consider the pattern $xxyy$.

- The morphism defined by $x \mapsto a$, $y \mapsto b$ is unambiguous w.r.t. $xxyy$.

Consider the pattern xyy.

- The morphism defined by $x \mapsto a$, $y \mapsto \varepsilon$ is unambiguous w.r.t. xyy.
- If h is a non-erasing morphism and g is defined by $g(x) = h(xyy)$, $g(y) = \varepsilon$, then $h(xyy) = g(xyy)$. Thus no non-erasing morphism is unambiguous w.r.t. xyy.

Consider the pattern $xyxy$.

- If h is a morphism and g_1 is defined by $g_1(x) = h(xy)$, $g_1(y) = \varepsilon$ and g_2 is defined by $g_2(x) = \varepsilon$, $g_2(y) = h(xy)$, then $h(xyxy) = g_1(xyxy) = g_2(xyxy)$ and at least one of g_1, g_2 is not h. Thus no morphism is unambiguous w.r.t. $xyxy$.

One Result

Theorem 2 (Freydenberger, Reidenbach and Schneider, 2006)

Let α be a pattern.
There is a non-erasing morphism that is unambiguous w.r.t. α
if and only if
α is not the fixed point of any nontrivial morphism.

Outline

Weak Unambiguity

A non-erasing morphism h is *weakly unambiguous w.r.t. a pattern* α if there is no other non-erasing morphism g such that $h(\alpha) = g(\alpha)$.

If h is not length-increasing, then h is weakly unambiguous w.r.t. every pattern. Thus the interesting question in this case is the following:

- For which patterns does there exist a length-increasing weakly unambiguous morphism?

Example 3

Consider the pattern xxy.
The morphism defined by $x \mapsto ab, y \mapsto ba$ is weakly unambiguous w.r.t. xxy, because no other non-erasing morphism maps xxy to $ababb$.
The morphism defined by $x \mapsto aa, y \mapsto b$ is not weakly unambiguous w.r.t. xxy, because also the morphism defined by $x \mapsto a, y \mapsto aab$ maps xxy to $aaaab$.

Neighbors

Let $\alpha = a_0 a_1 \dots a_n a_{n+1}$, where $a_0 = a_{n+1} = \varepsilon$ and $a_1, \dots, a_n \in \Xi$. The set of *left neighbors of x in α* is

$$L_\alpha(x) = \{a_i \mid 0 \leq i \leq n, a_{i+1} = x\},$$

and the set of *right neighbors of x in α* is

$$R_\alpha(x) = \{a_i \mid 1 \leq i \leq n+1, a_{i-1} = x\}.$$

Example 4

If $\alpha = xyzxy$, then $L_\alpha(x) = \{\varepsilon, z\}$ and $R_\alpha(x) = \{y\}$.

Loyal Neighbors

Given a pattern α, a variable x has *loyal neighbors in* α if at least one of the following two conditions is satisfied:

$$\varepsilon \notin L_\alpha(x) \text{ and } R_\alpha(y) = \{x\} \text{ for all } y \in L_\alpha(x),$$
$$\varepsilon \notin R_\alpha(x) \text{ and } L_\alpha(y) = \{x\} \text{ for all } y \in R_\alpha(x).$$

Example 5

Let $\alpha = xyzy$. The variable y has loyal neighbors in α because $L_\alpha(y) = \{x, z\}$ and $R_\alpha(x) = R_\alpha(z) = \{y\}$. The other variables do not have loyal neighbors in α:

- x does not, because $\varepsilon \in L_\alpha(x)$, and $R_\alpha(x) = \{y\}$ but $L_\alpha(y) \neq \{x\}$.
- z does not, because $L_\alpha(z) = \{y\}$ but $R_\alpha(y) \neq \{z\}$, and $R_\alpha(z) = \{y\}$ but $L_\alpha(y) \neq \{z\}$.

Result

Theorem 6 (Freydenberger, Nevisi and Reidenbach, 2012)

Let $\#\Sigma \geq 3$ and let α be a pattern.
There is a length-increasing morphism that is weakly unambiguous w.r.t. α if and only if
at least one variable does not have loyal neighbors in α.

- In the case $\#\Sigma = 1$ there is a different characterization.
- In the case $\#\Sigma = 2$ there are only partial results.

Outline

Basic Definitions

Let Σ be an alphabet of *constants* and Ξ an alphabet of *variables*.

- A word $\alpha \in (\Xi \cup \Sigma)^+$ is called a *pattern* (with constants).
- A *morphism* is a mapping $h : (\Xi \cup \Sigma)^* \to \Sigma^*$ such that

$$\begin{aligned} h(\alpha\beta) &= h(\alpha)h(\beta) && \text{for all } \alpha, \beta \in (\Xi \cup \Sigma)^* \text{ and} \\ h(a) &= a && \text{for all } a \in \Sigma. \end{aligned}$$

Thus all morphisms are assumed to be constant-preserving.

Weak Unambiguity

A non-erasing morphism h is *weakly unambiguous w.r.t. a pattern* α if there is no other non-erasing morphism g such that $h(\alpha) = g(\alpha)$.

If h is not length-increasing, then h is weakly unambiguous w.r.t. every pattern. Thus the interesting question in this case is the following:

- For which patterns does there exist a length-increasing weakly unambiguous morphism?

Example 7

Let $\Xi = \{x, y\}$ and $\Sigma = \{a, b\}$. Consider the pattern xay.
The morphism defined by $x \mapsto a, y \mapsto ba$ is weakly unambiguous w.r.t. xay, because no other non-erasing morphism maps xay to $aaba$.
The morphism defined by $x \mapsto a, y \mapsto ab$ is not weakly unambiguous w.r.t. xay, because also the morphism defined by $x \mapsto aa, y \mapsto b$ maps xay to $aaab$.

Neighbors

Let $\alpha = a_0 a_1 \dots a_n a_{n+1}$, where $a_0 = a_{n+1} = \varepsilon$ and $a_1, \dots, a_n \in \Xi \cup \Sigma$. The set of *left neighbors of* x *in* α is

$$L_\alpha(x) = \{a_i \mid 0 \leq i \leq n, a_{i+1} = x\},$$

and the set of *right neighbors of* x *in* α is

$$R_\alpha(x) = \{a_i \mid 1 \leq i \leq n+1, a_{i-1} = x\}.$$

Example 8

If $\alpha = xyaxy$, then $L_\alpha(x) = \{\varepsilon, a\}$ and $R_\alpha(x) = \{y\}$.

Loyal Neighbors

Given a pattern α with constants, a variable x has *loyal neighbors in* α if at least one of the following two conditions is satisfied:

$$L_\alpha(x) \subseteq \Xi \text{ and } R_\alpha(y) = \{x\} \text{ for all } y \in L_\alpha(x),$$
$$R_\alpha(x) \subseteq \Xi \text{ and } L_\alpha(y) = \{x\} \text{ for all } y \in R_\alpha(x).$$

Example 9

Let $\Xi = \{x, y, z, t\}$, $\Sigma = \{a\}$, and $\alpha = xayzyt$. The variable y has loyal neighbors in α because $R_\alpha(y) = \{z, t\}$ and $L_\alpha(z) = L_\alpha(t) = \{y\}$. The other variables do not have loyal neighbors in α:

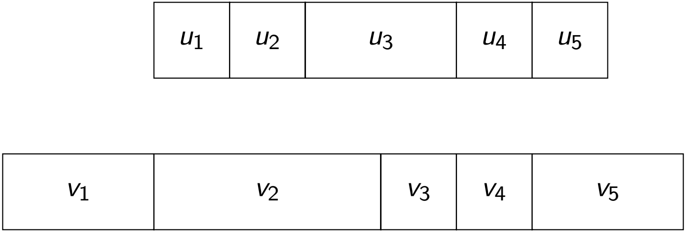
- x does not, because $\varepsilon \in L_\alpha(x)$ and $a \in R_\alpha(x)$.
- z does not, because $L_\alpha(z) = \{y\}$ but $R_\alpha(y) \neq \{z\}$, and $R_\alpha(z) = \{y\}$ but $L_\alpha(y) \neq \{z\}$.
- t does not, because $L_\alpha(t) = \{y\}$ but $R_\alpha(y) \neq \{t\}$, and $\varepsilon \in R_\alpha(t)$.

Outline

Lemmas

Lemma 10

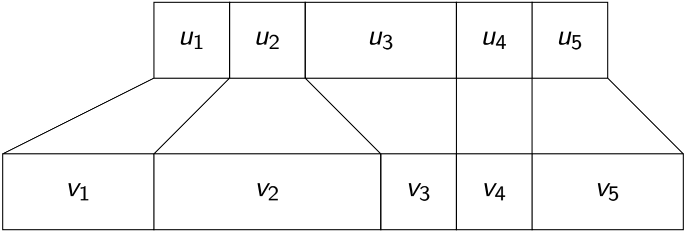
Let $u_1, \ldots, u_n, v_1, \ldots, v_n \in \Sigma^*$. If $u_1 \ldots u_n$ is a factor of $v_1 \ldots v_n$, then either $u_i = v_i$ for all i or u_i is a proper factor of v_i for some i.



Lemmas

Lemma 10

Let $u_1, \ldots, u_n, v_1, \ldots, v_n \in \Sigma^*$. If $u_1 \ldots u_n$ is a factor of $v_1 \ldots v_n$, then either $u_i = v_i$ for all i or u_i is a proper factor of v_i for some i.



Lemmas

Lemma 11

Let α be a pattern and h a non-erasing morphism. If there is $x \in \Xi$ such that $|h(x)| > 1$ and x has loyal neighbors in α, then h is not weakly unambiguous w.r.t. α.

Proof (sketch).

- WLOG, let $L_\alpha(x) \subseteq \Xi$ and $R_\alpha(y) = \{x\}$ for all $y \in L_\alpha(x)$.
- Let $h(x) = au$ where $a \in \Sigma$ and $u \in \Sigma^+$.
- Let g be defined by $g(x) = u$, $g(y) = h(y)a$ for all $y \in L_\alpha(x)$ and $g(z) = h(z)$ for all other $z \in \Xi$.
- Then $h(\alpha) = g(\alpha)$.

□

Lemmas

Lemma 12

Let α be a pattern and x a variable that does not have loyal neighbors in α. Let $a, b, c \in \Sigma$ be different letters such that $L_\alpha(x) \cap \Sigma \neq \{a\}$ and $R_\alpha(x) \cap \Sigma \neq \{b\}$. The morphism h defined by $h(x) = ab$ and $h(y) = c$ for all $y \in \Xi \setminus \{x\}$ is weakly unambiguous w.r.t. α.

Proof (sketch).

- Let $g \neq h$ be a length-increasing morphism and $h(\alpha) = g(\alpha)$.
- Lemma 10 implies that $g(x)$ is a proper factor of $h(x)$, say $h(x) = a$.
- $|g(y)|_a = 0$ for all $y \neq x$, because otherwise $|g(\alpha)|_a > |h(\alpha)|_a$.
- $g(y)$ begins with b for all $y \in R_\alpha(x)$ and $R_\alpha(x) \subseteq \Xi$.
- x does not have loyal neighbors, so $|g(\alpha)|_b > |h(\alpha)|_b$ (contradiction).

$\square$

Result

Theorem 13

Let $\#\Sigma \geq 3$ and let α be a pattern.
There is a length-increasing morphism that is weakly unambiguous w.r.t. α if and only if
at least one variable does not have loyal neighbors in α.

Proof.

Assume first that all variables have loyal neighbors in α and h is a length-increasing morphism. Then some variable x satisfies the conditions of Lemma 11, so h is not weakly unambiguous w.r.t. α.
Assume then that a variable x does not have loyal neighbors in α. Because $\#\Sigma \geq 3$, the three letters of Lemma 12 exist, and there is a length-increasing morphism that is weakly unambiguous w.r.t. α. □

Outline

Result

Theorem 14

Let $\#\Sigma \geq n + 2$ and let $\alpha_1, \ldots, \alpha_n$ be patterns.
There is a morphism h that is length-increasing w.r.t. every α_i and weakly unambiguous w.r.t. every α_i
if and only if
at least one variable does not have loyal neighbors in any α_i.

The assumption $\#\Sigma \geq n + 2$ is necessary.

Example

Example 15

Let $\Xi = \{x, y_1, y_2, z_1, z_2, t_1, t_2\}$ and $\Sigma = \{a_1, \dots, a_n, b\}$. Let $a_0 = a_n$ and

$$\alpha_i = y_1 y_2 a_i x z_1 z_2 x a_{i+1} t_1 t_2$$

for $i \in \{0, \dots, n-1\}$. The variable x does not have loyal neighbors in any α_i, but there does not exist a length-increasing morphism that would be weakly unambiguous w.r.t. every α_i:

- If h would be a length-increasing morphism that is weakly unambiguous w.r.t. α_0, then $|h(x)| > 1$ by Lemma 11.
- If $h(x)$ starts with a_i, then h is not weakly unambiguous w.r.t. α_i.
- If $h(x)$ ends with a_{i+1}, then h is not weakly unambiguous w.r.t. α_i.
- If $h(x) = bub$, then h is not weakly unambiguous w.r.t. any α_i.

Outline

Further Questions

- What about the case $\#\Sigma = 2$?
  (Open for constant-free patterns also.)
- What about strong unambiguity?
  (More important, potential for applications.)

Thank You!